Locally stable kidney exchanges

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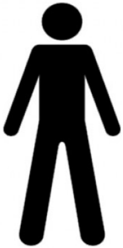
Kidney Exchanges

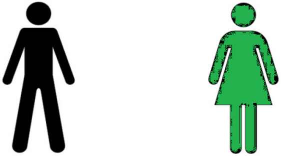
Introduction



Patient 1

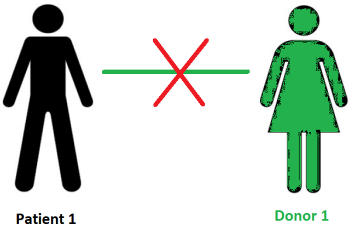
Patient with a kidney disease that requires a kidney transplant.

- Dialysis
- Deceased donor waiting list
- Willing donor



Patient 1 Donor 1

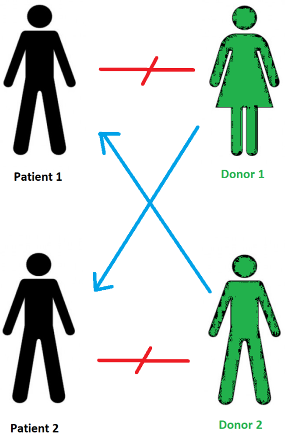
Kidney exchange problem



Might not be compatible with its donor:

- Blood incompatibility
- Antigens incompatibility

Kidney exchange problem



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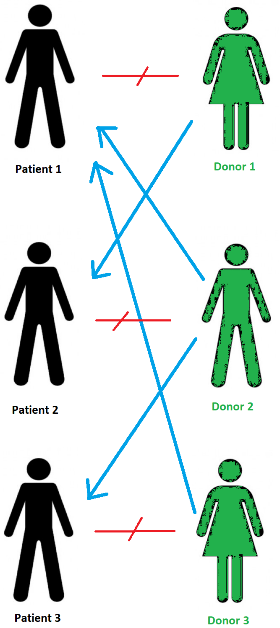
Locally stable kidney exchanges

Marie Baratto

Kidney exchange problem



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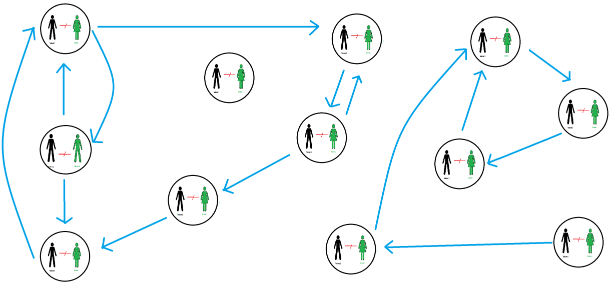
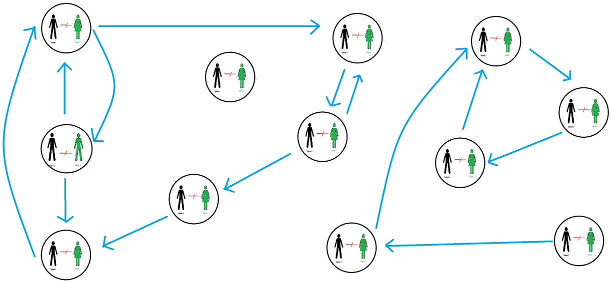
Pool of incompatible pairs



Compatibility graph



G=(V,A,w) where:

- $V = \{1, ..., n\}$ set of vertices, consisting of all patient-donor pairs.
- A, the set of arcs, designating compatibilities between the vertices. Two vertices i and j are connected by arc (i, j) if the donor in pair i is compatible with the patient in pair j.

We denote by $\mathcal{C}_K(G)$ be the set of feasible cycles of length at most K for $G = (V, A)$.

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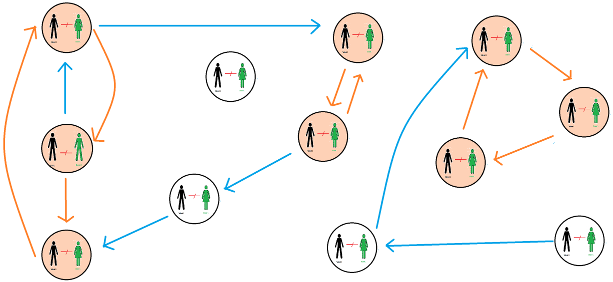
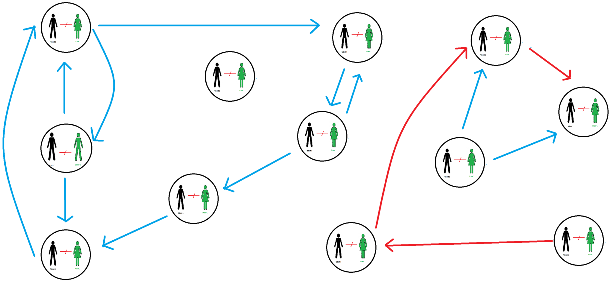
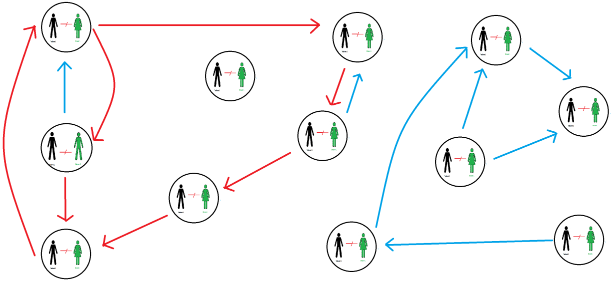
Possible exchanges

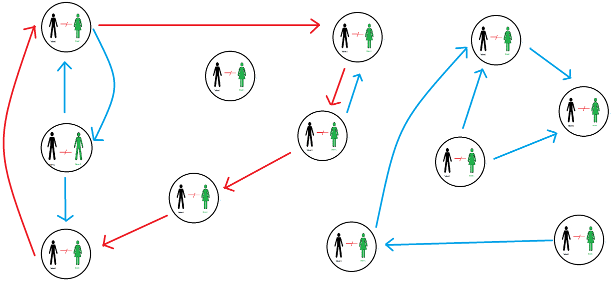
Definition

An **exchange** is a set of disjoint cycles in the directed graph such that every cycle length does not exceed a given limit K.









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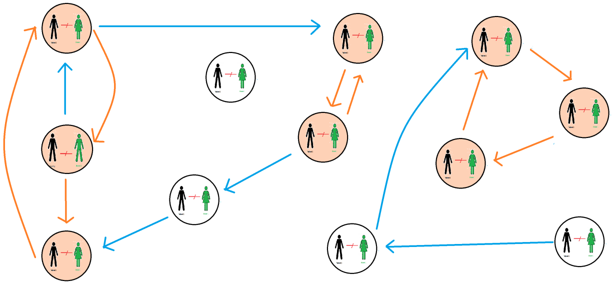


→ Objective is to **maximize the number of patients transplanted**

Formulations

1. **Arc-based models** with small number of variables but exponentially many constraints
   - Edge formulation
2. **Cycle-based models** with a small number of constraints but exponentially many variables
   - Cycle formulation
3. **Arc based compact models** that create multiple clones of the directed graph
   - Extended edge formulation
   - EE-MTZ and SPLIT-MTZ formulations
   - Position-indexed formulations (PIEF, PICEF and HPIF)

Cycle formulation

Let $C_K(K)$ be the set of all cycles in graph $G = (V, A)$ with length at most K.

Variables

$$y_u = \begin{cases} 1 \text{ if cycle u is selected;} \\ 0 \text{ otherwise} \end{cases} \quad \forall u \in C_K(K)$$

Objective function

$$\max \sum_{u \in C_K(K)} w_u \, y_u$$

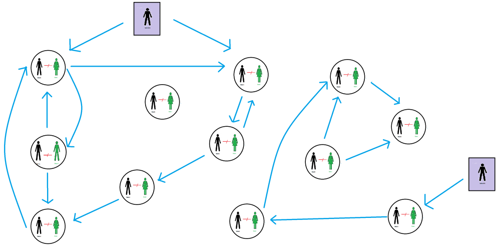
Constraints

$$\sum_{u \in C_K(K): i \in u} y_u \leq 1 \qquad \forall i \in V$$

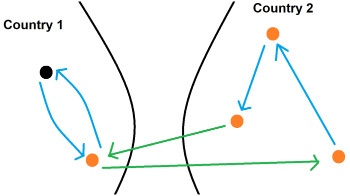
$$y_u \in \{0, 1\} \forall u \in C_K(K)$$

Variants

- Altruistic donors



- International exchanges

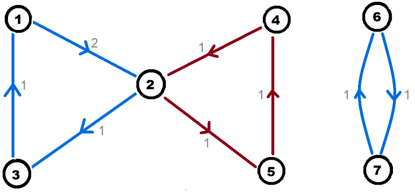


Stable Exchanges

Stable exchange

For each vertex $i \in V$, a preference order is given on the set of its in-neighbors. That is $N^{-}(i) := \{j : (j, i) \in A\}$.

For a given **node** i**, a preference** p **associated to node** j means that the recipient of the pair i ranks the donor of pair j at position p in its preferences list of acceptable donors.

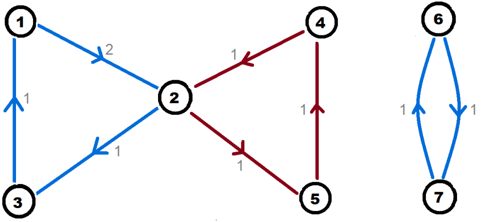


Stable exchange

Definition

A **blocking cycle** u for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(u)$, i prefers u to $\mathcal{M}$.
We say that vertex i *prefers* the cycle u to the exchange $\mathcal{M}$ if either

- $i \notin V(\mathcal{M})$, or
- $i \in V(\mathcal{M})$, $(k,i) \in A(u)$, $(k',i) \in A(\mathcal{M})$, and i prefers k to k'.



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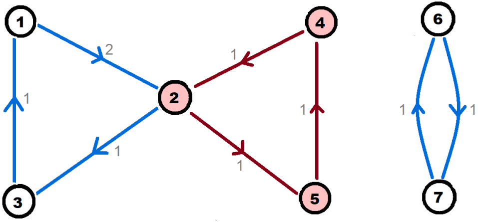

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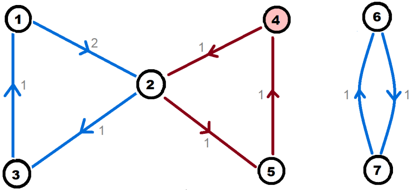

→ Vertex 4 is unmatched in exchange $\mathcal{M}$ in blue

Stable exchange

Definition

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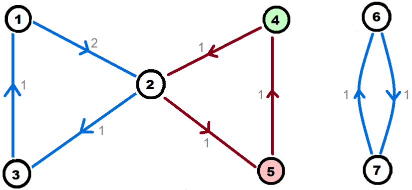

→ Vertex 5 is unmatched in exchange $\mathcal{M}$ in blue

Stable exchange

Definition

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We say that vertex i *prefers* the cycle u to the exchange $\mathcal{M}$ if either

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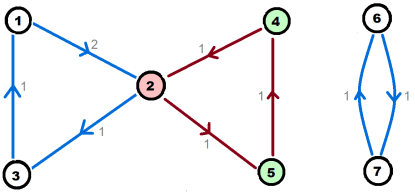

$\rightarrow$ Vertex 2 prefers cycle red because the donor of pair 4 is number one on its preference list and donor of pair 1 is at at the second position of its preference list.

Stable exchange

Definition

A **blocking cycle** u for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(u)$, i prefers u to $\mathcal{M}$.
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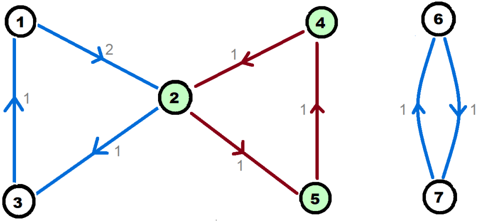


Definitions - Stability

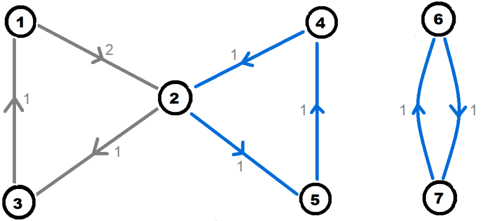
Definition

A **blocking cycle** u for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(u)$, i prefers u to $\mathcal{M}$.
We say that vertex i *prefers* the cycle u to the exchange $\mathcal{M}$ if either

- $i \notin V(\mathcal{M})$, or
- $i \in V(\mathcal{M})$, $(k, i) \in A(u)$, $(k', i) \in A(\mathcal{M})$, and i prefers k to k'.

Definition

Given a directed graph $G = (V, A)$, an exchange is called **stable** if no blocking cycle c exists for $\mathcal{M}$.

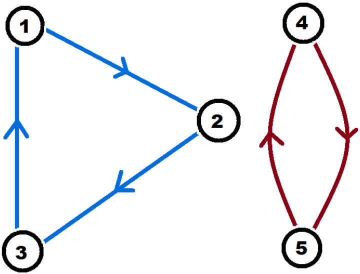


Stable exchange - Drawback

Definition

A **blocking cycle** u for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(u)$, i prefers u to $\mathcal{M}$.
We say that vertex i *prefers* the cycle u to the exchange $\mathcal{M}$ if either

- $i \notin V(\mathcal{M})$, or
- $i \in V(\mathcal{M})$, $(k, i) \in A(u)$, $(k', i) \in A(\mathcal{M})$, and i prefers k to k'.



Locally stable exchange

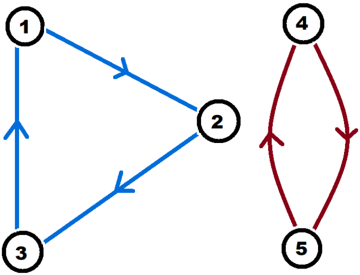
Definition

A **locally blocking cycle** u for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ **but has a vertex in common with** $\mathcal{M}$ and such that, for every vertex $i \in V(u)$, i prefers u to $\mathcal{M}$.
We say that vertex i *prefers* the cycle u to the exchange $\mathcal{M}$ if either

- $i \notin V(\mathcal{M})$, or
- $i \in V(\mathcal{M})$, $(k,i) \in A(u)$, $(k',i) \in A(\mathcal{M})$, and i prefers k to k'.

Definition

Given a directed graph $G = (V, A)$, an exchange is called **locally stable (L-stable)** if no L-blocking cycle c exists for $\mathcal{M}$.



Aim

Stability

known notion, stable marriage problem, stable rommate problem

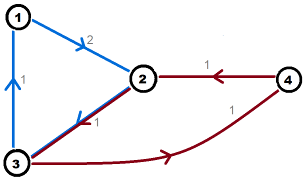
vs.

Local Stability ...

- In the context of KE, local stability seems more relevant
- Our aim is to better know that notion
- Formulation(s)
- Comparison with stability
- ...

For each cycle u, one can define two sets of cycles:

- $\mathcal{B}(u) :=$ the set of cycles blocking $\{u\}$
- $\mathcal{F}(u) :=$ the set of cycles not blocking u and not blocked by $\{u\}$.



If two cycles u and v intersect, three situations can occur:

- u is blocking $\{v\}$:= $u \in \mathcal{B}(v)$
- v is blocking $\{u\}$:= $v \in \mathcal{B}(u)$
- u is not blocking v and v is not blocking u := $u \in \mathcal{F}(v)$ and $v \in \mathcal{F}(u)$

Formulation

Variables

$$y_u = \begin{cases} 1 \text{ if cycle u is selected;} \\ 0 \text{ otherwise} \end{cases} \quad \forall u \in C(K)$$

Objective function

$$\max \sum_{u \in C(K)} w_u y_u$$

Constraints

$$y_u + y_v \leq 1 \qquad \forall u \in \mathcal{C}_K(G), \forall v \in \mathcal{B}(u) \cup \mathcal{F}(u) \qquad (1)$$

$$y_u \leq \sum_{w \in \mathcal{B}(v) \cup \mathcal{F}(v)} y_w \qquad \forall u \in \mathcal{C}_K(G), \forall v \in \mathcal{B}(u) \qquad (2)$$

$$y_v \in \{0, 1\} \qquad \forall v \in \mathcal{C}_K(G) \qquad (3)$$

1. Constraints (4) ensure the independence of the cycles selected
2. Constraints (5) ensure the local stability of the exchange

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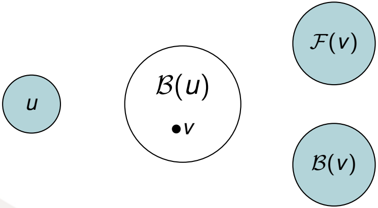
Constraints

$$y_u + y_v \leq 1 \qquad \forall u \in \mathcal{C}_K(G), \forall v \in \mathcal{B}(u) \cup \mathcal{F}(u) \tag{4}$$

$$y_u \leq \sum_{w \in \mathcal{B}(v) \cup \mathcal{F}(v)} y_w \qquad \forall u \in \mathcal{C}_K(G), \forall v \in \mathcal{B}(u) \tag{5}$$

$$y_v \in \{0, 1\} \qquad \forall v \in \mathcal{C}_K(G) \tag{6}$$

1. Constraints (4) ensure the independence of the cycles selected
2. Constraints (5) ensure the local stability of the exchange

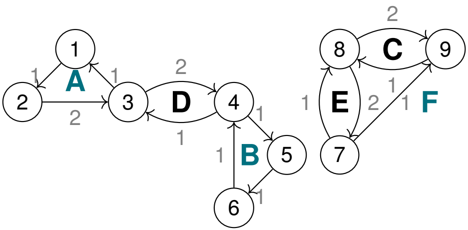


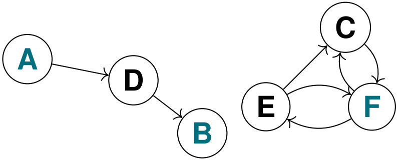
Definition

Starting from the initial directed graph $G = (V, A)$, lets construct a directed graph $G' = (V', A')$ such that:

- For each $v \in \mathcal{C}_K(G)$ there is a vertex v in V' representing that cycle.
- An arc $(u, v) \in A'$ if $v \in \mathcal{B}(u)$ or if $v \in \mathcal{F}(u)$.

G' is the **blocking directed graph** associated to G.





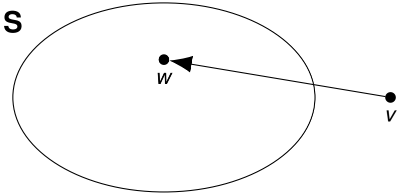
- Independent
- Absorbing

(Local) Kernel - Definitions

Definition

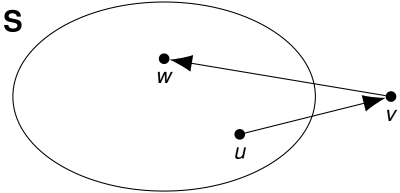
Given a directed graph $G' = (V', A')$, subset $S \subseteq V'$ is a **kernel** of G' if it is independent and absorbing. That is:

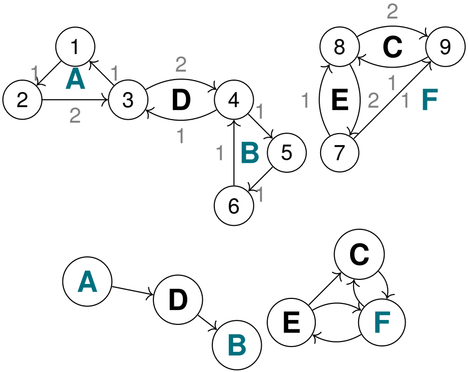
- for all $(u, v) \in A'$ either $u \notin S$ or $v \notin S$
- for every $v \notin S$ there exists a vertex $u \in S$ such that $(v, u) \in A'$



Definition

A **local kernel** of G is an independent subset S of vertices such that every neighbor (or out-neighbor) of S is absorbed by S.





Result

A L-kernel in G' defines a L-stable exchange in G.

and a kernel in G' defines a stable exchange in G.

Result

The empty set $S = \emptyset$ is an L-kernel. So, every directed graph has an L-kernel (but not necessarily a not empty one).

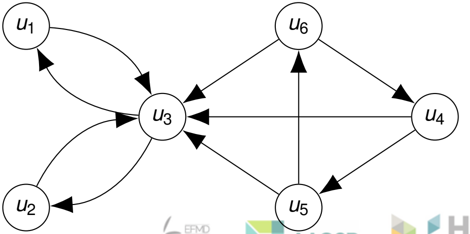
Result

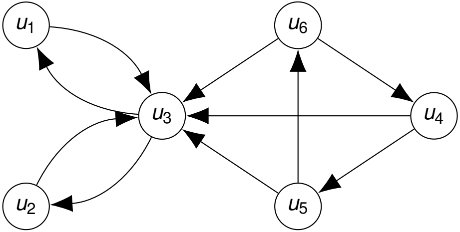
Given a directed graph $G = (V, A)$, deciding whether G has a nonempty local kernel is NP-complete.

Reduction from SAT

Result

Given a directed graph,the cardinality of its maximum L-kernel is greater or equal than the cardinality of its maximum kernel (if there is one).

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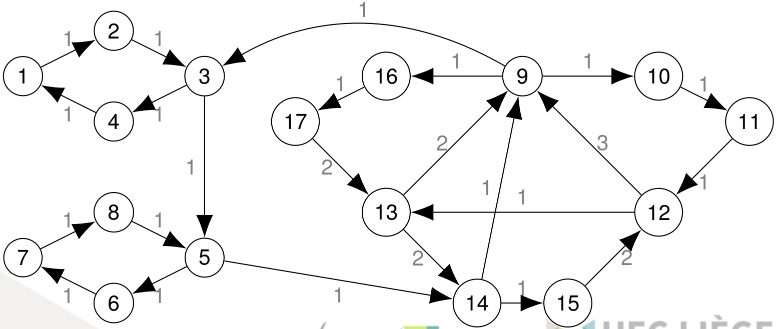
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Formulation (bis)

$$y_u + y_v \leq 1 \qquad \forall (u,v) \in A' \qquad (7)$$

$$y_u \leq \sum_{w \in N^+(v)} y_w \qquad \forall (u,v) \in A' \qquad (8)$$

$$y_v \in \{0,1\} \qquad \forall v \in V' \qquad (9)$$

- **Independence** constraint (7) can be replaced by

$$\sum_{u \in \mathcal{C}_K(G):i \in V(u)} y_u \leq 1 \qquad \forall i \in V \qquad (10)$$

- **Stability/Absorbing** constraint (8) can be replaced by fixing v and adding each constraint above for all $(u,v) \in A'$:

$$\sum_{w \in N^-(v)} y_w \leq |N^-(v)| \sum_{w \in N^+(v)} y_w \qquad \forall v \in V' \qquad (11)$$

where $N^-(v) = |\{w : (w,v) \in A'\}|$.

Numerical Tests

Comparison LP formulations

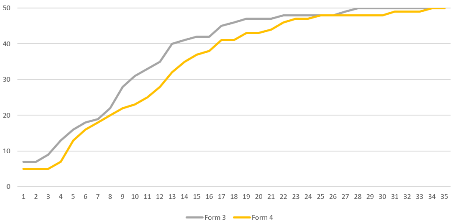
Comparison between 4 formulations:

1. Initial formulation for L-stable exchange (Form-LS) (7)-(8)
2. Form-LS with stability constraints modified (7)-(11)
3. Form-LS with independence constraints modified (10)-(8)
4. Form-LS with independence constraints and stability constraints modified (10)-(11)

Integrality gap:

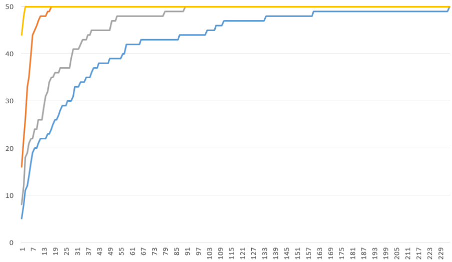
$$Gap_{LP}^{k} = 100 \times \frac{z_{LP}^{k} - z^{*}}{z^{*}}, \qquad \in [0; +\infty[$$

- Formulation 1 and 2 $Gap_{LP} \in [112; 6094]$
- Formulation 3 and 4 $Gap_{LP} < 35\%$



Comparison IP formulations

1. Initial formulation for L-stable exchange (Form-LS) (7)-(8)
2. Form-LS with stability constraints modified (7)-(11)
3. Form-LS with independence constraints modified (10)-(8)
4. Form-LS with independence constraints and stability constraints modified (10)-(11)



Best IP formulation for L-stable exchanges in terms of computation time:

$$\sum_{v\in\mathcal{C}_K(G):i\in V(c)} y_v \leq 1 \qquad \forall i \in V$$

$$\sum_{w\in N^-(v)} y_w \leq |N^-(v)| \sum_{w\in N^+(v)} y_w \qquad \forall v \in V'$$

$$y_v \in \{0,1\} \qquad \forall v \in V'$$

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Aim

Stability

vs.

Local Stability ...

Stability vs *Local* stability

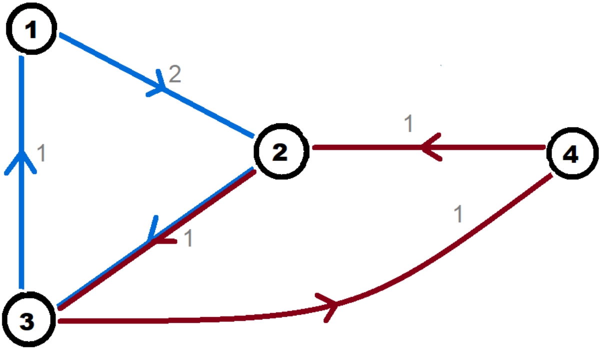
- Problem of maximum stable exchange(SE) and problem of maximum L-stable exchange (LSE) are not the same problems (not the same set of feasible solutions) **BUT** we can compare the objective value of both problems

◇ **SE** problem: Some instances do not have an optimal solution ($K = 2$, 72 out of 600 tested - 12%)

◇ **LSE** problem: All instances have an optimal solution. ($K = 2$, 1 out of 600 tested has a solution of cardinality zero 0.2%)

- for N=200,
  1. 45 out of 50 instances have a stable exchange
  2. 50 out of 50 instances have a locally stable exchange > 0
  3. 45 instances stable exchange = locally stable exchange
  4. 45: average optimal value is 105.8
  5. 5: average optimal value is 43.8

Local Strong Stability



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Conclusion

New notion of stability: **Local stability**

- Revelant in the context of kidney exchanges
- IP Formulations
- Link between (local) stable exchanges and (local) kernels in an associated digraph
- Many digraphs have a local stable exchange (> 0) without having a stable exchange

Further research

- Investigate the relevance of local stability for different classes of matching problems
- Strengthen the formulations
- ...

Locally stable exchanges

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