Invariant Kalman filtering with state equality constraints

Sven Goffin, Silvère Bonnabel, Olivier Brüls, and Pierre Sacré

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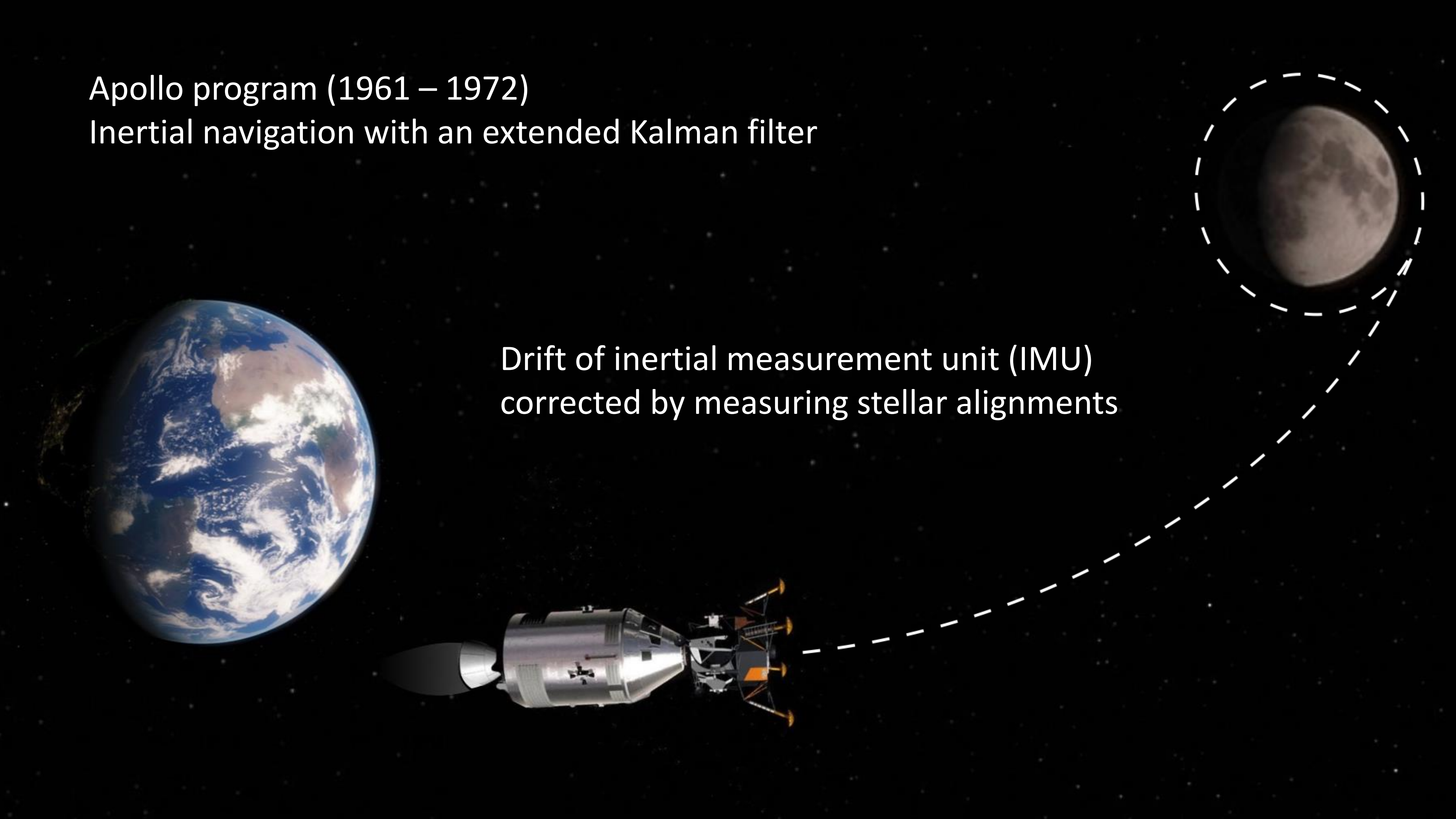
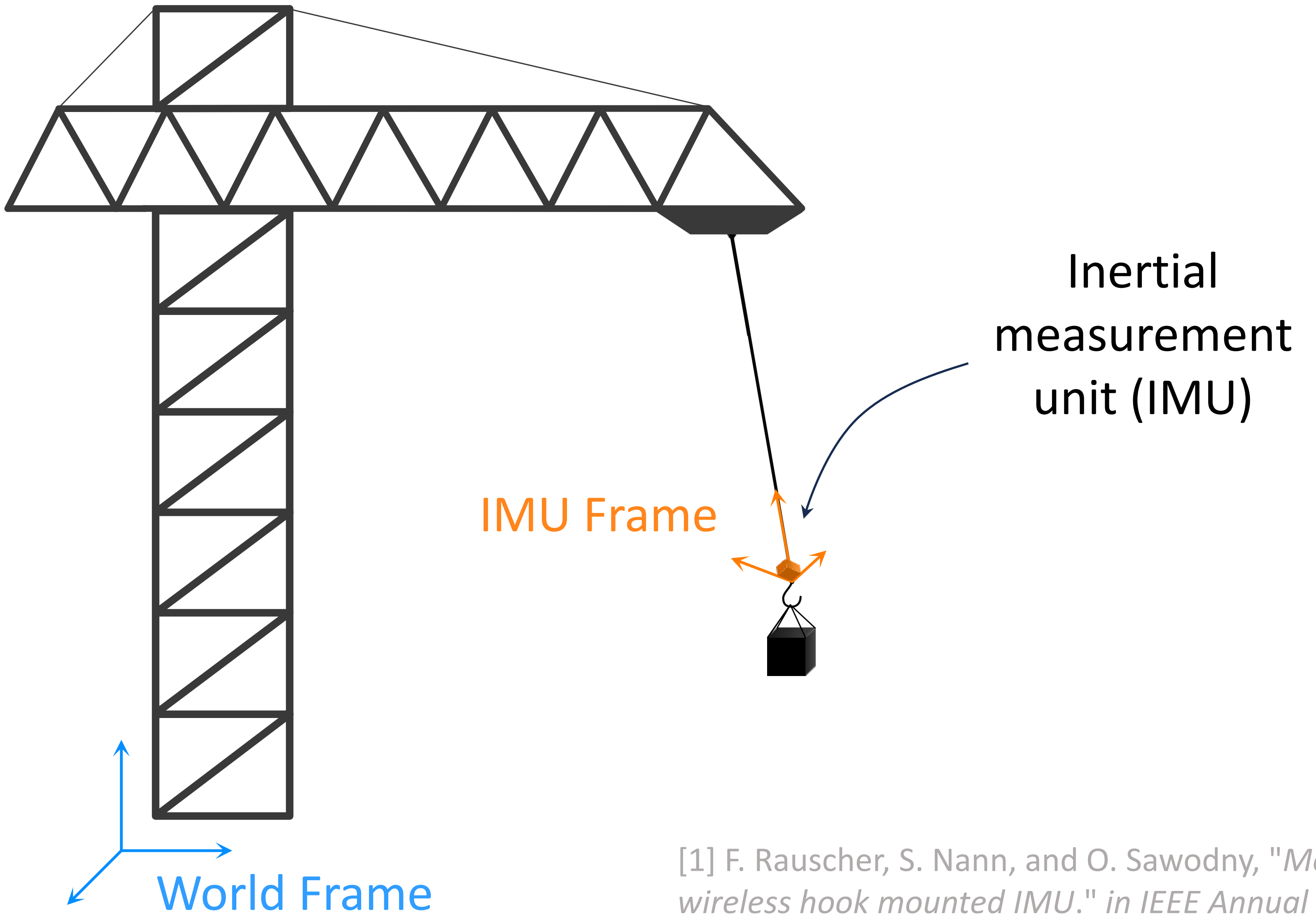
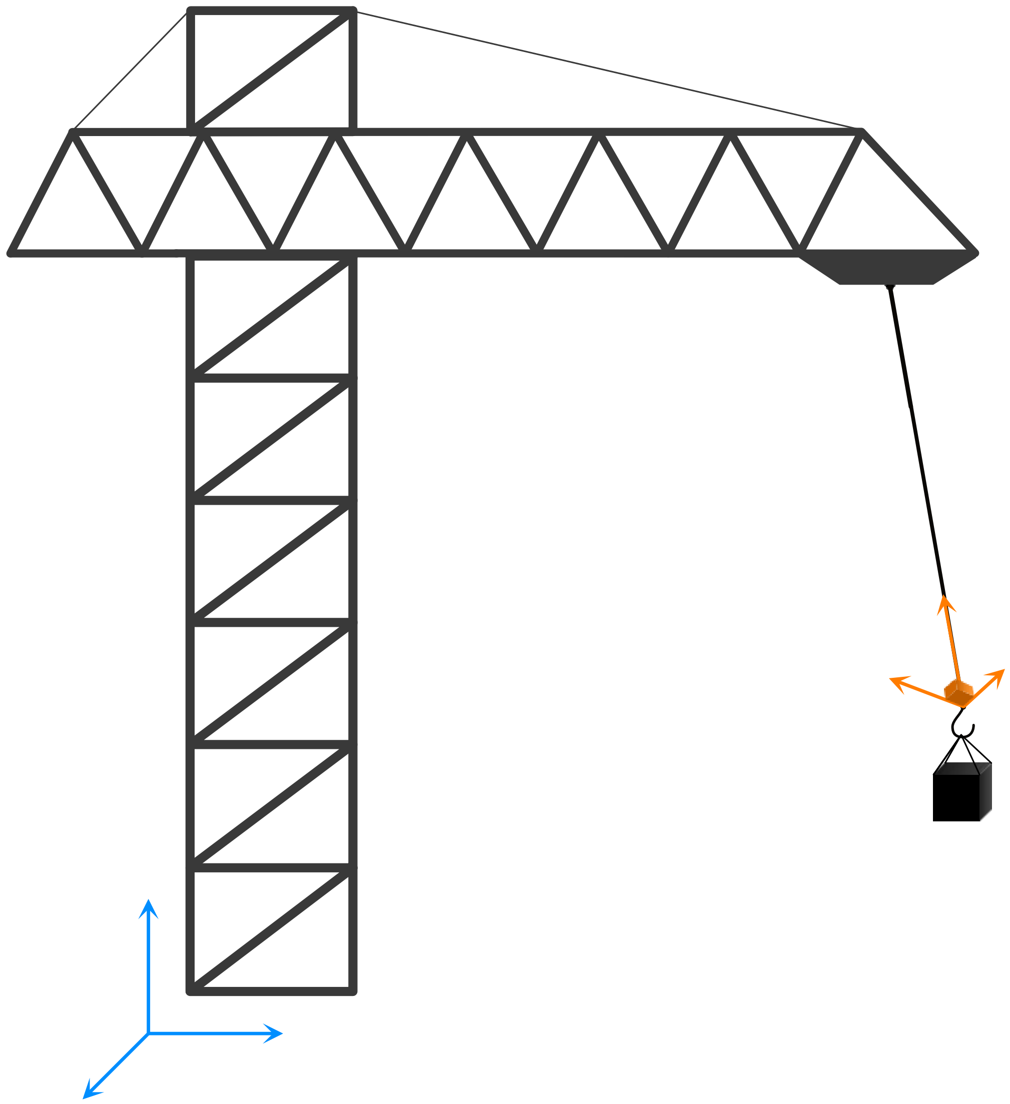

Apollo program (1961 – 1972)
Inertial navigation with an extended Kalman filter
Drift of inertial measurement unit (IMU)
corrected by measuring stellar alignments

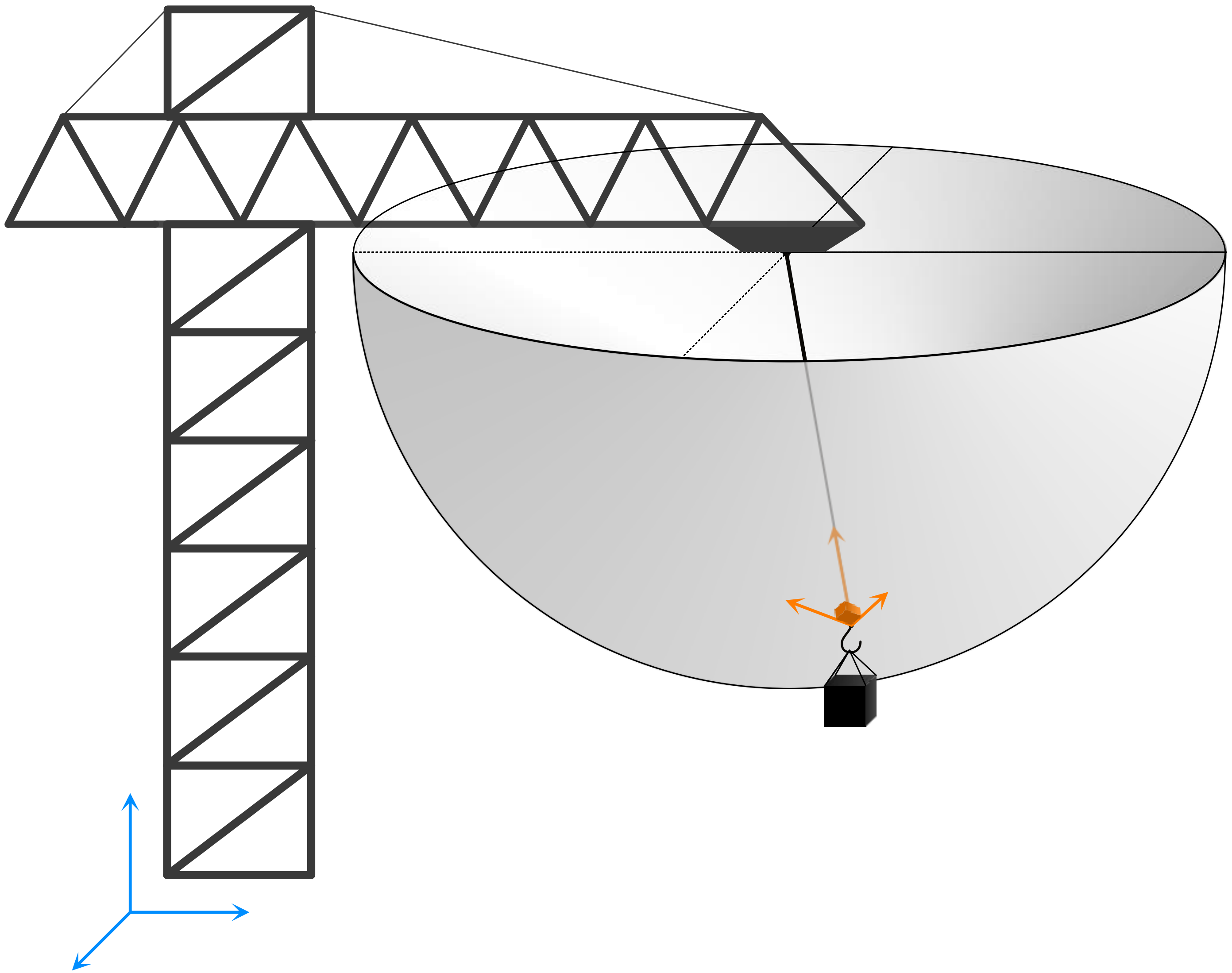


[1] F. Rauscher, S. Nann, and O. Sawodny, "*Motion control of an overhead crane using a wireless hook mounted IMU.*" *in IEEE Annual American Control Conference (ACC)*, 2018.



Information about the crane :

- Measured cable length
- Cable straight at all time



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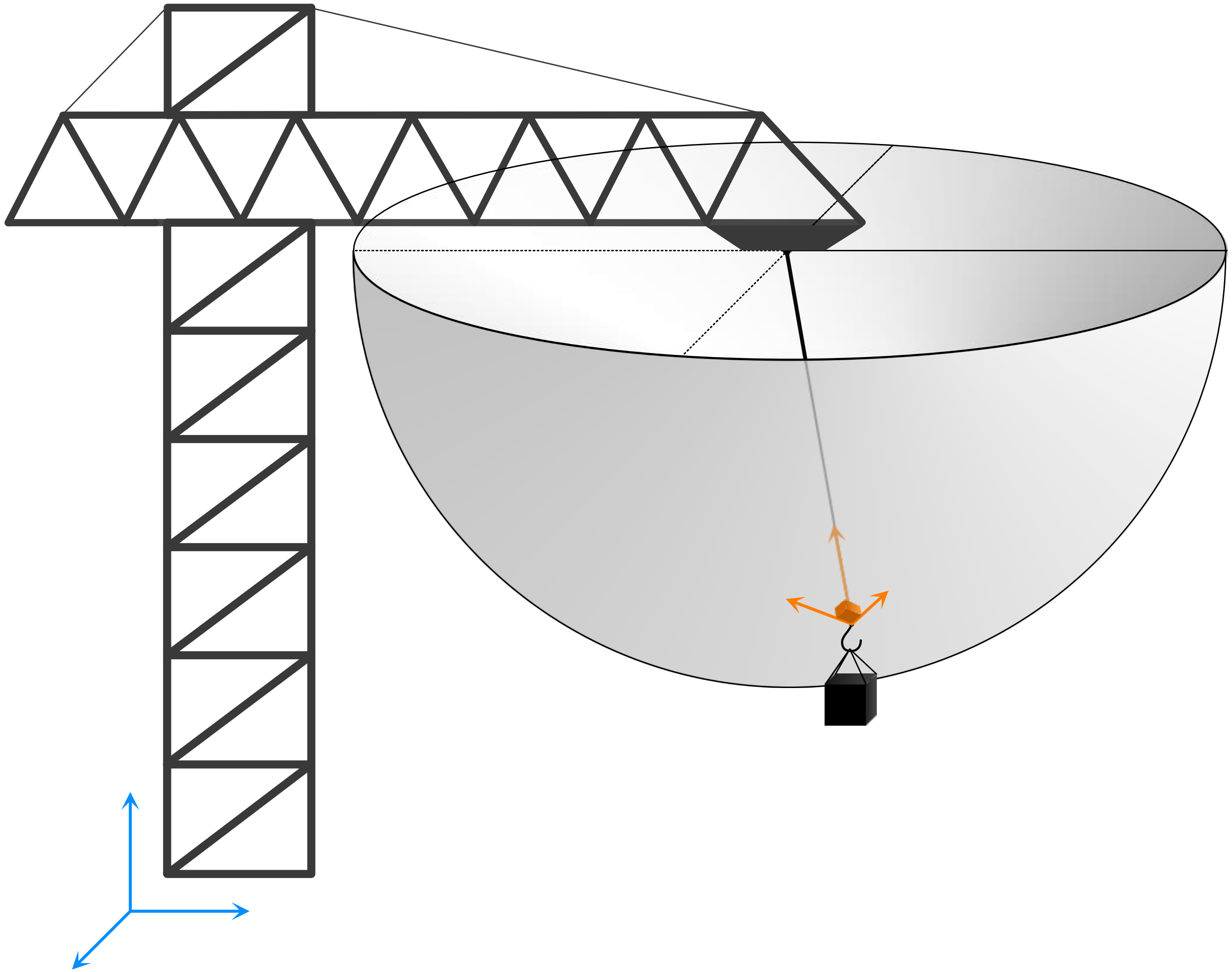
Equality constraint for state x_k :

$$h(x_k) = y_k$$



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Equality constraint for state x_k :

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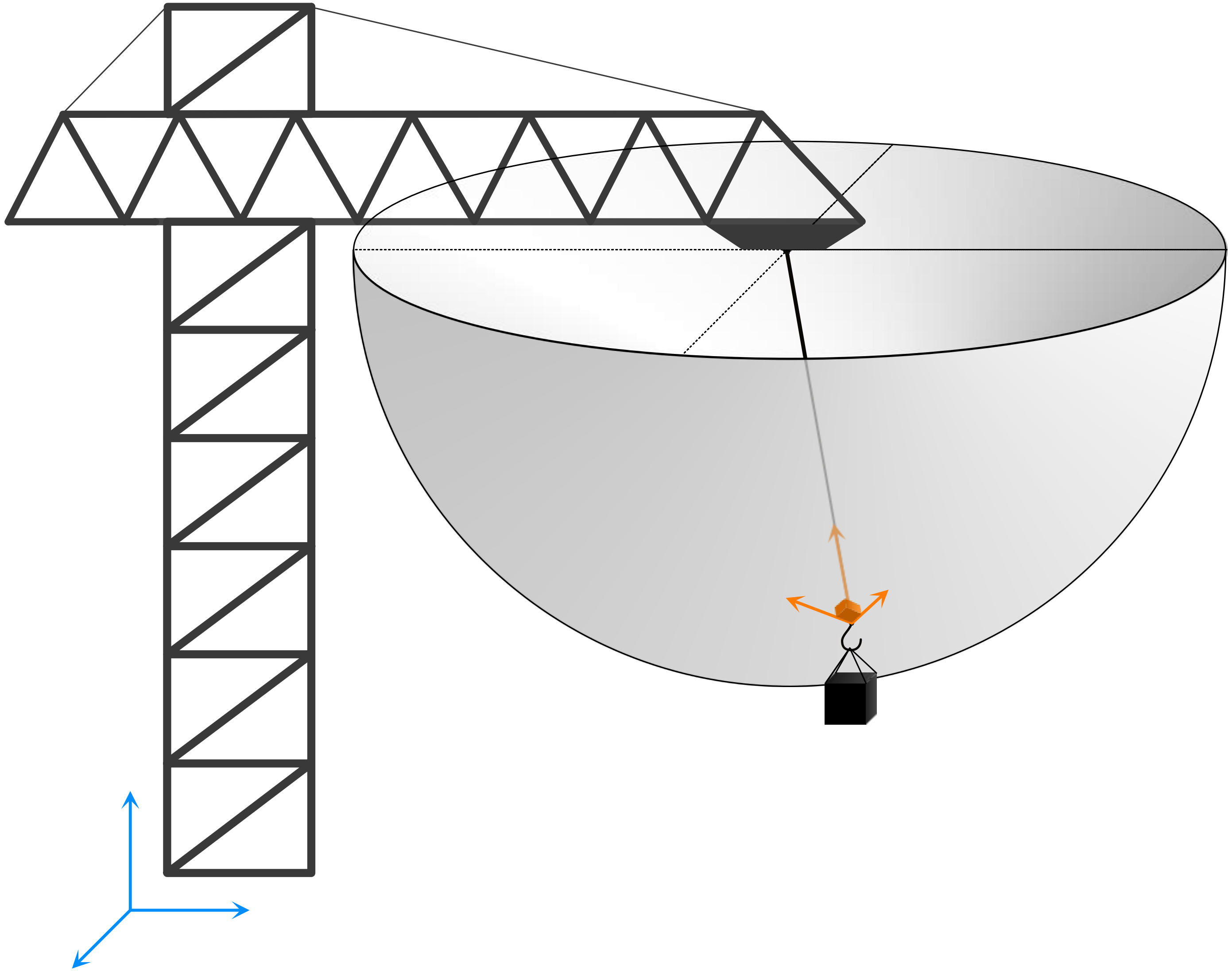
→ Kalman noise-free pseudo-measurement



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Noise-free pseudo-measurements in extended Kalman filtering:

- Use a classical EKF with small noise covariance

  → Error distribution not consistent with the constraint

  → Constraint almost satisfied depending on noise variance

Noise-free pseudo-measurements in extended Kalman filtering:

- Use a classical EKF with small noise covariance

  → Error distribution not consistent with the constraint

  → Constraint almost satisfied depending on noise variance

- EKF with linearized equality constraint + state projection on the equality constraint (D. Simon et al., 2002)

  → Error distribution not consistent with the constraint

  → Arbitrary choice in projection method

[2] D. Simon and Tien Li Chia, "Kalman filtering with state equality constraints“, in IEEE Transactions on Aerospace and Electronic Systems, vol. 38, no. 1, pp. 128-136, Jan. 2002.

Design an Iterated Invariant Extended Kalman Filter (IIEKF)
able to incorporate equality constraints
as noise-free pseudo-measurements

Preview

1. Noise-free pseudo-measurements in extended Kalman filtering
2. Noise-free pseudo-measurements in invariant filtering:
   - Invariant filtering in a nutshell
   - Handling noise-free pseudo-measurements with an IEKF
3. Application to IMU pose estimation for the hook of a crane

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Noise-free pseudo-measurements in extended Kalman filtering

First problem:

Kalman gain with noisy measurements :

$$K_k = P_k H_k^T (H_k P_k H_k^T + N_k)^{-1} \quad \text{with} \quad H_k = \frac{\partial h}{\partial x}\Big|_{\hat{x}_k}$$

Noise-free pseudo-measurements in extended Kalman filtering

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Measurement noise covariance

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N_k → Measurement noise covariance

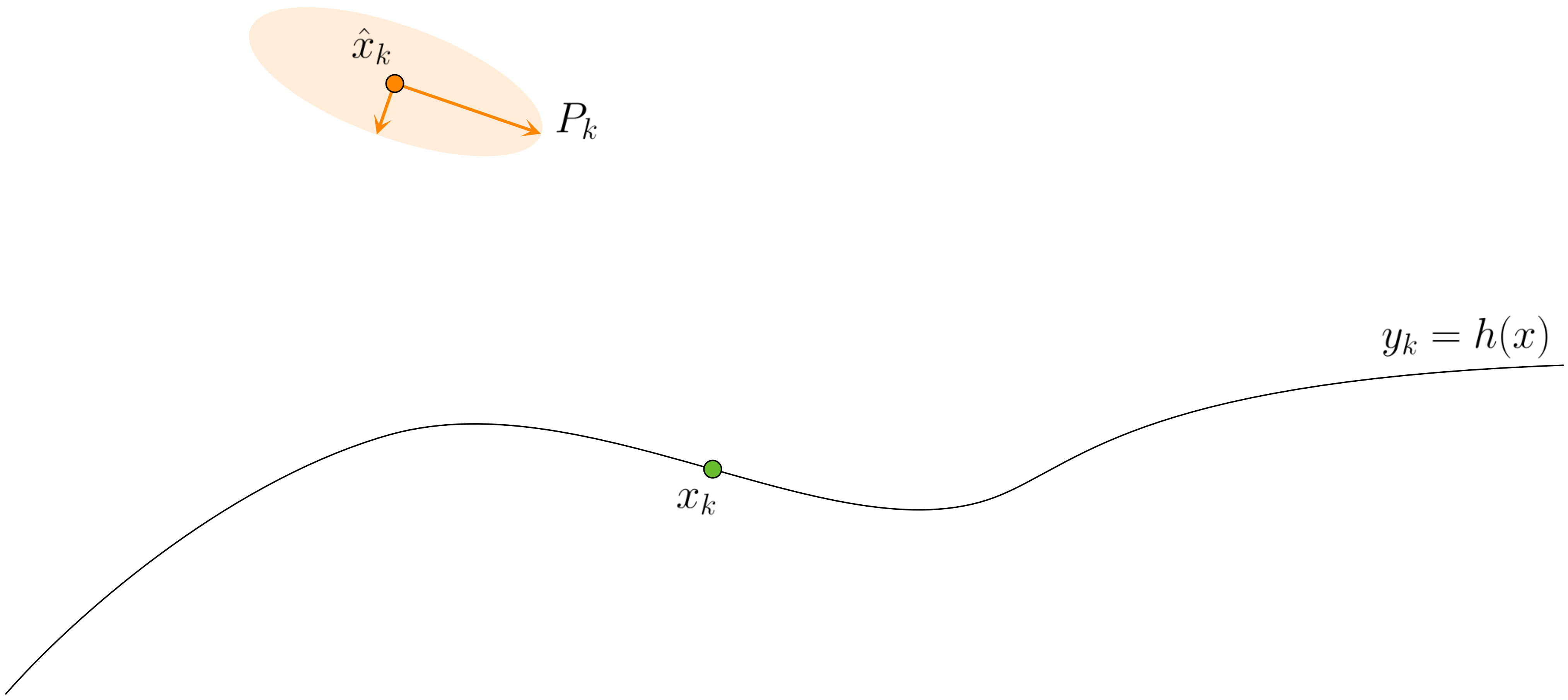
Kalman gain with noise-free measurements :

$$K_k = P_k H_k^T (H_k P_k H_k^T)^{-1}$$

$H_k P_k H_k^T$ → can be rank deficient

Noise-free pseudo-measurements in extended Kalman filtering

Second problem:



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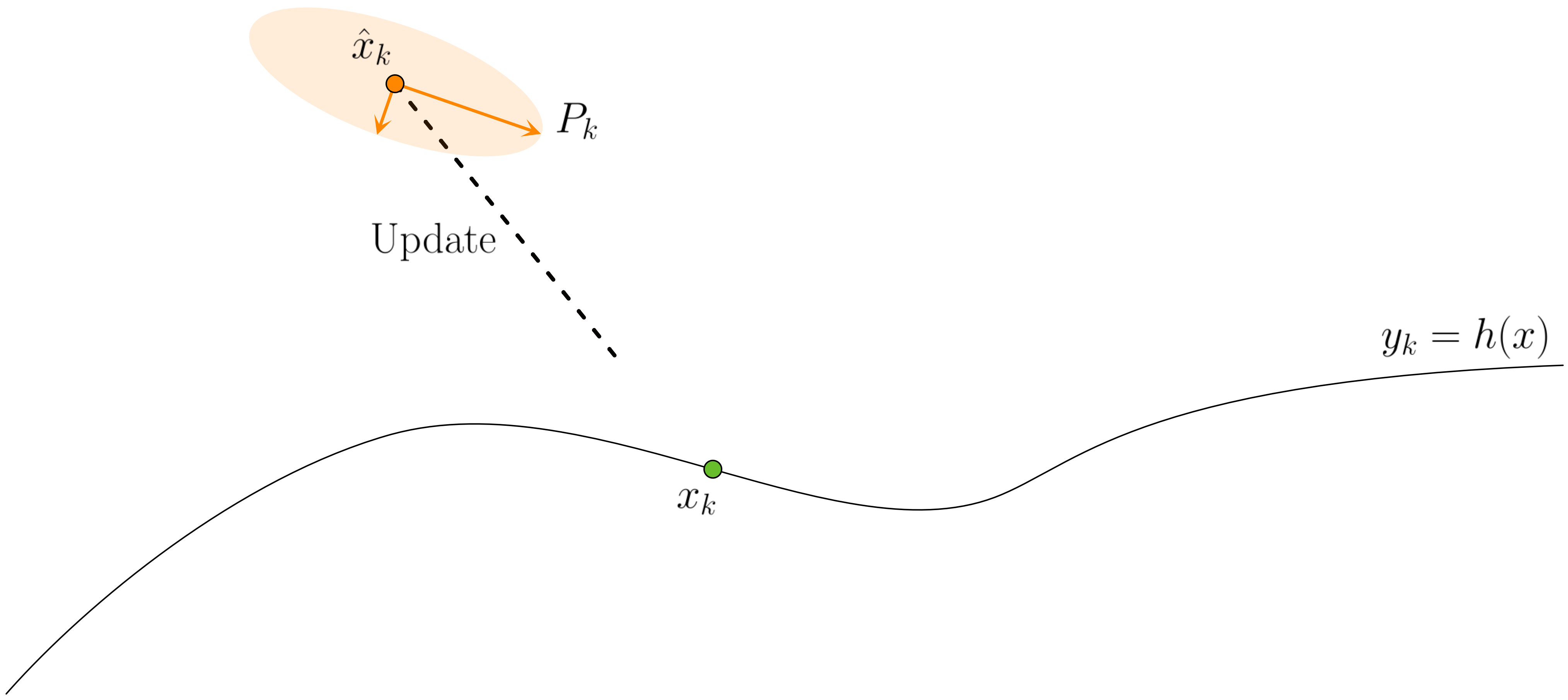

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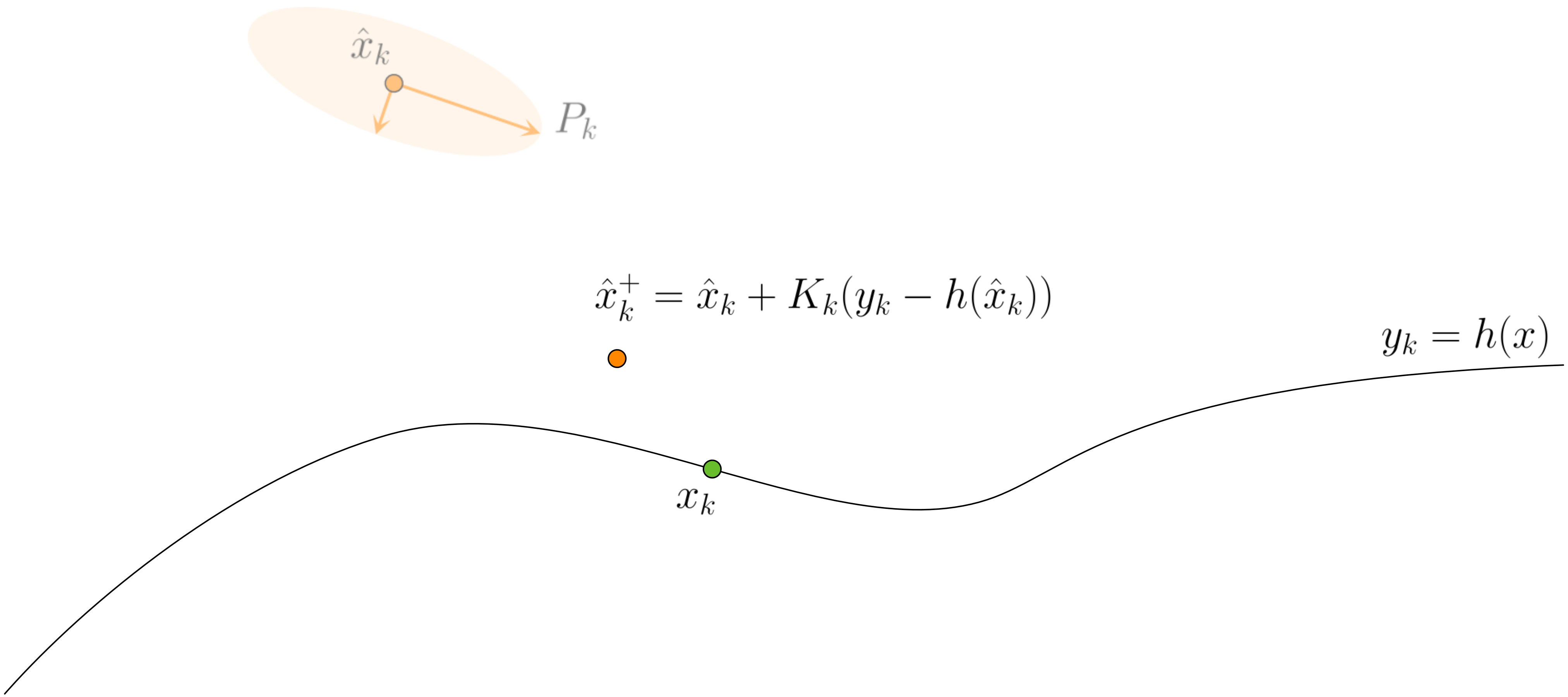
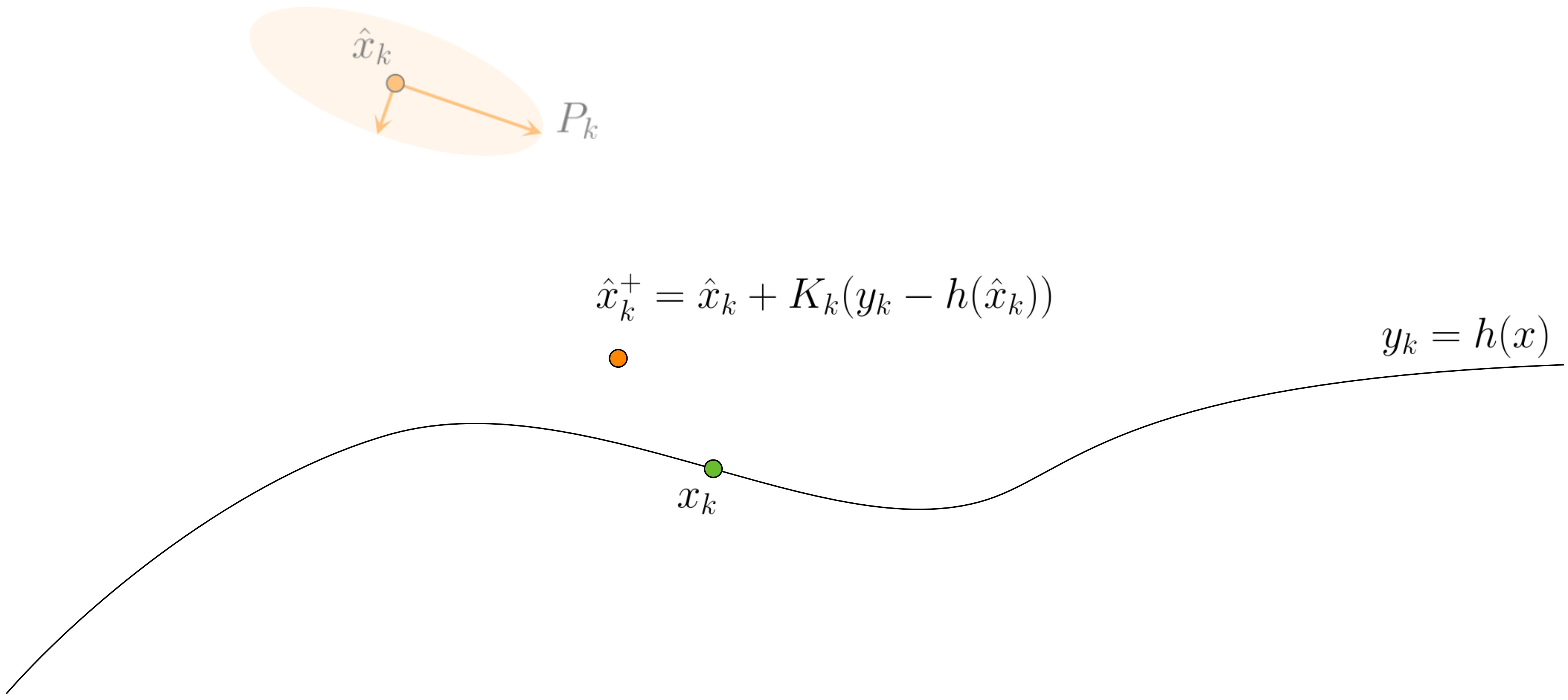


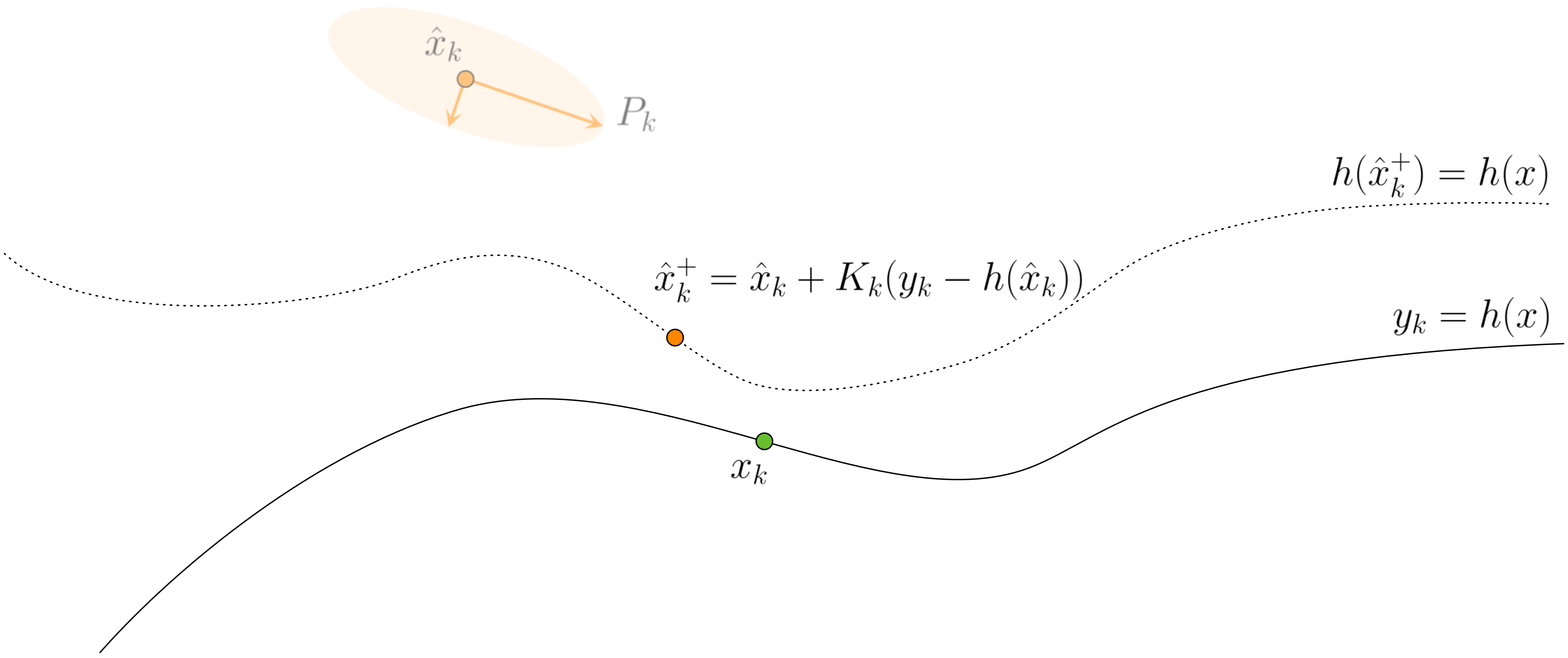
Noise-free pseudo-measurements in extended Kalman filtering

Second problem: state update inconsistency $\longrightarrow$ $h(\hat{x}_k^+) \neq y_k$



Noise-free pseudo-measurements in extended Kalman filtering

Third problem:



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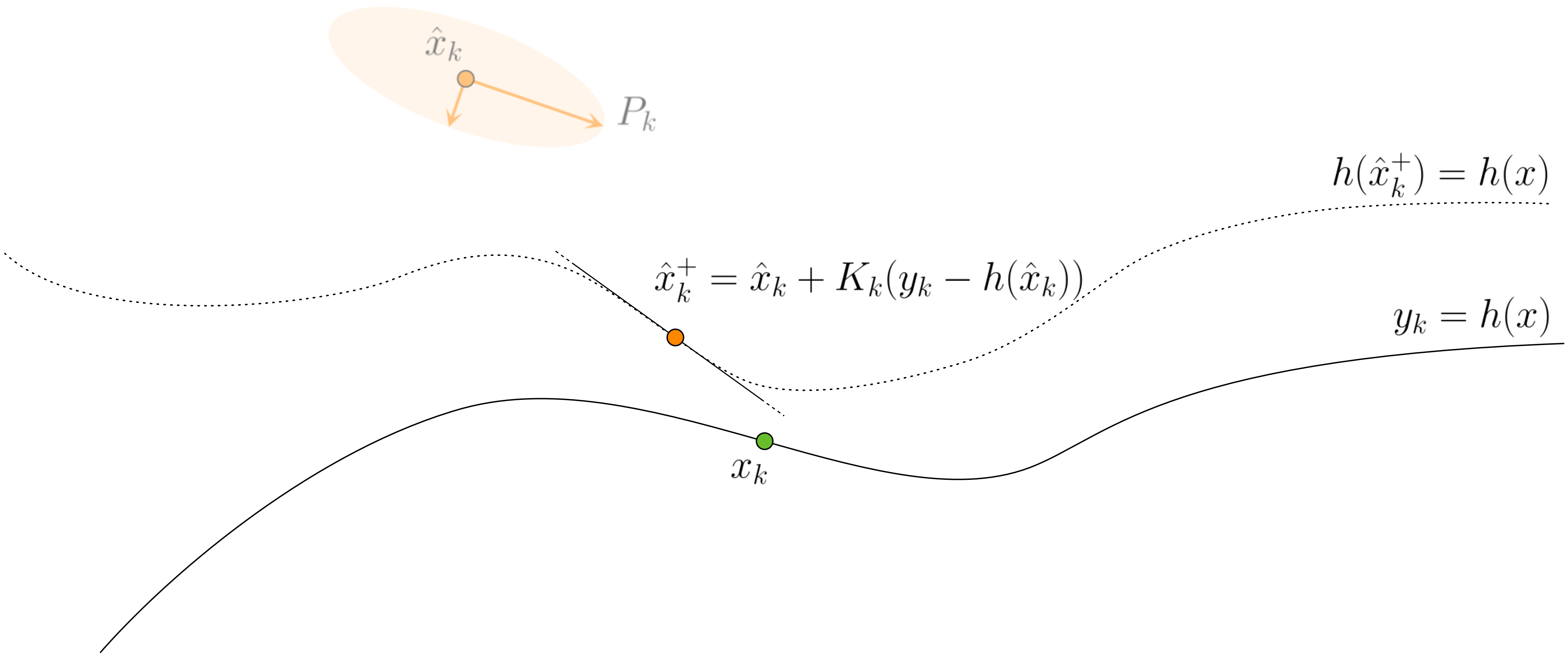

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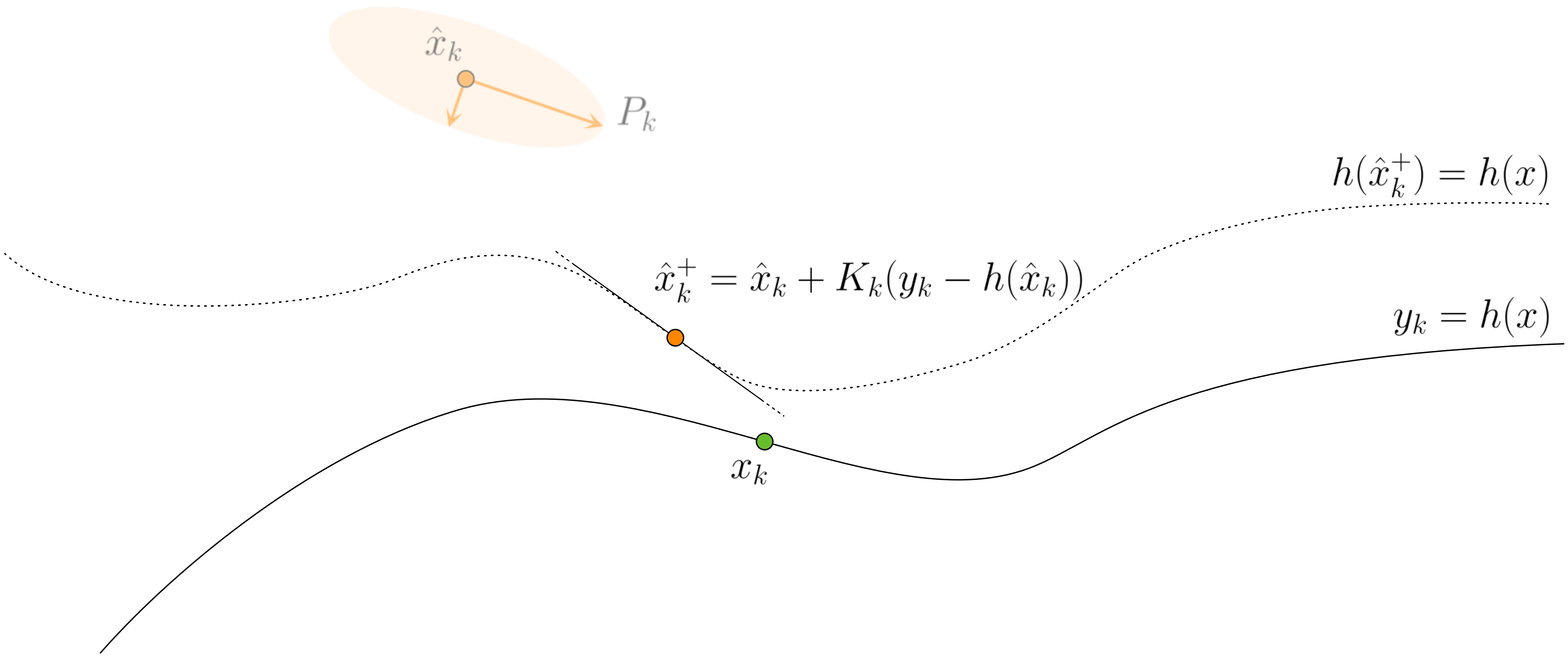
Third problem:

$$H_k = \frac{\partial h}{\partial x}\Big|_{\hat{x}_k} \neq \frac{\partial h}{\partial x}\Big|_{\hat{x}_k^+}$$



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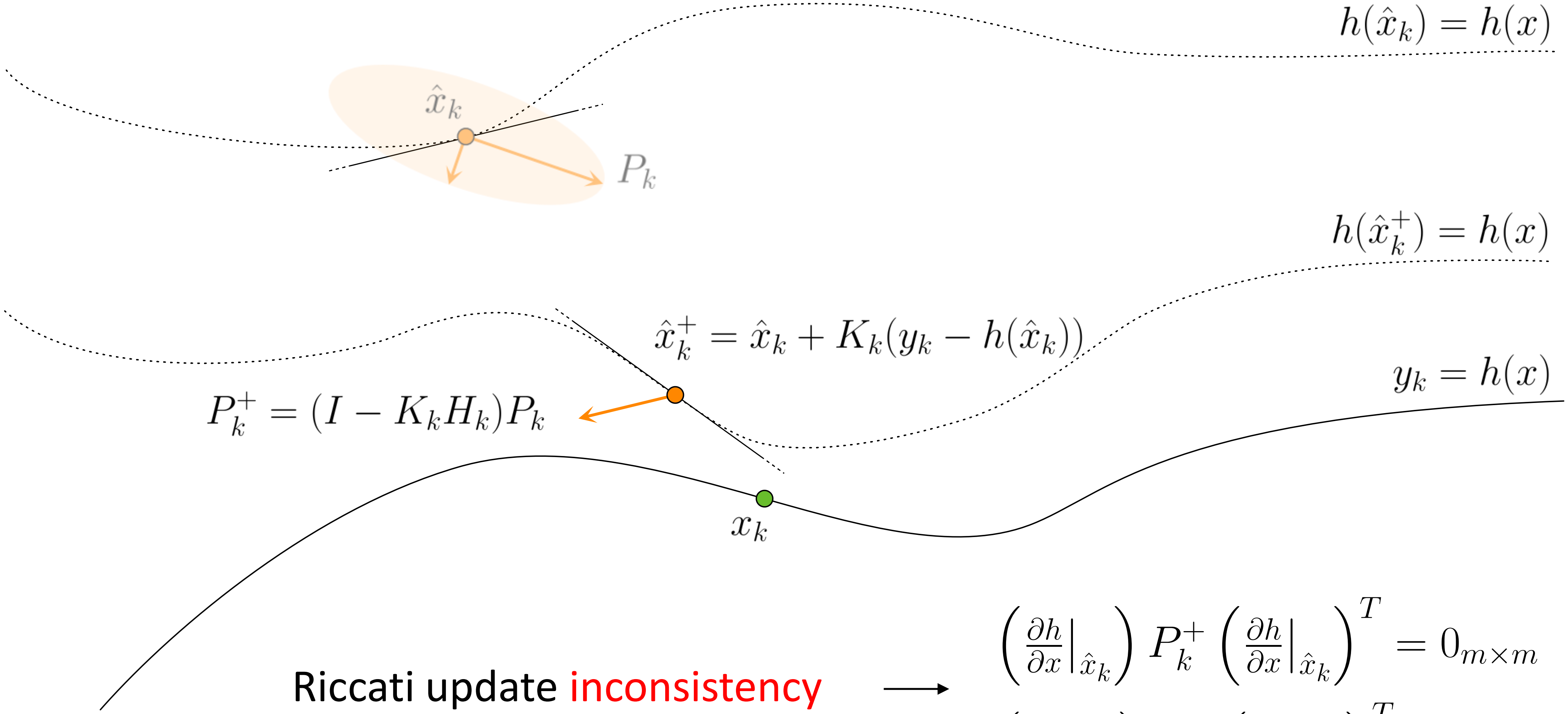

Riccati update inconsistency →

$$\left(\frac{\partial h}{\partial x}\Big|_{\hat{x}_k}\right) P_k^+ \left(\frac{\partial h}{\partial x}\Big|_{\hat{x}_k}\right)^T = 0_{m\times m} \quad \text{but}$$

$$\left(\frac{\partial h}{\partial x}\Big|_{\hat{x}_k^+}\right) P_k^+ \left(\frac{\partial h}{\partial x}\Big|_{\hat{x}_k^+}\right)^T \neq 0_{m\times m}$$

Noise-free pseudo-measurements in extended Kalman filtering

Problem 1: $H_k P_k H_k^T$ rank deficiency

Problem 2: state update inconsistency $\longrightarrow$ $h(\hat{x}_k^+) \neq y_k$

Problem 3: Riccati update inconsistency $\longrightarrow$ $\left(\frac{\partial h}{\partial x}\Big|_{\hat{x}_k^+} \right) P_k^+ \left(\frac{\partial h}{\partial x}\Big|_{\hat{x}_k^+} \right)^T \neq 0$

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Invariant filtering in a nutshell

State embedded into a matrix Lie group (Axel Barrau and Silvère Bonnabel, 2016):

$$\chi_k \in G$$

[3] A. Barrau and S. Bonnabel, "Invariant Kalman filtering", in *Annual Review of Control, Robotics, and Autonomous Systems*, vol. 1, p. 237-257, 2018.

Invariant filtering in a nutshell

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$$\chi_k \in G$$

Left- or right-invariant estimation error :

$$\eta_k = \hat{\chi}_k^{-1} \chi_k \qquad \text{or} \qquad \eta_k = \chi_k \hat{\chi}_k^{-1}$$

$$\Rightarrow \quad \underset{\in G}{\underset{\downarrow}{\eta_k}} = \exp(\underset{\in \mathbb{R}^n}{\underset{\downarrow}{\xi_k}}) \quad \text{with} \quad \xi_k \sim \mathcal{N}(0_{n \times 1}, P_k)$$

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Invariant filtering in a nutshell

Output function
in left- or right-invariant form:
$y_k = \chi_k d_k$ or $y_k = \chi_k^{-1} d_k$

$\Rightarrow$

Invariant filtering Jacobian H_k
independent from the trajectory

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Example (left-invariant case):

$$
\begin{aligned}
z_k &= \hat{\chi}_k^{-1} y_k - d_k, \\
&= \hat{\chi}_k^{-1} \chi_k d_k - d_k, \\
&= \exp(\xi_k) d_k - d_k,
\end{aligned}
$$

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&= \boxed{\exp(\xi_k) d_k - d_k,} \longrightarrow \text{Linearization around } \xi_k = 0_{n\times 1} \\
&\approx H_k \xi_k.
\end{aligned}
$$

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Handling noise-free pseudo-measurements with an IEKF

- Solving problem 1 (rank deficiency of $H_k P_k H_k^T$)

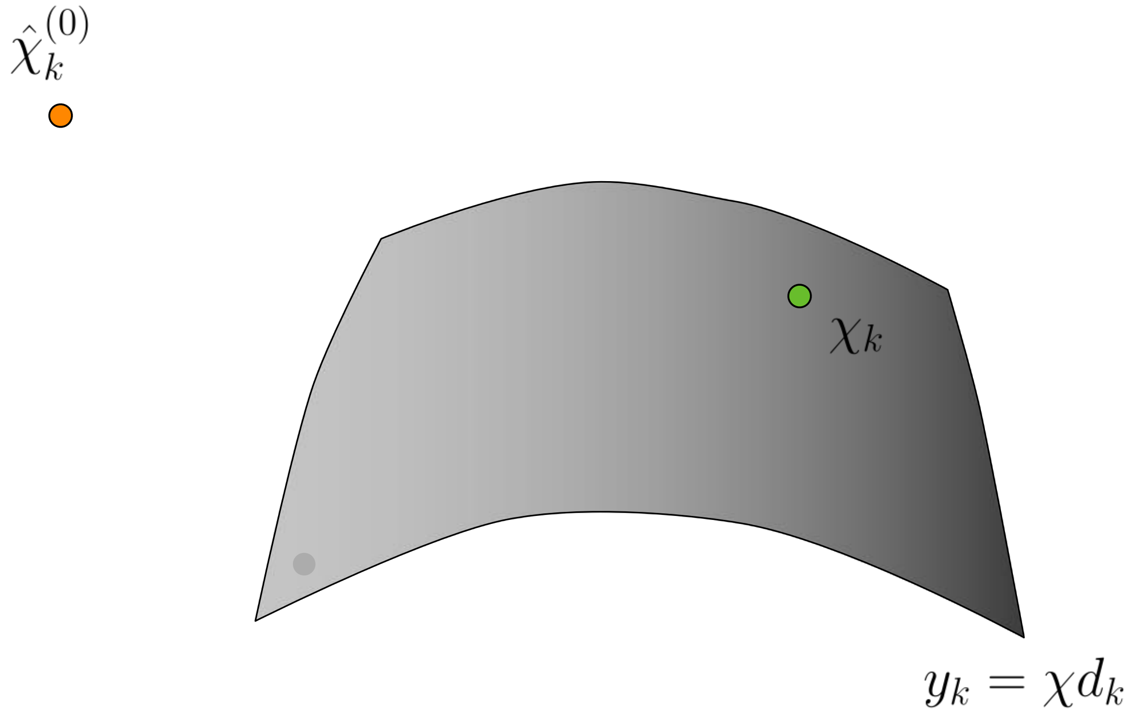
$$\lim_{\delta \to 0} K_k = \lim_{\delta \to 0} P_k H_k^T (H_k P_k H_k^T + \delta I)^{-1},$$
$$= L_k (H_k L_k)^{\dagger},$$
$$= K_k^{\mathrm{nf}},$$

with $(\cdot)^{\dagger}$ the Moore-Penrose pseudo-inverse and $P_k = L_k L_k^T$.

Handling noise-free pseudo-measurements with an IEKF

- Solving problem 2 (state update inconsistency)

  Use the Gauss-Newton algorithm to bring the estimate onto the right subgroup.



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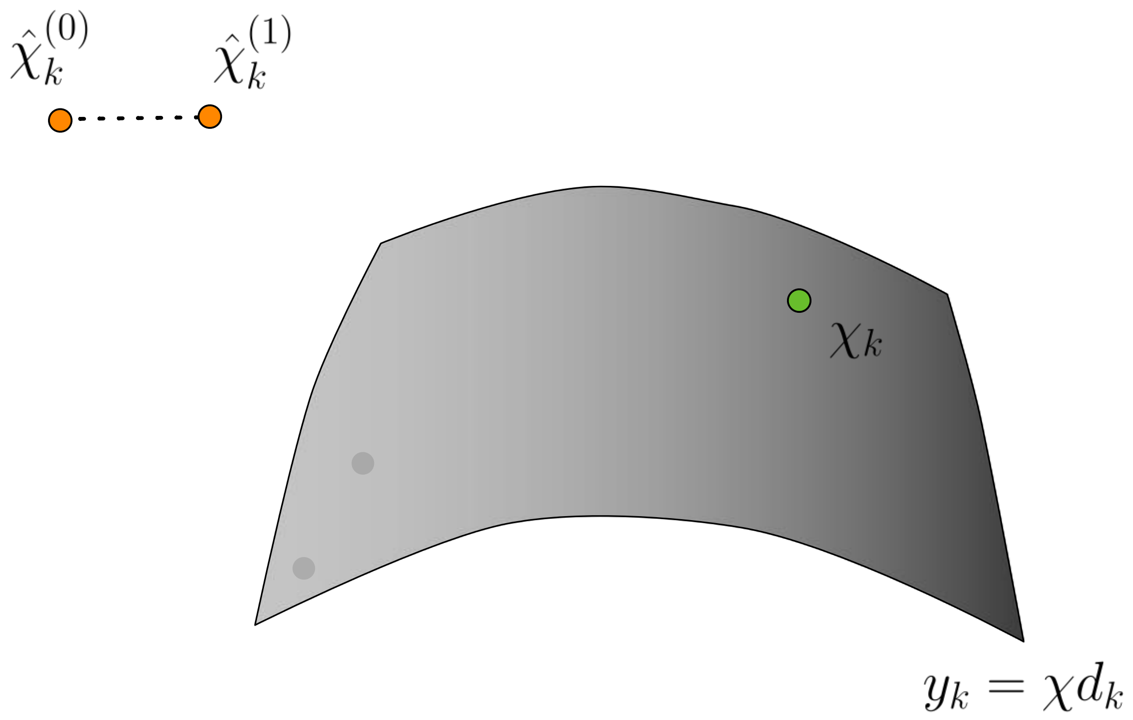

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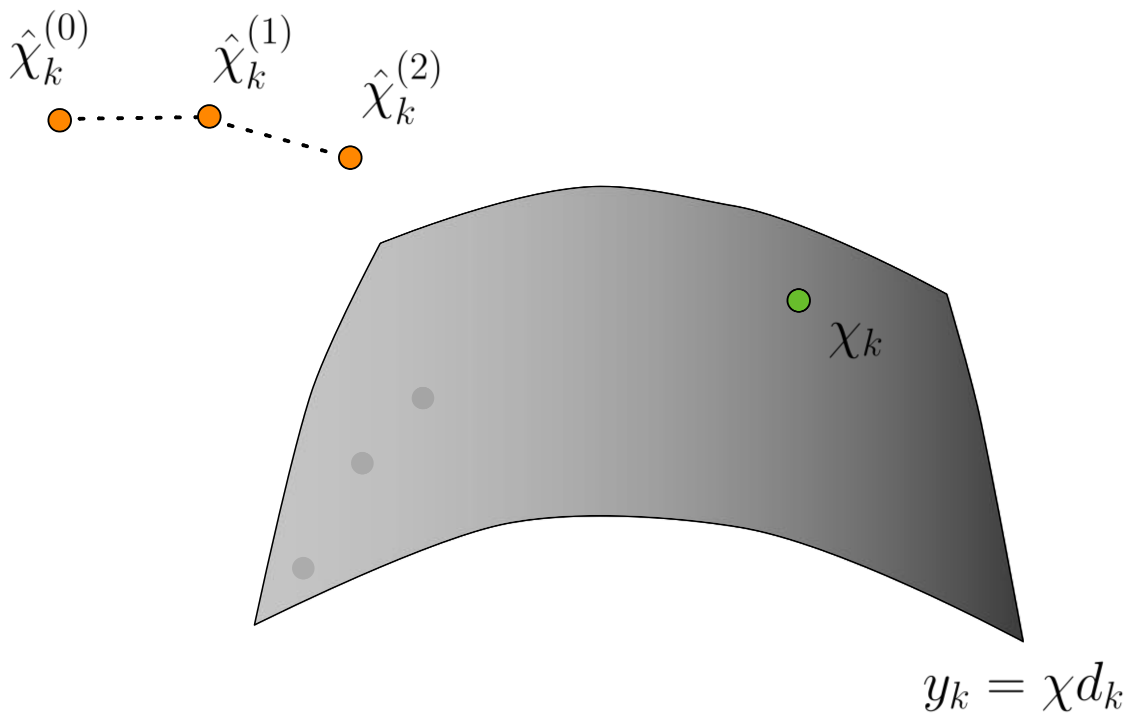

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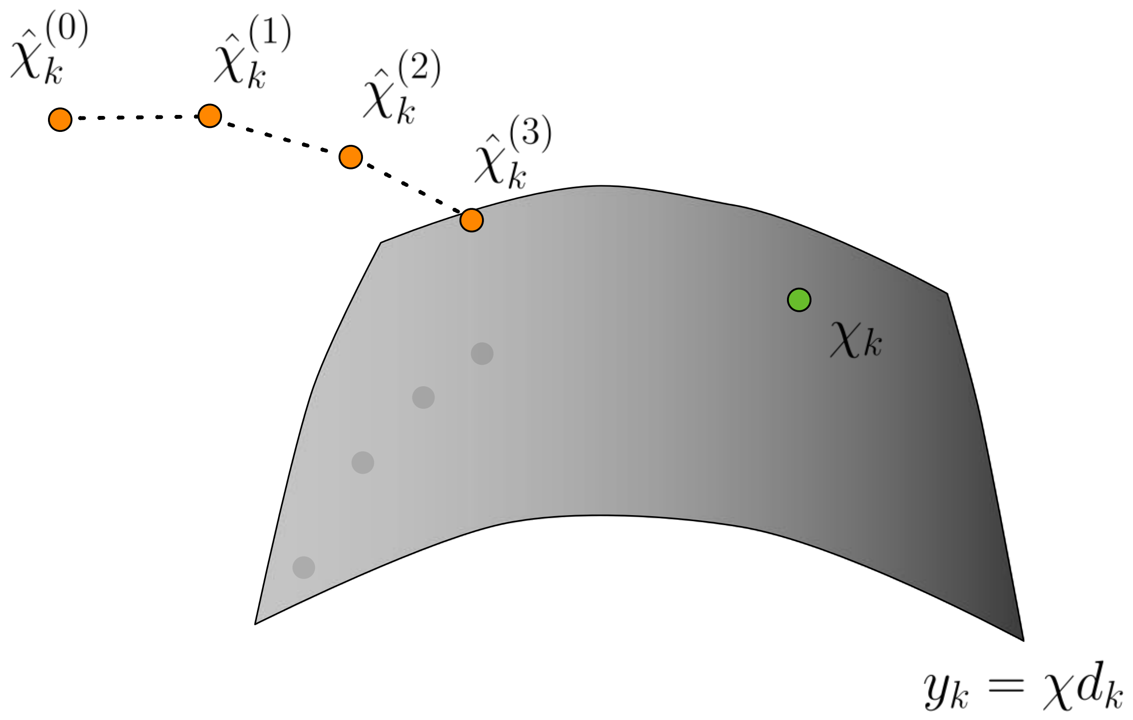

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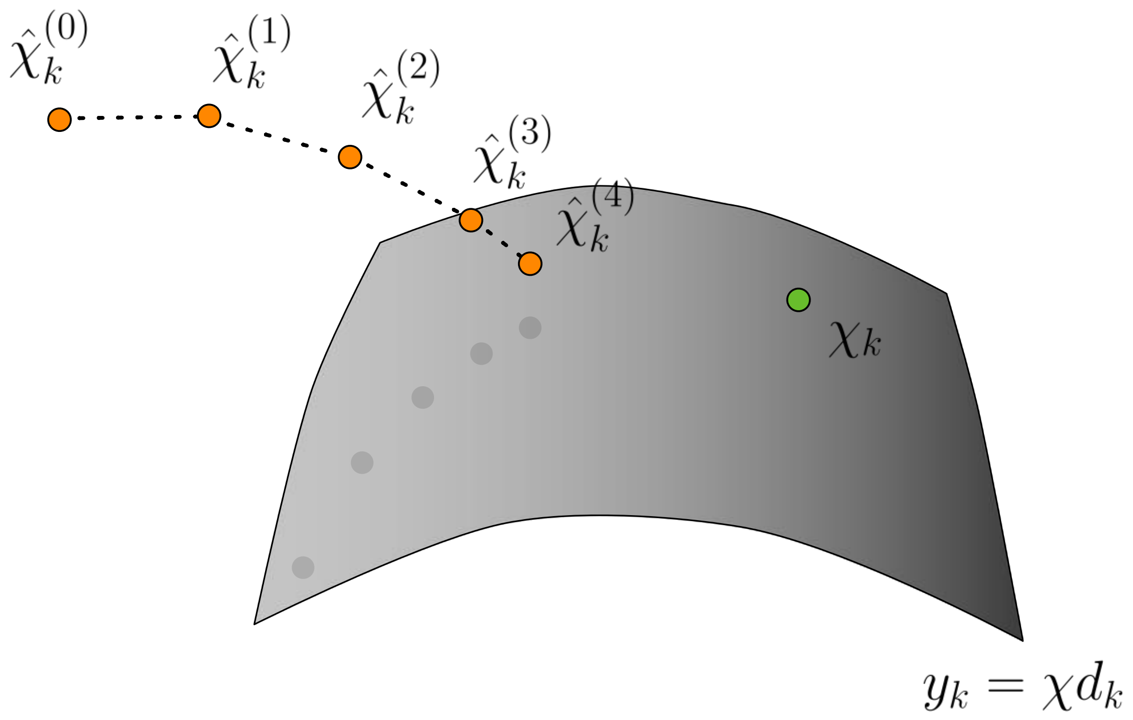

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➢ Solving problem 2 (state update inconsistency)

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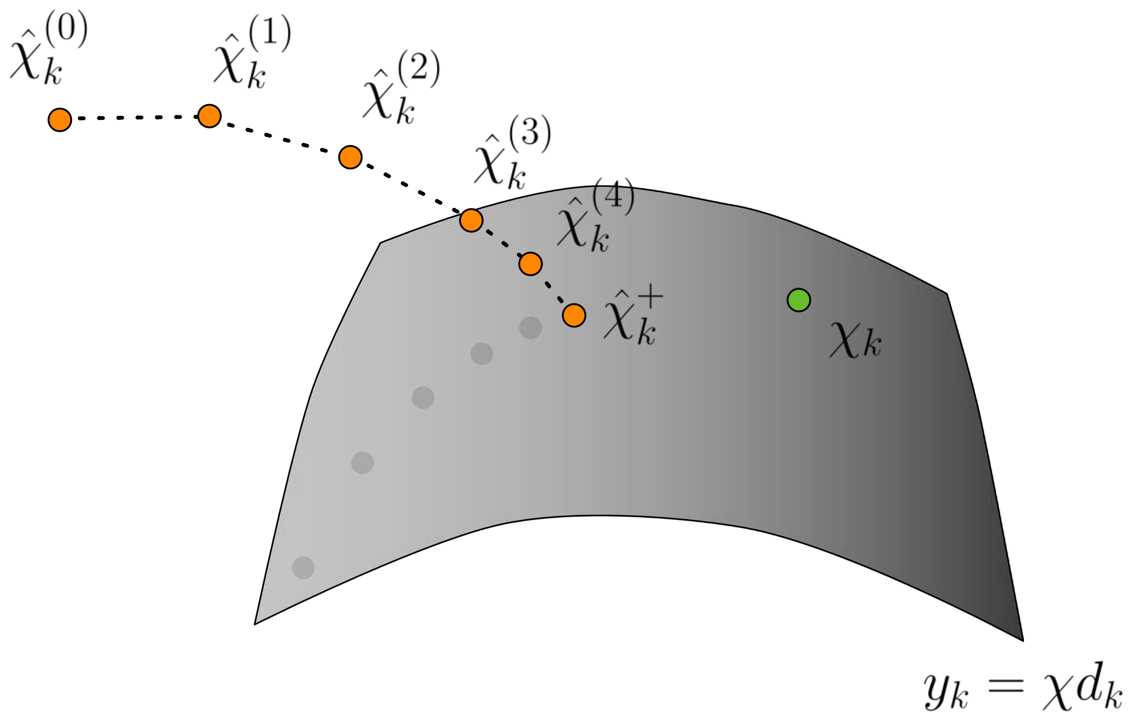

Handling noise-free pseudo-measurements with an IEKF

- Solving problem 3 (Riccati update inconsistency)

Invariant framework

$$\Downarrow$$

H_k independent from the current estimate

$$\Downarrow$$

$H_k P_k^+ H_k^T = 0_{m\times m}$ enforced at χ_k and χ_k^+

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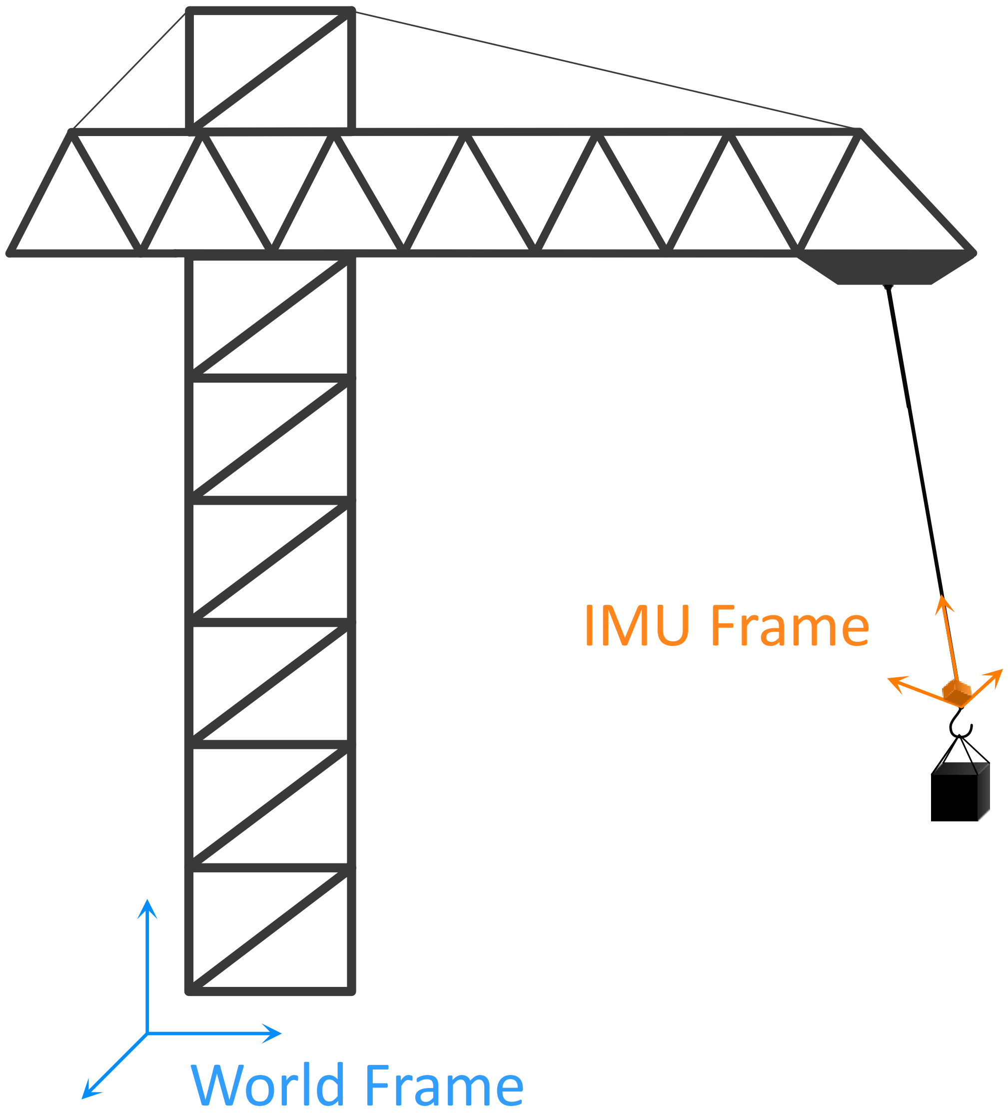
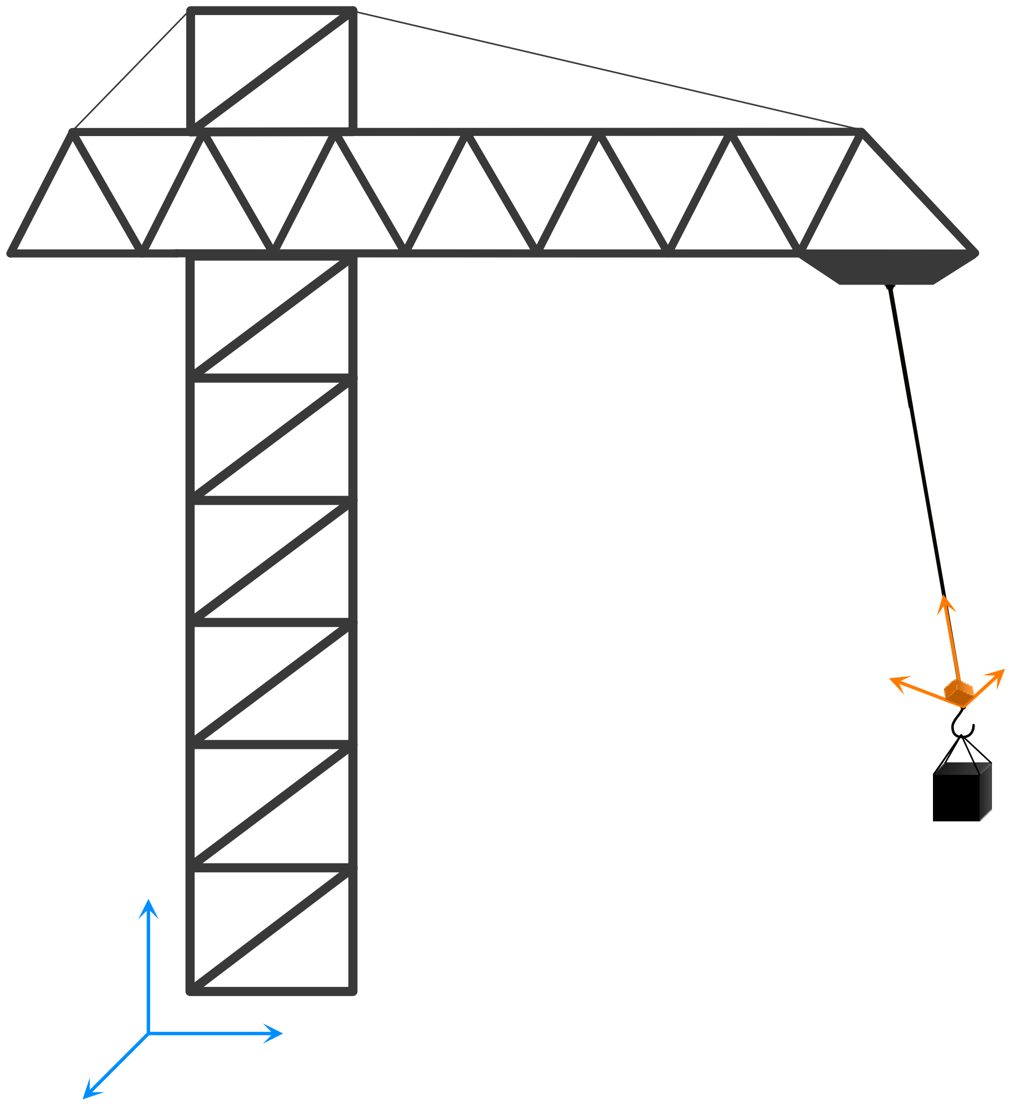


Hook pose: $\chi_k = \begin{bmatrix} R_k & v_k & p_k \\ 0_{1\times 3} & 1 & 0 \\ 0_{1\times 3} & 0 & 1 \end{bmatrix} \in SE_2(3)$

R_k	rotation matrix (IMU to world frame)
v_k	IMU velocity vector in world frame
p_k	IMU position vector in world frame

[5] Axel Barrau, *"Non-linear state error based extended Kalman filters with applications to navigation"*, Diss. Mines Paristech, 2015.

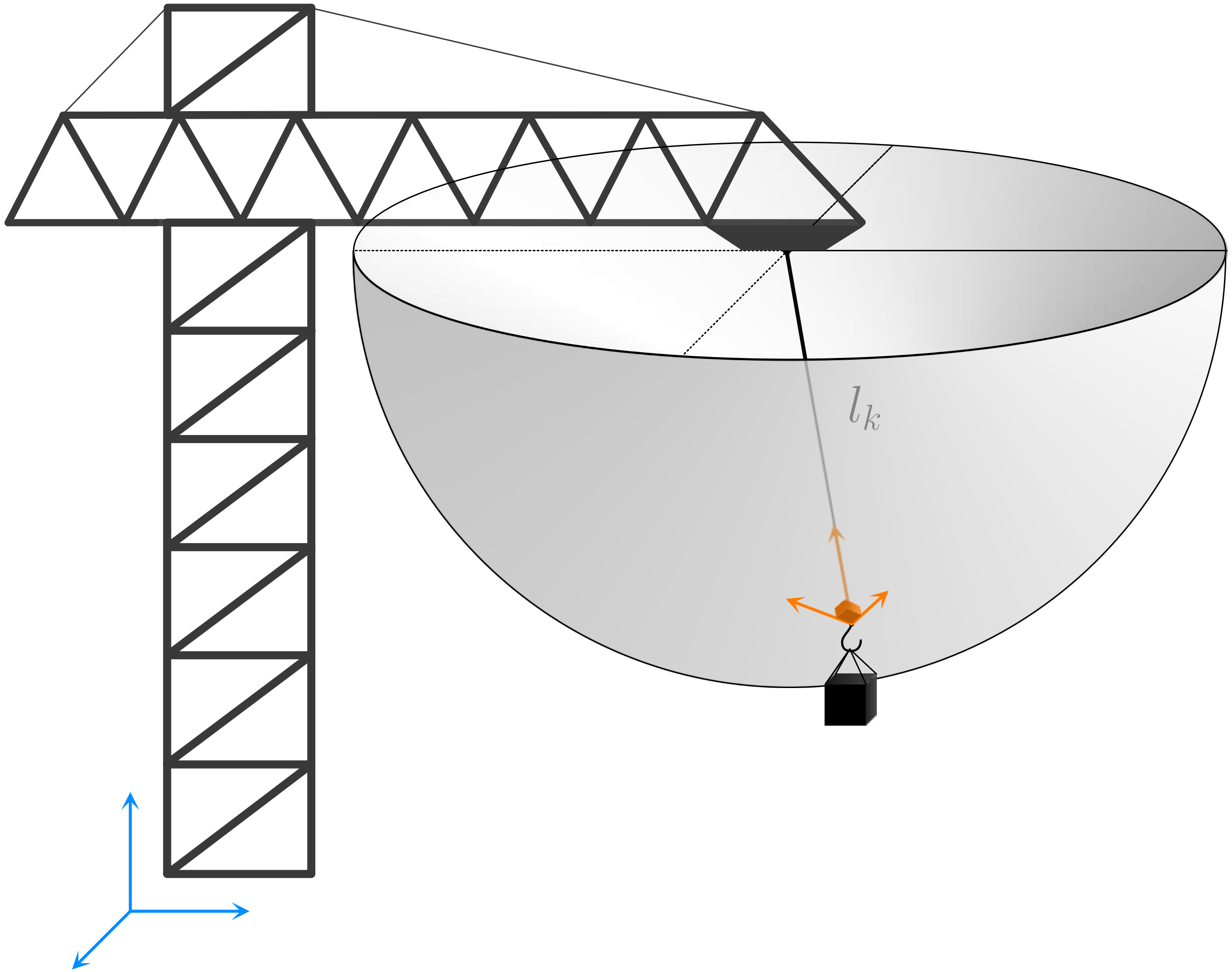
Application to IMU pose estimation for the hook of a crane



System dynamics:

$$\begin{cases} R_{k+1} = R_k \exp((\omega_k + w_k^{\omega})_{\times}\, dt), \\ v_{k+1} = v_k + (R_k(a_k + w_k^{a}) + g)\ dt, \\ p_{k+1} = p_k + v_k\, dt, \end{cases}$$

Application to IMU pose estimation for the hook of a crane



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v_{k+1} = v_k + \left(R_k(a_k + w_k^a) + g\right) \, dt, \\
p_{k+1} = p_k + v_k \, dt,
\end{cases}
$$

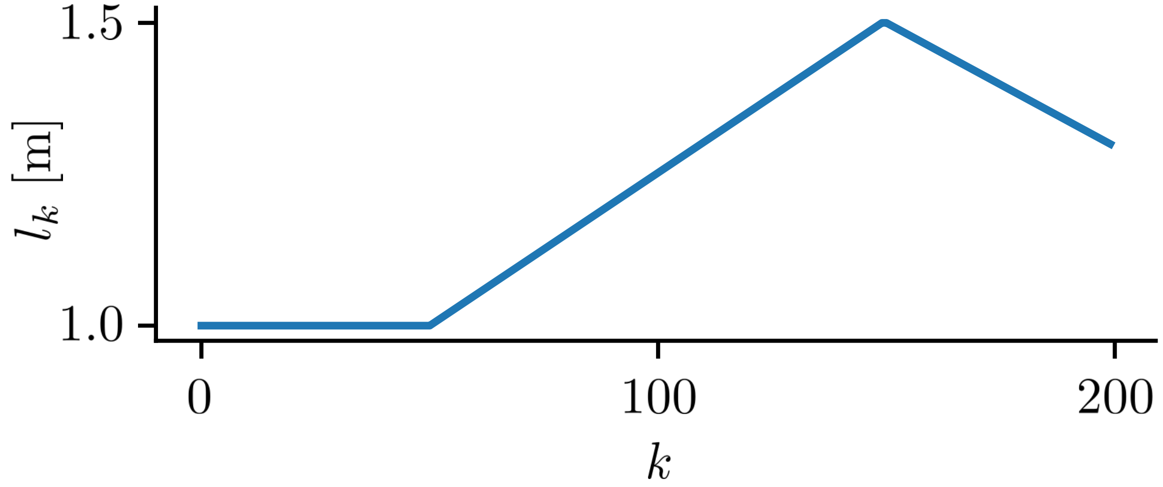
Pseudo-measurement :

$$
\underbrace{\chi_k \begin{bmatrix} 0 \\ 0 \\ l_k \\ 0 \\ 1 \end{bmatrix}}_{\chi_k d_k} = \underbrace{\begin{bmatrix} q_k \\ 0 \\ 1 \end{bmatrix}}_{y_k}
$$

with l_k the cable length and q_k the position of the hang-up point in the world frame.

Application to IMU pose estimation for the hook of a crane

Length profile :

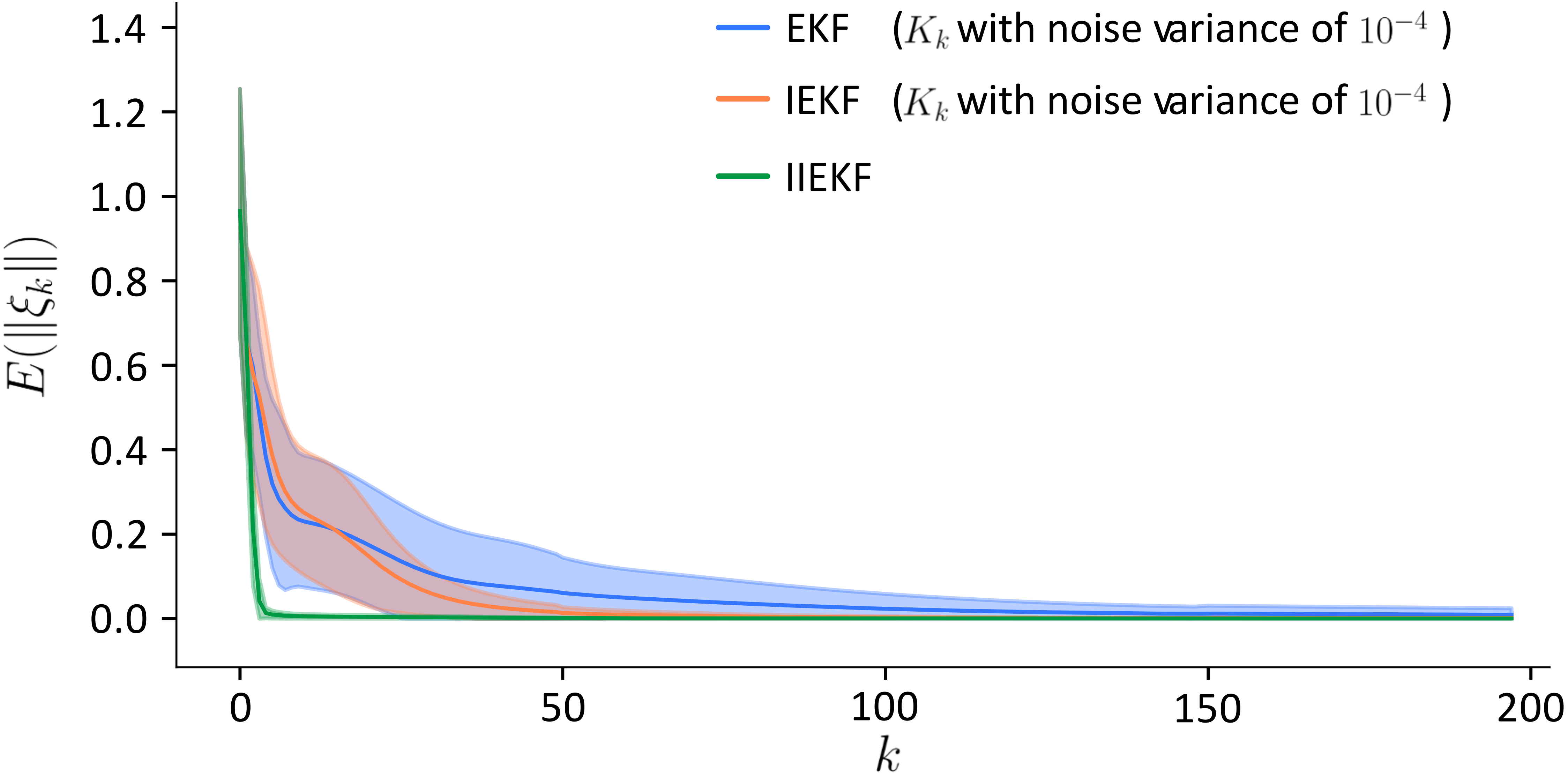


Simulation :

- Hook initial inclination is 20° w.r.t. the vertical (ground truth)
- No initial angular velocities and known initial azimuthal angle
  $\Rightarrow$ motion in a plane to avoid observability issues
- Random initial estimation error

Application to IMU pose estimation for the hook of a crane

Average and standard deviation of the norm of the linearized error over 30 simulations



Conclusion

- Equality constraints can be seen as noise-free pseudo-measurements

- We tackled the issues stemming from the noise-free nature of pseudo-measurments and developed a filter that can be applied to a wide range of problems involving equality constraints:

  - Rank deficiency (Kalman gain) $\longrightarrow$ Noise-free gain K_k^{nf}
  - State update inconsistency $\longrightarrow$ Iterative algorithm
  - Riccati update inconsistency $\longrightarrow$ IEKF framework

- The noise-free IEKF outperformed the other filters in a simple pose estimation simulation.

References

[1] F. Rauscher, S. Nann, and O. Sawodny, "*Motion control of an overhead crane using a wireless hook mounted IMU*." *in IEEE Annual American Control Conference (ACC)*, 2018.

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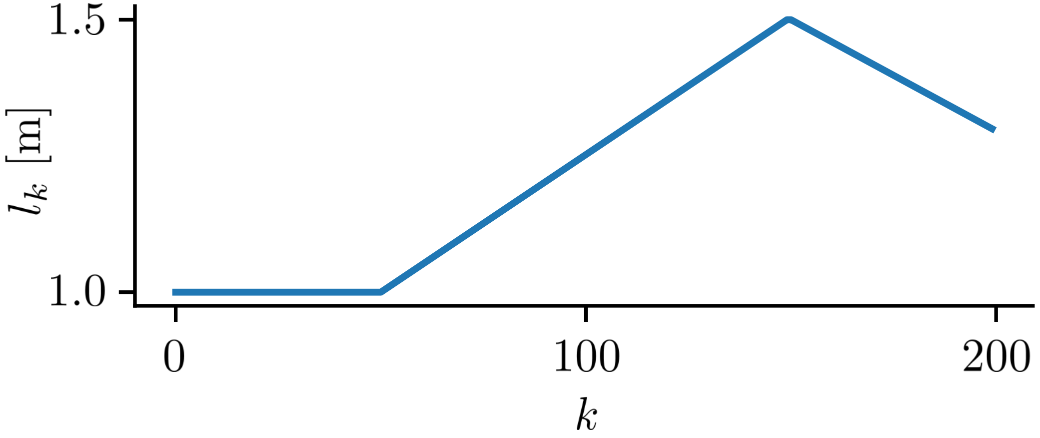
Back-up slides

Simulation : Crane state estimation

Cable length :



Estimation error (en back-up):

$$\xi_k = \begin{bmatrix} \log(\hat{R}_k^T R_k)^\vee \\ J_{\xi_k^R} \hat{R}_k^T (v_k - \hat{v}_k) \\ J_{\xi_k^R} \hat{R}_k^T (p_k - \hat{p}_k) \end{bmatrix}$$

with

- $\xi_k^R = \log(\hat{R}_k^T R_k)^\vee$
- $J_{\xi_k^R}$ the left-Jacobian of group SO(3) evaluated at ξ_k^R