Maximum entanglement of symmetric states under unitary gates

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Outline of the talk

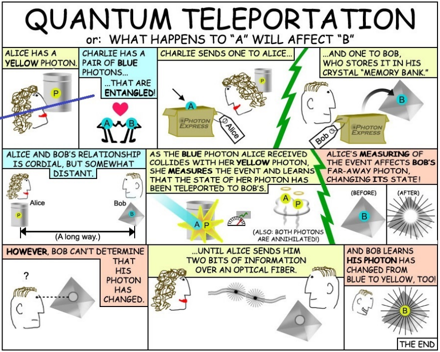
1. Statement of the problem
2. Results
   a. Symmetric 2-qubit system (Analytical results)
   b. Applications: Hamiltonian at finite temperatures
   c. Symmetric 3-qubit system (Numerical results)
3. Conclusions

Quantum technology

Quantum Technology: The Second Quantum Revolution, Dowling and Milburn (2003)

The goal of quantum technology is to deliver useful devices and processes that are based on quantum principles: entanglement, quantum superposition, etc.

Maximally entangled pure states



In most of the cases, the best states to create/enhance a technology are those which maximize a quantum correlation: **entanglement**, QFI, anticoherence, fidelity, etc.

Optimal quantum states

Pure states, $\mathcal{H}$

$$\rho = |\psi\rangle\langle\psi|$$

Mixed states

$$\rho = \sum_{\alpha=1}^{d} \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

$\lambda_\alpha \geqslant 0, \qquad \sum_\alpha \lambda_\alpha = 1$

$\{\psi_\alpha\}$ orthonormal basis

Entanglement is convex under mixture of states

$$E\left(\sum_\alpha \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|\right) \leqslant \sum_\alpha \lambda_\alpha E\left(|\psi_\alpha\rangle\langle\psi_\alpha|\right) ,$$

which means that the states that maximize a quantum correlation are pure states. However...

Pure vs mixed states

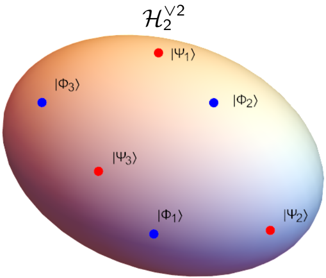
Ideal vs Reality

Pure states of a quantum system are ideal scenarios. They face many experimental challenges

Mixedness

- Open quantum systems (Decoherence) $|\psi\rangle \Rightarrow_t \rho$
- Finite temperatures $\rho = \frac{e^{-H/k_B T}}{Z}$
- Marginal densities $\rho_{\text{spin}} = \text{Tr}_{\text{spatial}}\left(|\Psi\rangle\langle\Psi|\right)$
- Particle loss $\rho = \text{Tr}_1(|\Psi\rangle\langle\Psi|)$

Maximizing a quantum correlation for a mixed state



Global unitary transformation

$$\rho = \sum_{k=1}^{d} \lambda_k |\phi_k\rangle\langle\phi_k| \,,$$

$$U\rho U^\dagger = \sum_{k=1}^{d} \lambda_k |\psi_k\rangle\langle\psi_k| \,,$$

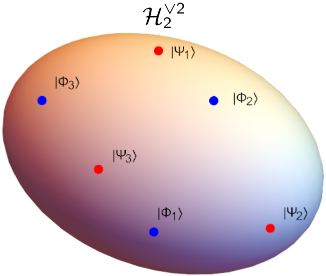
Maximum entanglement by spectrum

$$\max_{U \in SU(d)} E\left(U\rho U^\dagger\right)$$

Global unitary transformations = Group of Schrödinger evolutions
= Set of thermal states with fixed T and fixed eigenenergies
= Group of unitary gates of a quantum protocol

Entanglement (Pure state case)

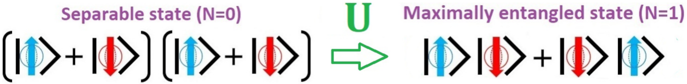
Not-invariant under **global** unitary transformations $SU(3)$



Global unitary transformation

$$\rho_S = \sum_{k=0}^{2} \lambda_k |\phi_k\rangle\langle\phi_k| \,,$$

$$U_S \rho_S U_S^\dagger = \sum_{k=0}^{2} \lambda_k |\psi_k\rangle\langle\psi_k| \,,$$

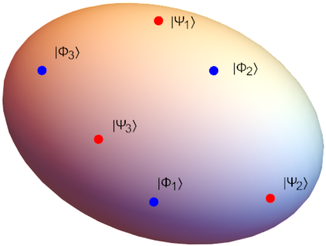


Pure state ρ_{pure}

$$\lambda_0 = 1 \,, \lambda_1 = \lambda_2 = 0 \,, \qquad \max_{U_S \in SU(3)} \mathcal{N}(U_S \rho_{pure} U_S^\dagger) = 1 \,,$$

Maximum entanglement

Not-invariant under **global** unitary transformations $SU(3)$



Global unitary transformation

$$\rho_S = \sum_{k=0}^{2} \lambda_k |\phi_k\rangle\langle\phi_k| \,,$$

$$U_S \rho U_S^\dagger = \sum_{k=0}^{2} \lambda_k |\psi_k\rangle\langle\psi_k| \,,$$

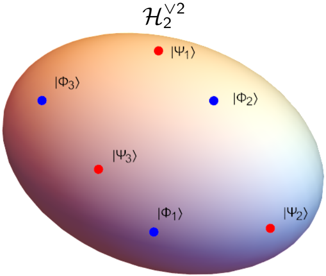
$$\rho_0 = U_S \rho_0 U_S^\dagger = \frac{1}{3}\mathbb{1} = \frac{1}{4\pi}\int_{S^2} |\boldsymbol{n}\rangle|\boldsymbol{n}\rangle\langle\boldsymbol{n}|\langle\boldsymbol{n}| \,\mathrm{d}^2\boldsymbol{n} \,.$$

Maximally mixed state ρ_0

$$\lambda_0 = \lambda_1 = \lambda_2 = 1/3 \,, \qquad \max_{U_S \in SU(3)} \mathcal{N}(U_S \rho_0 U_S^\dagger) = 0 \,,$$

Entanglement (Maximally mixed state case)

Not-invariant under **global** unitary transformations $SU(3)$



Global unitary transformation

$$\rho_S = \sum_{k=0}^{2} \lambda_k |\phi_k\rangle\langle\phi_k| \, ,$$

$$U_S \rho_S U_S^\dagger = \sum_{k=0}^{2} \lambda_k |\psi_k\rangle\langle\psi_k| \, ,$$

Questions

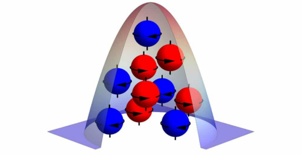
- For a generic mixture, $(\lambda_0, \lambda_1, \lambda_2)$, what is the maximum entanglement of ρ_S attained in its $SU(3)$-orbit?

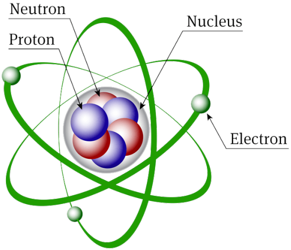
$$\max_{U_S \in SU(3)} \mathcal{N}(U_S \rho_S U_S^\dagger)$$

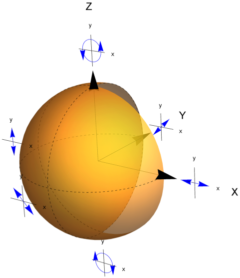
- Is ρ_0 the unique state that is symmetric absolutely separable (SAS) over all its unitary orbit?

Symmetric case

BEC, spin-j systems, multiphotonic systems, etc.







Symmetric states for qubit-qubit systems

$$|1,1\rangle = (|\uparrow\rangle|\uparrow\rangle)$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)$$
$$|1,-1\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle|\downarrow\rangle)$$
$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \text{ (Forbidden state)}$$

Symmetric bipartite systems

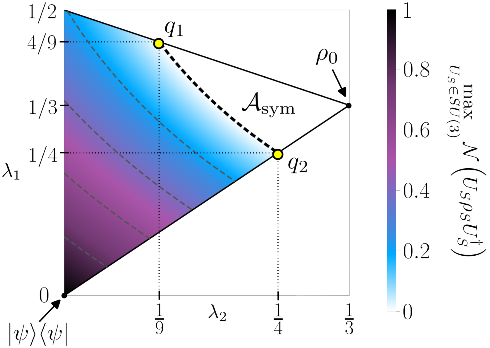
Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$	Symmetric 2-qubit system $\mathcal{H}_2^{\vee 2}$
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$	ρ_S-spectrum: $(\lambda_0, \lambda_1, \lambda_2, 0)$
$\max_{U \in SU(4)} \mathcal{N}\left(U\rho U^\dagger\right)$	$\max_{U_S \in SU(3)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right)$
Qubit-qutrit system $\mathcal{H}_2 \otimes \mathcal{H}_3$	**Symmetric 3-qubit system $\mathcal{H}_2^{\vee 3}$**
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$	ρ_S-spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, 0, 0)$
$\max_{U \in SU(6)} \mathcal{N}\left(U\rho U^\dagger\right)$	$\max_{U_S \in SU(4)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right)$

Symmetric 2-qubit system

Theorem [ESE, Martin (2023)]

Let $\rho_S \in \mathcal{B}(\mathcal{H}_2^{\vee 2})$ with spectrum $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2$. It holds that

$$\max_{U_S \in SU(3)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right) = \max\left(0, \sqrt{\lambda_0^2 + (\lambda_1 - \lambda_2)^2} - \lambda_1 - \lambda_2\right).$$



Maximally entangled state

$$\rho_S = \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

Corollary [ESE, Martin (2023)]

ρ_S is SAS iff

$$\sqrt{\lambda_1} + \sqrt{\lambda_2} \geq 1\,.$$

Applications

Symmetric qubit-qubit system at finite temperature

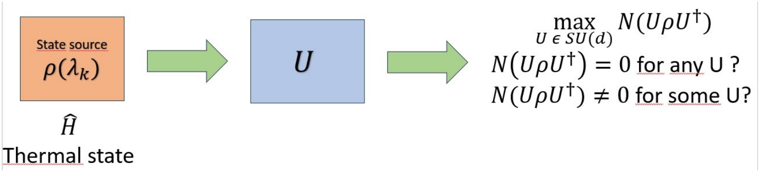
Hamiltonian: BEC [Ribeiro, Vidal Mosseri (2007)], Lipkin-Meshkov-Glick model (1965)

$$H = gJ_z + \gamma_x J_x^2 + \gamma_z J_z^2 \,,$$

with eigenenergies ϵ_j.

State at finite temperature T

$$\lambda_k = \frac{e^{-\beta \epsilon_{2s+2-k}}}{Z} \,, \quad \text{with} \quad Z = \mathrm{Tr}\left(e^{-\beta H}\right) \,.$$

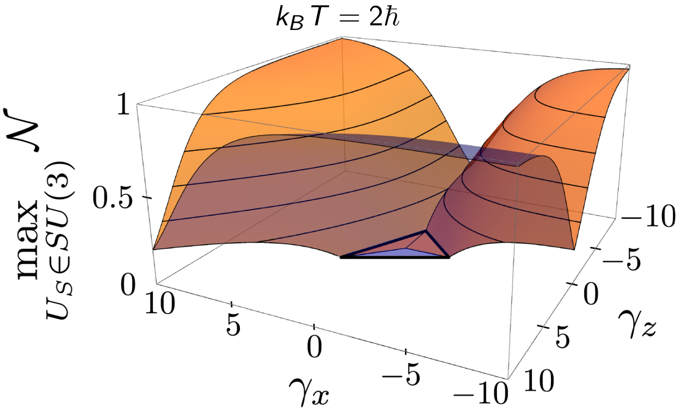


Maximum entanglement

Spectrum ϵ_j of H

For $g = 0$

$$\{\gamma_x\,, \gamma_z\,, \gamma_x + \gamma_z\}\;.$$



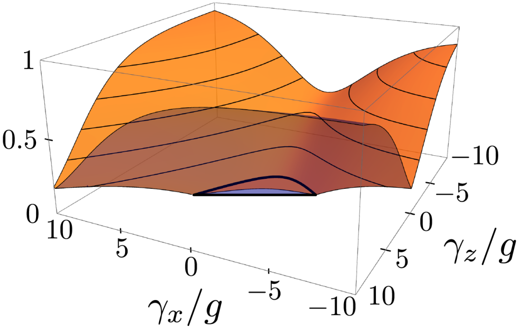
Maximum entanglement

Spectrum ϵ_j of H

For $g \neq 0$

$$\left\{ \gamma_x \, , \frac{1}{2}\left(\gamma_x + 2\gamma_z - \sqrt{4g^2 + \gamma_x^2}\right) \, , \frac{1}{2}\left(\gamma_x + 2\gamma_z + \sqrt{4g^2 + \gamma_x^2}\right) \right\} .$$

$k_B T = 3\hbar$



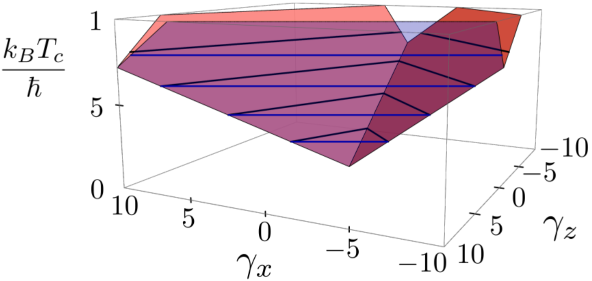
Applications

Symmetric qubit-qubit system at finite temperatures

Condition of SAS states

$$\rho \in \mathcal{A}_{\mathrm{sym}} \Leftrightarrow \frac{k_B T}{\hbar} \geqslant \frac{\epsilon_1 + \epsilon_2 - 2\epsilon_3}{2\ln 2},$$

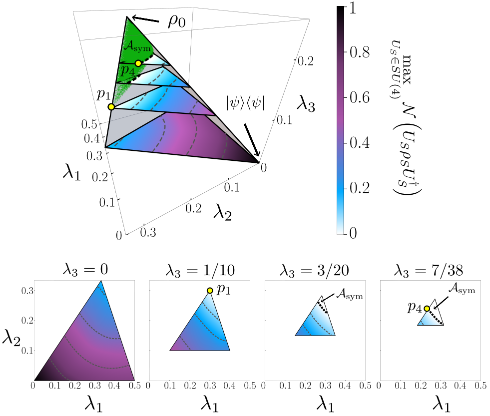
$g = 0$



Symmetric 3-qubit system

Symmetric 3-qubit system

$\mathcal{H}_2^{\vee 3} \subset \mathcal{H}_2 \otimes \mathcal{H}_3$, numerical results



Maximally entangled states in the $SU(4)$-orbit

$$\rho_S = \begin{pmatrix} \lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix},$$

$$\rho_S = \begin{pmatrix} \frac{\lambda_0+\lambda_3}{2} & 0 & 0 & \frac{\lambda_0-\lambda_3}{2} \\ 0 & \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1-\lambda_2}{2} & 0 \\ 0 & \frac{\lambda_1-\lambda_2}{2} & \frac{\lambda_1+\lambda_2}{2} & 0 \\ \frac{\lambda_0-\lambda_3}{2} & 0 & 0 & \frac{\lambda_0+\lambda_3}{2} \end{pmatrix}$$

Conclusions

Maximum entanglement (negativity) over the unitary orbit for $N = 2, 3$

- Characterization of the SAS states for symmetric 2-qubit system
- Numerical study of the SAS states for symmetric 3-qubit system

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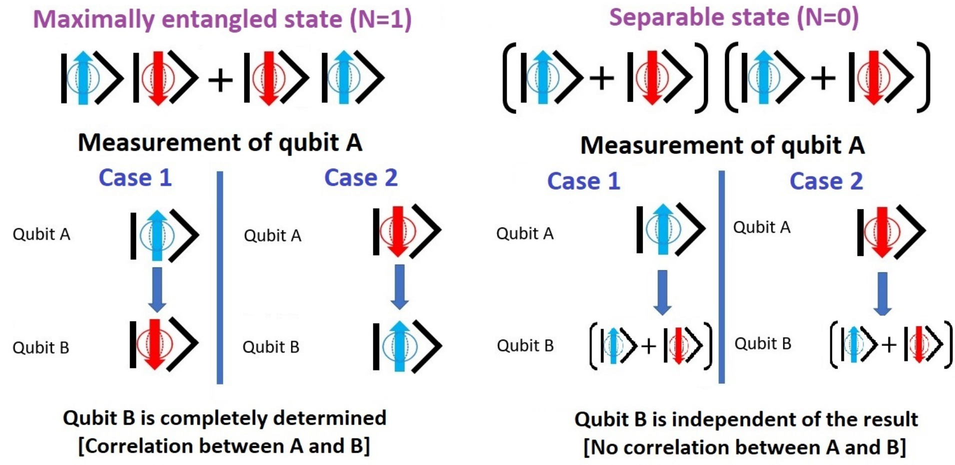
Thank you very much for your attention!

Entanglement

(Symmetric) Qubit-qubit system $\mathcal{H}_2^{\vee 2}$



Entanglement of mixed states

Mixed states

$$\rho = \sum_{k=0}^{2} \lambda_k |\phi_k\rangle\langle\phi_k| \, .$$

Separable mixed states [Werner (1989)]

ρ is separable if

$$\rho = \int_{S^2} P(\boldsymbol{n}) |\boldsymbol{n}\rangle \otimes |\boldsymbol{n}\rangle\langle\boldsymbol{n}| \otimes \langle\boldsymbol{n}| \, \mathrm{d}\boldsymbol{n} \, .$$

with $P(\boldsymbol{n}) \geq 0$. Otherwise is entangled.

Measure of entanglement

- $E(\rho) = 0$ if and only if ρ is separable.
- Invariant under local unitary transformations.
- Other properties...

For qubit-qubit and qubit-qutrit: **Negativity** [Peres (1996)], [Horodecki et al (1996)]

The sum of the negative eigenvalues Λ_k of ρ^{T_A}

$$\mathcal{N}(\rho) = -2 \sum_{\Lambda_k < 0} \Lambda_k \, ,$$

Overview of the proof

Maximum entanglement in $\mathcal{H}_2^{\otimes 2}$ [Verstraete et al (2001)]

Observation 1

$$\mathcal{N}(\rho) = -2 \sum_{\Lambda_k < 0} \Lambda_k = -2\,(0\,,\Lambda_{\min})\ ,$$

Observation 2

$$\min_{U \in SU(4)} \Lambda_{\min} = \min_{U \in SU(4)} \min_{|\psi\rangle \in \mathcal{H}_2^{\otimes 2}} \mathrm{Tr}\left[\rho U^\dagger (|\psi\rangle\langle\psi|)^{T_A} U\right]$$

Observation 3

$$\min_{U \in SU(4)} \Lambda_{\min} = \min_{V \in SU(4)} \min_{\alpha \in [0,\pi]} \mathrm{Tr}\left[\rho V D V^\dagger\right]$$

$$\left(\rho_{jk} = \lambda_j \delta_{jk}\,, D_{jk} = \sigma_j(\alpha)\delta_{jk}\right) = \min_{B \in \mathcal{U}} \min_{\alpha \in [0,\pi]} \boldsymbol{\lambda}^T B \boldsymbol{\sigma}\,,$$

B an unistochastic matrix, $B \in \mathcal{U} \subset \mathcal{B}$.

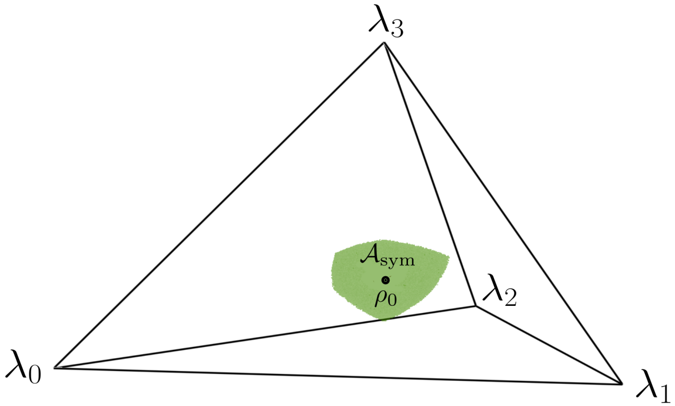
Observation 4 (Birkhoff's Theorem)

Permutations matrices achieve extremal values of a convex function $f(B)$

$$\min_{U \in SU(4)} \Lambda_{\min} = \min_{\Pi \in S_3} \min_{\alpha} \sum_{j=1}^{4} \boldsymbol{\lambda}^T \Pi \boldsymbol{\sigma}\,.$$

Symmetric 3-qubit system

$\mathcal{H}_2^{\vee 3} \subset \mathcal{H}_2 \otimes \mathcal{H}_3$, numerical results



Set of SAS states in the spectra polytope of symmetric 3-qubit states.