Absolute separability witnesses for symmetric multiqubit states

E. Serrano-Ensástiga*, J. Denis, and J. Martin

Institut de Physique Nucléaire, Atomique et de Spectroscopie, CESAM, University of Liège, Liège, Belgique

*ed.ensastiga@uliege.be

Abstract

Entanglement is a valuable resource for quantum applications, and a well-established method for creating entangled multiqubit symmetric states in a controlled manner is the application of a global unitary operation. However, certain states, called symmetric absolutely separable (SAS), remain unentangled after any unitary gate preserving permutation invariance in the constituents of the system. In this work, we develop criteria for detecting SAS states of any number of qubits [1, 2]. Our approach is based on the Glauber-Sudarshan P representation for finite-dimensional quantum systems. We introduce families of linear and non-linear SAS witnesses formulated respectively as algebraic inequalities or a quadratic optimization problem. These witnesses are capable of identifying more SAS states than previously known counterparts [3].

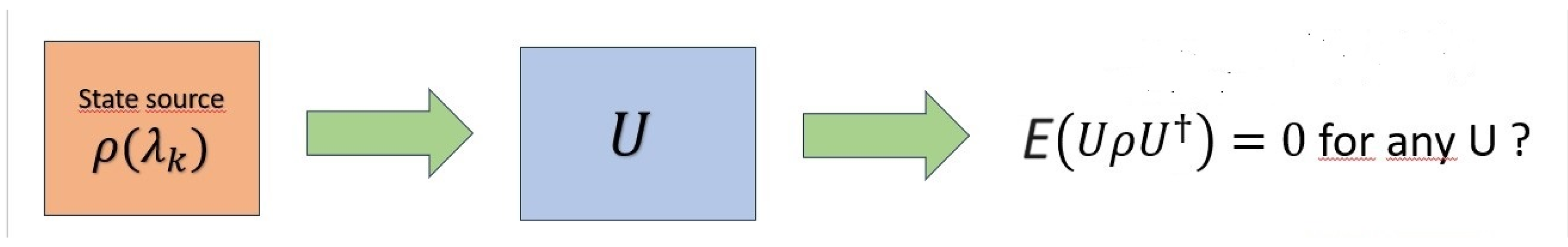
Motivation

We use quantum correlations of a quantum system to create/enhanced technology. For a particular task, the optimal quantum states of a system are those who maximize a particular quantum correlation (entanglement, QFI, fidelity, ...), which are in most of the cases maximized by pure states, *i.e.*, by ensembles of 100% of the same state, $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \dots) = (1, 0, 0, \dots)$. However, pure states of a quantum system are ideal scenarios. They face many experimental challenges: decoherence, finite temperatures, marginal densities, ...

Problem statement

For a symmetric state of N qubits ρ, which mixtures $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_N)$ are absolutely separable (not entangled) states? We called these states **symmetric absolutely separable** (SAS). A way to solve this problem is using:

1) The unitary orbit of the state $U\rho U^\dagger$, with $U \in SU(N+1)$



2) The formal definition of separability for symmetric, which is given by the Glauber-Sudarshan P representation: A state ρ is SAS if there exits a function $P(U\rho U^\dagger, \Omega)$ on the sphere such that

$$U\rho U^\dagger = \int_{S^2} P\left(U\rho U^\dagger, \Omega\right) \left(D(\Omega) \left|\hat{z}\right\rangle \left\langle\hat{z}\right| D^\dagger(\Omega)\right)^{\otimes N} \mathrm{d}\Omega \,.$$

with $\Omega = (\theta, \phi)$ and

$$P\left(U\rho U^\dagger, \Omega\right) \geqslant 0 \qquad \forall\, \Omega \in S^2, \quad U \in SU(N+1).$$

We take advantage of the non-uniqueness of the P-function [3]

$$P(\rho\,, \boldsymbol{n}) = \underbrace{\sum_{L=0}^{N} \sum_{M=-L}^{L} y_{LM} Y_{LM}(\boldsymbol{n})}_{P_0 = \mathrm{Tr}(\rho\omega^{(1)}(\boldsymbol{n}))\ ,\ \text{unique for } \rho} + \underbrace{\sum_{L=N+1}^{\infty} \sum_{M=-L}^{L} y_{LM} Y_{LM}(\boldsymbol{n})}_{P'\ ,\ \text{arbitrary } y_{LM}} \,,$$

to build functions that are easy to study over the unitary orbit.

SAS witnesses

$\mathcal{W}_1$: ρ is SAS if $\boldsymbol{\lambda}^{\downarrow} \boldsymbol{\Delta}^{\uparrow T} \geqslant 0\,, \qquad \Delta_k = (-1)^{N-k} \binom{N+1}{k}\,,$

$\mathcal{W}_2(\{y_L\})$: ρ is SAS if for some values of $\{y_L\}$

$$\min_{\substack{\boldsymbol{\lambda}B \\ B \in \mathcal{B}_{N+1}}} P_{LB}(y_L; \boldsymbol{\lambda}B) = \min_{\substack{\boldsymbol{\lambda}B \\ B \in \mathcal{B}_{N+1}}} f + \sum_{L=1}^{2j} \left[g_L \boldsymbol{\lambda} B\, \mathbf{t}_L^T + h_L \left(\boldsymbol{\lambda} B\, \mathbf{t}_L^T\right)^2 \right] \geqslant 0\,,$$

where $\mathcal{B}_{N+1}$ is the set of bistochastic matrices and f, g_L and h_L are functions dependent only on the eigenspectrum $\boldsymbol{\lambda}$.

$\mathcal{W}_3$: ρ is SAS if

$$r^2 \leqslant \frac{1}{(2j+1)^2} \left(\sum_{L=1}^{N} \frac{g_L^2}{1 - 2\Theta\left(L - \frac{N}{2}\right) \frac{F(L,0)}{F(L,1)}} \right)^{-1} ,$$

where $r^2 \equiv \mathrm{Tr}(\rho^2) - (N+1)^{-1} = \sum_k \lambda_k^2 - (N+1)^{-1}$ and $F(L,k)$ constant numbers. See more details in [2].

SAS witnesses

Number of qubits $N = 2j$	Witness $\mathcal{W}_1$ / Witness $\mathcal{W}_3$
2	$\boldsymbol{\lambda}\,(-3,\,1,\,3)^T \geqslant 0$ $r^2 \leqslant \frac{1}{78} \approx 0.01282$
3	$\boldsymbol{\lambda}\,(-6,\,-1,\,4,\,4)^T \geqslant 0$ $r^2 \leqslant \frac{1}{354} \approx 0.002825$
4	$\boldsymbol{\lambda}\,(-10,\,-5,\,1,\,5,\,10)^T \geqslant 0$ $r^2 \leqslant \frac{11}{25390} \approx 0.0004332$
5	$\boldsymbol{\lambda}\,(-15,\,-15,\,-1,\,6,\,6,\,20)^T \geqslant 0$ $r^2 \leqslant \frac{1595}{16058598} \approx 0.00009932$

Table 1. SAS witnesses $\mathcal{W}_1$ and $\mathcal{W}_3$ for a state with eigenspectrum $\boldsymbol{\lambda}$ sorted in descending order $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_N$.

Dark Brown = $\mathcal{S}_2(\{y_L\})$ Orange surface = Bound of $\mathcal{S}_3$
Light Brown = $\mathcal{S}_1$ Blue surface = Bound of $\mathcal{S}$ [3]
Green = Unwitnessed SAS states by $\mathcal{W}_k$

Case $N = 2$

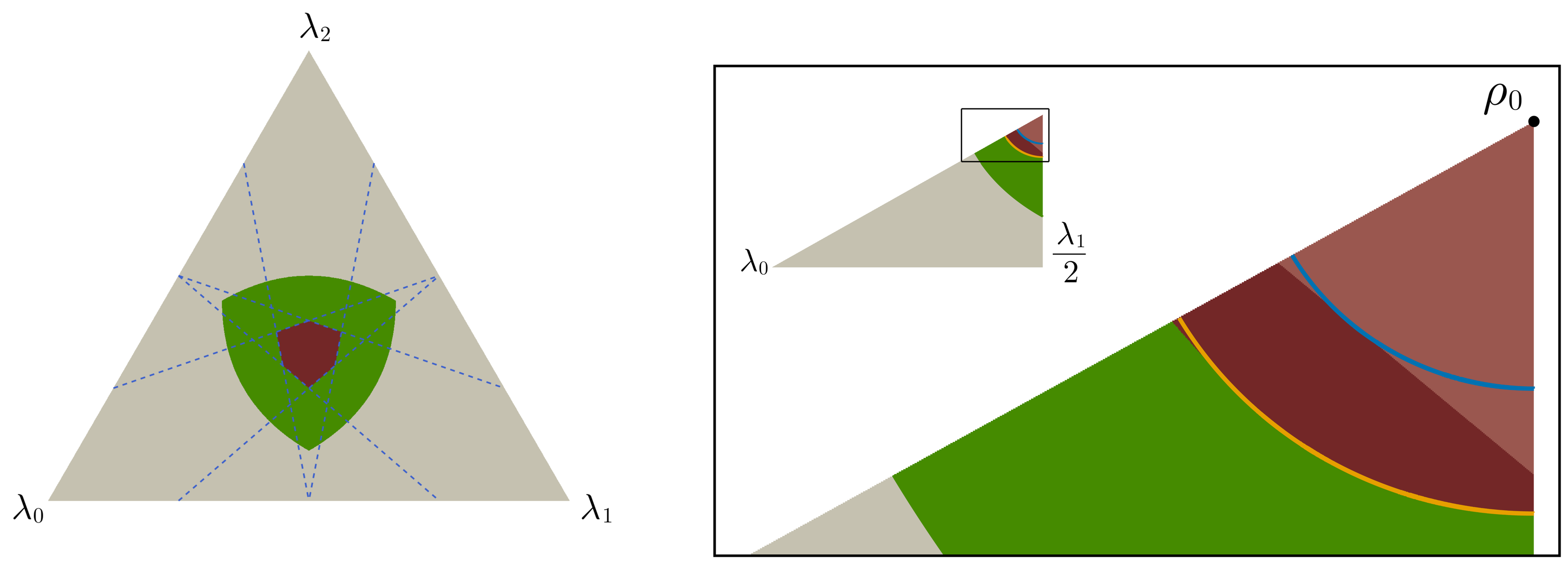


FIG. 1. SAS states witnessed by $\mathcal{W}_k$ for N = 2 in the simplex of eigenvalues $\boldsymbol{\lambda}$ in barycentric coordinates. The full set of SAS states was characterized in Corollary 1 of Ref. [1]: ρ is SAS if and only if $\sqrt{\lambda_1} + \sqrt{\lambda_2} \geqslant 1$.

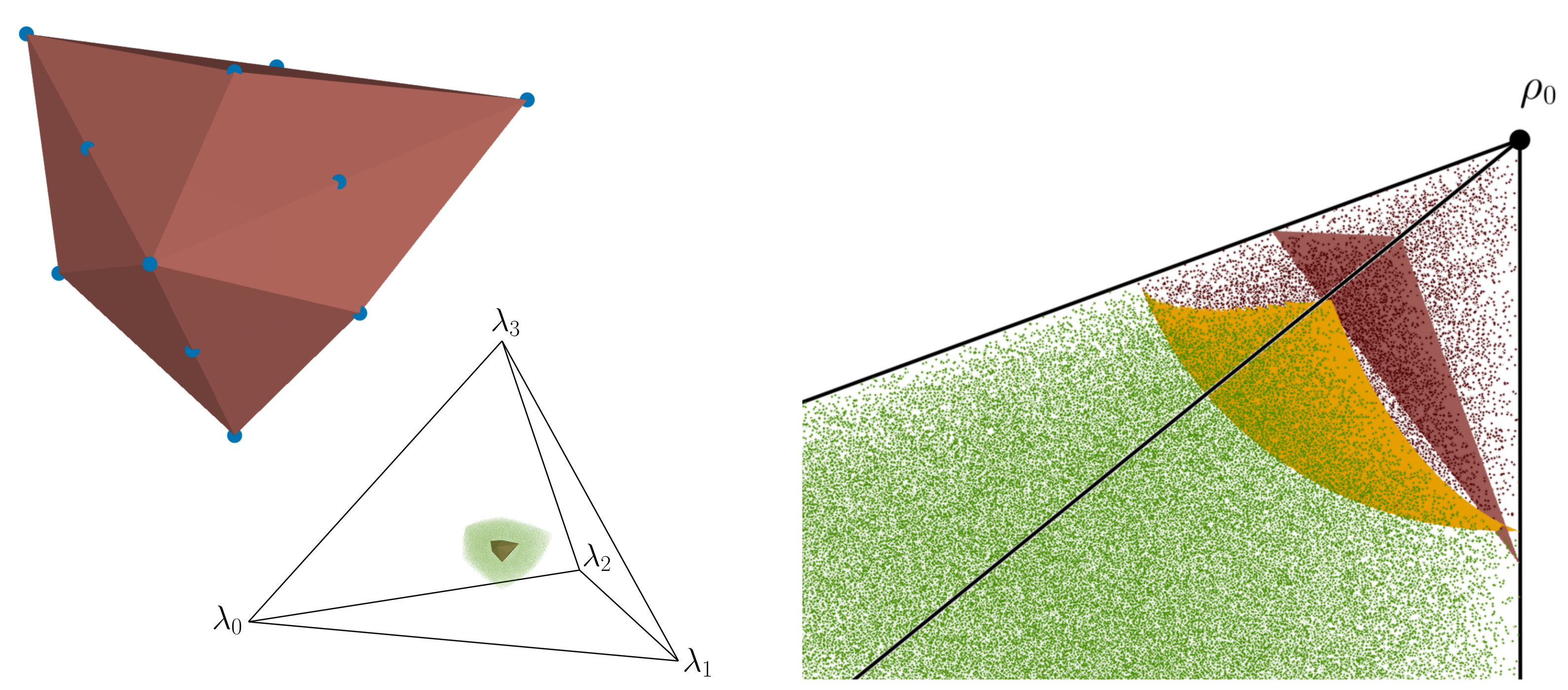


FIG. 2. SAS states witnessed by $\mathcal{W}_k$ for N = 3. The set of SAS states was calculated numerically in Ref. [1].

Bibliography

[1] E. Serrano-Ensástiga and J. Martin, SciPost Phys. **15**, 120 (2023).

[2] E. Serrano-Ensástiga, J. Denis, and J. Martin, Phys. Rev. A **109**, 022430 (2024).

[3] F. Bohnet-Waldraff, O. Giraud, and D. Braun, Phys. Rev. A **95**, 012318 (2017).