

B. Fortz · A. R. Mahjoub · S. T. McCormick · P. Pesneau

Two-edge connected subgraphs with bounded rings: Polyhedral results and Branch-and-Cut

Received: July 11, 2003 / Accepted: January 18, 2005
Published online: March 14, 2005 – © Springer-Verlag 2005

Abstract. We consider the network design problem which consists in determining at minimum cost a 2-edge connected network such that the shortest cycle (a “ring”) to which each edge belongs, does not exceed a given length K . We identify a class of inequalities, called cycle inequalities, valid for the problem and show that these inequalities together with the so-called cut inequalities yield an integer programming formulation of the problem in the space of the natural design variables. We then study the polytope associated with that problem and describe further classes of valid inequalities. We give necessary and sufficient conditions for these inequalities to be facet defining. We study the separation problem associated with these inequalities. In particular, we show that the cycle inequalities can be separated in polynomial time when $K \leq 4$. We develop a Branch-and-Cut algorithm based on these results and present extensive computational results.

1. Introduction

A graph $G = (V, E)$ is said to be k -edge connected if between every pair of nodes of G , there are at least k edge-disjoint paths. Given a graph $G = (V, E)$, with edge costs $c_e \geq 0$, for all $e \in E$, and an integer $K \geq 3$, the 2-edge connected subgraph with bounded rings problem (2ECSBR) is to find a minimum cost 2-edge connected subgraph $H = (V, F)$ such that each edge of F belongs to a cycle of length less than or equal to K . Such a cycle is called a *feasible cycle*. This problem has applications in the design of survivable telecommunication networks. The bound on the length of the cycles comes from limitations of the routing equipment in some networks such as SDH/SONET networks and WDM ring technologies [23].

Fortz et al. [8] consider the node version of 2ECSBR, that is the problem of finding a minimum cost 2-node connected spanning subgraph where every edge belongs to a cycle of length less than or equal to K . They derive valid and facet defining inequalities for the associated polytope, and devise separation procedures. They also present a cutting plane algorithm along with experimental results. In [9], Fortz and Labbé give a formulation

B. Fortz: Institut d’Administration et de Gestion, Université Catholique de Louvain, Louvain-la-Neuve, Belgique. e-mail: fortz@poms.ucl.ac.be

A. R. Mahjoub: LIMOS CNRS, Université Blaise Pascal, Clermont-Ferrand, France.
e-mail: Ridha.Mahjoub@math.univ-bpclermont.fr

S. T. McCormick: Sauder School of Business, University of British Columbia, Vancouver, Canada.
e-mail: stmv@adk.commerce.ubc.ca

P. Pesneau: LIMOS CNRS, Université Blaise Pascal, Clermont-Ferrand, France and IAG, Université Catholique de Louvain, Louvain-la-Neuve, Belgique. e-mail: pesneau@poms.ucl.ac.be

Mathematics Subject Classification (1991): 90B10, 90C27, 90C57

for the problem based on a set covering approach. They provide further classes of facets and discuss the associated separation problems. They also report computational results with a cutting plane algorithm. Related work can also be found in [11]. For an extensive survey of this problem, see Fortz [7].

In this paper we study the 2ECSBR. We identify a class of valid inequalities, called cycle inequalities, and we show that these inequalities together with the cut and the integrality constraints yield a 0 – 1 integer programming formulation for the problem in the space of the natural design variables. We show that some facet defining inequalities for the 2-node connected polytope with bounded rings are also facet defining inequalities for the 2ECSBR polytope. We describe new classes of valid inequalities for the 2ECSBR polytope and give necessary and sufficient conditions for most of these inequalities to be facet defining. We show that the separation problem for the cycle inequalities when $K \leq 4$ can be reduced to solving a max-flow problem and then can be solved in polynomial time, and we give a heuristic for solving this problem for $K \geq 5$. We also devise separation procedures for the other classes of inequalities. These are used in a Branch-and-Cut algorithm for which we report computational results on random and real-world problems.

In addition to the fact that we study here the edge connectivity version of the problem, which is also of practical interest, we bring several contributions that were not part of the previous work of Fortz et al. on the node connectivity version. The major new contribution is the introduction of cycle inequalities that are sufficient, with cut constraints, to formulate the problem as a 0 – 1 integer program. Cycle partition inequalities are also new. The characterization of facet defining cyclomatic inequalities extend the work in [11] to the full class of inequalities, while only a special case was considered in [11]. Another major difference is the fact that we consider the polytope associated to the problem and not its dominant as in [9, 11].

The 2ECSBR problem is a generalization of the classical minimum 2-edge connected subgraph problem. In fact, the latter is nothing but the 2ECSBR when $K \geq |V|$. The minimum 2-edge connected subgraph problem has been the subject of extensive research in the last ten years. In [21] Mahjoub gave a complete description of the 2-edge connected polytope when the underlying graph is series-parallel. In [3] Barahona and Mahjoub characterized this polytope in the class of Halin graphs. Further polyhedral results related to k -edge (k -node) connected graphs and more general survivable network design problems can be found in [4] [6] [15] [19].

The paper is organized as follows. In the following section we give a 0 – 1 integer programming formulation for the 2ECSBR and study the dimension of the associated polytope. In Sections 3 and 4 we describe various classes of valid inequalities for this polytope and give necessary and sufficient conditions for these inequalities to be facet defining. We also discuss separation algorithms for these inequalities. In section 5 we devise a Branch-and-Cut algorithm based on these results for the 2ECSBR. In Section 6 we present some computational results and in Section 7 we give some concluding remarks.

The rest of this section is devoted to more definitions and notations. The graphs we consider are undirected and loopless. We denote a graph by $G = (V, E)$ where V is the node set and E the edge set. Throughout this paper, $n := |V|$ and $m := |E|$ will denote the number of nodes and edges of G .

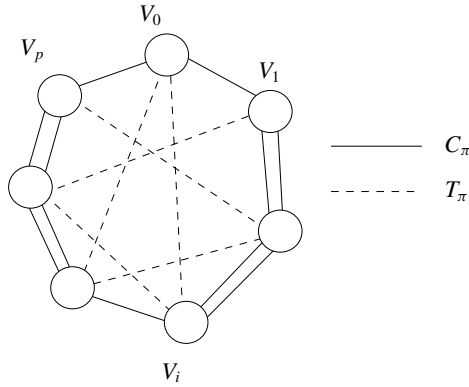


Fig. 1. C_π and T_π

Given a graph $G = (V, E)$ and $W \subset V$, the edge set

$$\delta(W) := \{ij \in E \mid i \in W, j \in V \setminus W\}$$

is called the *cut* induced by W . If V_0, \dots, V_p are pairwise disjoint node subsets of V , then $\delta(V_0, \dots, V_p)$ denotes the set of edges having one node in V_i and the other in V_j for $i \neq j$. If $\pi = (V_0, \dots, V_p)$ is a partition of V , then we let $C_\pi = \cup_{i=0}^{p-1} \delta(V_i, V_{i+1}) \cup \delta(V_0, V_p)$ and $T_\pi = \delta(V_0, \dots, V_p) \setminus C_\pi$ (see Figure 1).

The *degree* of a node v is the cardinality of $\delta(v)$. Given a node subset W of V , we denote by $E(W)$ the set of edges having both endnodes in W . If $F \subseteq E$ is an edge subset, we denote by $V(F)$ the set of nodes incident to edges of F . We denote by $G(W) = (W, E(W))$ the subgraph induced by edges having both endnodes in W . G/W is the graph obtained from G by contracting the nodes in W to a new node w (retaining parallel edges). Contracting an edge set F consists in contracting the nodes of $V(F)$.

We denote by $V - z := V \setminus \{z\}$ and $E - e := E \setminus \{e\}$ the subsets obtained by removing one node or one edge from the set of nodes or edges, and $G - z$ denotes the graph $G(V - z)$, i.e. the graph obtained by removing a node z and its incident edges from G .

If $x \in \mathbb{R}^m$, the support graph of x , denoted by G_x , is the graph with vertex set V and edge set $\{e \mid x_e > 0\}$.

2. Formulation and associated polyhedron

In this section, we formulate 2ECSBR as a 0 – 1 integer program in the space of the natural variables x . We also determine the dimension of the associated polyhedron. To this end, we first introduce some notation and a family of valid inequalities for 2ECSBR.

We denote by $\mathcal{F}(G)$ the set of edge-sets that induce feasible solutions of 2ECSBR. Given an edge subset $F \subset E$, we denote by $x^F \in \mathbb{R}^m$ the *incidence vector* of F , given by:

$$x_e^F = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, each vector $x \in \{0, 1\}^m$ induces a subset

$$F^x := \{e \in E \mid x_e = 1\}.$$

The convex hull of the incidence vectors of all the solutions of 2ECSBR is

$$\mathcal{P}(G, K) := \text{conv}\{x \in \{0, 1\}^m \mid F^x \in \mathcal{F}(G)\}.$$

Finally, for any subset of edges $F \subseteq E$ we define $x(F) := \sum_{e \in F} x_e$.

Now let $G = (V, E)$ be a graph and $K \geq 3$. Let $\pi = (V_0, \dots, V_p)$ be a partition of V such that $p \geq K$ and let $e \in \delta(V_0, V_p)$. Consider the following inequality

$$x(T_\pi^e) \geq x_e, \quad (2.1)$$

with $T_\pi^e := T_\pi \cup (\delta(V_0, V_p) \setminus \{e\})$. We have the following.

Theorem 1. *Let $F \subseteq E$ be an edge subset inducing a 2-edge connected subgraph. Then $(V(F), F)$ is a solution of 2ECSBR if and only if for any partition $\pi = (V_0, \dots, V_p)$, with $p \geq K$, and edge $e \in \delta(V_0, V_p)$, inequality (2.1), associated with π and e , is satisfied by x^F .*

Proof. Suppose that $(V(F), F)$ is a solution of the 2ECSBR. Suppose there are a partition $\pi = (V_0, \dots, V_p)$ and an edge $e \in \delta(V_0, V_p)$ such that the associated inequality (2.1) is violated by x^F . Then $e \in F$ and $F \cap (T_\pi^e) = \emptyset$. By the definition of T_π^e , any cycle in F using e is completely included in C_π , and is therefore of length greater than p . As $p \geq K$, e does not belong to a feasible cycle, which leads to a contradiction since F induces a solution and $e \in F$.

Conversely, suppose there is an edge $e := uv \in F$ that does not belong to a feasible cycle of $(V(F), F)$, that is every cycle of $(V(F), F)$ containing e is of length $\geq K + 1$. Let $d(u, i)$ denote the distance from u to i in $F \setminus \{e\}$. Then $d(u, v) \geq K$ and the inequality (2.1) associated with the partition

$$\begin{aligned} V_i &= \{w \in V \mid d(u, w) = i\} \quad i = 0, \dots, K - 1, \\ V_K &= \{w \in V \mid d(u, w) \geq K\} \end{aligned}$$

together with edge e is violated by x^F , which ends the proof of the theorem. \square

From Theorem 1, it follows that inequalities (2.1) are valid for $\mathcal{P}(G, K)$. These inequalities will be called *cycle inequalities*. An immediate consequence of Theorem 1, is the following.

Corollary 1. *Let $G = (V, E)$ be a graph and $K \geq 3$. 2ECSBR is equivalent to the following integer programming problem:*

$$\begin{aligned} \text{Min} \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & x(\delta(W)) \geq 2, \quad \text{for all } W \subset V, \emptyset \neq W \neq V, \end{aligned} \quad (2.2)$$

$$\begin{aligned} & x(T_\pi^e) \geq x_e, \quad \text{for all partition } \pi = (V_0, \dots, V_p) \\ & \quad \text{with } p \geq K \text{ and } e \in \delta(V_0, V_K), \end{aligned}$$

$$0 \leq x_e \leq 1, \quad \text{for all } e \in E, \quad (2.3)$$

$$x_e \in \{0, 1\}, \quad \text{for all } e \in E. \quad (2.4)$$

Inequalities (2.2) are called *cut inequalities*, and inequalities (2.3) are called *trivial inequalities*.

Note that, by replacing in the formulation given by Corollary 1 the constraints (2.2) by

$$x(\delta_{G-v}(W)) \geq 1, \quad W \subset V \setminus \{v\}, \quad v \in V,$$

we obtain a formulation for the 2-node connected subgraph with bounded rings problem.

By Corollary 1, it follows that

$$\mathcal{P}(G, K) = \text{conv}\{x \in \mathbb{R}^m \mid x \text{ verifies (2.1)–(2.4)}\}.$$

We now study the dimension of $\mathcal{P}(G, K)$. To this aim, we need the following concepts. Given a graph $G = (V, E)$, a constant $K \geq 3$, and a subset of edges $F \subseteq E$, the *restriction of F to bounded rings* (Fortz and Labbé [9]) is defined as

$$F_K := \{e \in F \mid e \text{ belongs to at least one feasible cycle in } F\}.$$

By extension, the subgraph $G_K = (V, E_K)$ is called the *restriction of G to bounded rings*. Remark that an edge $e \in E \setminus E_K$ will never belong to a feasible solution of 2ECSBR and is called an *infeasible edge*.

Given two edges $e, f \in E_K$, ($e \neq f$), we say that e *depends* on f , denoted by $e < f$, if and only if

$$e \in F \Rightarrow f \in F, \quad \text{for all } F \in \mathcal{F}(G),$$

or equivalently,

$$x_e^F \leq x_f^F, \quad \text{for all } F \in \mathcal{F}(G).$$

Note that the dependence relation $<$ is transitive and reflexive, and therefore defines a semiorder on the edges of E . We also define $\Gamma(e)$ as the set of edges that depend on e , i.e.

$$\Gamma(e) = \{f \in E \mid f < e\}.$$

If $e < f$ and $f < e$, then e and f are *equivalent* and $x_e^F = x_f^F$ for all $F \in \mathcal{F}(G)$. A maximal subset T (with respect to inclusion) of equivalent edges is an *equivalence class* for the semiorder $<$, and $<$ induces a partition of any set of edges into equivalence classes. As $\Gamma(e) = \Gamma(f)$ if e and f are equivalent, we extend the notation to a subset of equivalent edges T by setting $\Gamma(T) := \Gamma(e)$, $e \in T$. Given two equivalence classes T_i and T_j , we say that T_i depends on T_j if there are two edges $e_i \in T_i$ and $e_j \in T_j$ such that e_i depends on e_j . Finally, an edge $e \in E$ is *essential* if e belongs to every solution $F \in \mathcal{F}(G)$.

The dimension of $\mathcal{P}(G, K)$ is stated in the following theorem.

Theorem 2. *Let $G = (V, E)$ be a graph. Let E^* be the set of essential edges. Suppose $<$ induces a partition of $E_K \setminus E^*$ into l equivalence classes. Then*

$$\dim(\mathcal{P}(G, K)) = l.$$

Proof. First of all, if $E_K \notin \mathcal{F}(G)$, then $\mathcal{P}(G, K) = \emptyset$ and $\dim(\mathcal{P}(G, K)) = l = 0$. So suppose that E_K is a solution of $\mathcal{F}(G)$. Suppose the l equivalence classes of $E_K \setminus E^*$ are $T_i := \{e_1^i, \dots, e_{k_i}^i\}$ of cardinality k_i for $i = 1, \dots, l$. Since the sets $T_i, i = 1, \dots, l$, form a partition of $E_K \setminus E^*$,

$$\sum_{i=1}^l k_i = |E_K| - |E^*|.$$

Let x be the incidence vector of a feasible solution of 2ECSBR. From the definition of E_K, E^* and \prec , it is easy to see that x is a solution of the system

$$(S) \begin{cases} x_e = 0 & e \in E \setminus E_K, \\ x_e = 1 & e \in E^*, \\ x(e_j^i) - x(e_{j+1}^i) = 0 & j = 1, \dots, k_i - 1, \quad i = 1, \dots, l. \end{cases}$$

As $E \setminus E_K, E^*$ and $T_i, i = 1, \dots, l$ are obviously pairwise disjoint, the equalities in (S) are linearly independent and

$$\dim(\mathcal{P}(G, K)) \leq |E_K| - |E^*| - \sum_{i=1}^l (k_i - 1) = l.$$

Now consider the $l + 1$ subsets $F_0 := E_K$ and $F_i := E_K \setminus \Gamma(T_i), i = 1, \dots, l$.

We claim that $F_i \in \mathcal{F}(G)$ for $i = 1, \dots, l$. Assume, by contradiction, this is not the case. Then, there is some $i \in \{1, \dots, l\}$ such that, in the graph induced by F_i , either there exists a cut inequality which is not satisfied, or there is an edge which does not belong to a cycle of length $\leq K$. Suppose there is a cut in $G(F_i)$ with $|\delta(W) \cap F_i| \leq 1$. Since $E_K \in \mathcal{F}(G)$, $\Gamma(T_i)$ must intersect $\delta(W)$. Hence T_i must be contained in every solution of $\mathcal{F}(G)$. But this implies that the edges of T_i are essential, a contradiction. Now suppose there is an edge $f \in F_i$ that does not belong to any cycle of $G(F_i)$ of length $\leq K$. As $E_K \in \mathcal{F}(G)$, any cycle of length $\leq K$ in $G(E_K)$ containing f must intersect $\Gamma(T_i)$. Hence any solution of $\mathcal{F}(G)$ that contains f , intersects $\Gamma(T_i)$. It follows that any solution of $\mathcal{F}(G)$ containing f , also contains the edges of T_i . But this implies that $f \in \Gamma(T_i)$, which is impossible.

Claim. There exists a permutation σ of $1, \dots, l$ such that for all $i = 1, \dots, l - 1, T_{\sigma(i)} \cap \Gamma(T_{\sigma(j)}) = \emptyset$ for $j = i + 1, \dots, l$.

Proof. First we show that there is a T_i that does not depend on any other $T_j, j \in \{1, \dots, l\} \setminus \{i\}$. Indeed, suppose this is not the case. Then T_1 must depend on a set, say T_{i_1} , with $i_1 \in \{2, \dots, l\}$. As T_1 and T_{i_1} are two different equivalence classes, there must exist $i_2 \in \{1, \dots, l\} \setminus \{1, i_1\}$ such that T_{i_1} depends on T_{i_2} . Now by iterating the process we get an infinite sequence of different sets $T_1 = T_{i_0}, T_{i_1}, \dots, T_{i_s}, \dots$ such that T_{i_j} depends on $T_{i_{j+1}}$ for $j = 0, \dots, s, \dots$. As l is finite, this is impossible.

So we may suppose that T_1 does not depend on any other $T_i, i = 2, \dots, l$. Let $\sigma(1) = 1$. Now suppose, by induction, that we have constructed the desired permutation up to $\sigma(k)$. Consider \prec restricted to $E' = E_K \setminus (E^* \cup \bigcup_{i=1}^k T_{\sigma(i)})$. The sets $T_j, j \in \{1, \dots, l\} \setminus \{\sigma(1), \dots, \sigma(k)\}$ are the equivalence classes for \prec on E' . By applying the reasoning

above on E' , we obtain a set T_r , $r \in \{1, \dots, l\} \setminus \{\sigma(1), \dots, \sigma(k)\}$ that does not depend on any set T_j , $j \in \{1, \dots, l\} \setminus \{\sigma(1), \dots, \sigma(k), r\}$. We can then set $\sigma(k+1) = r$. \diamond

By the Claim above, there exists a permutation σ of $1, \dots, l$ such that for all $i = 1, \dots, l-1$, there exists $e \in T_{\sigma(i)}$ such that $e \notin \Gamma(T_{\sigma(j)})$, for $j = i+1, \dots, l$.

Therefore the incidence vectors $x^{\Gamma(T_i)}$ are linearly independent for $i = 1, \dots, l$. Noting that $x^{\Gamma(T_i)} = x^{F_0} - x^{F_i}$, it follows that the incidence vectors of F_0, \dots, F_l are affinely independent, which implies that $\dim(\mathcal{P}(G, K)) \geq l$. \square

Corollary 2. *If $G = (V, E)$ is a complete graph with $|V| \geq 4$, then $\mathcal{P}(G, K)$ is full dimensional.*

Proof. It is easy to see that E and $E - e$, for all $e \in E$, induce feasible solutions of 2ECSBR. Therefore, $E = E_K$, $E^* = \emptyset$ and no edge depends on another edge. It follows that there are no equivalent edges, and $E_K \setminus E^* = E$ is partitioned in m equivalence classes. The result follows by Theorem 2. \square

In the remainder of the paper, we assume that $G = (V, E)$ is a complete graph. This assumption is not restrictive, since the problem in an incomplete graph can be reduced to the problem in a complete graph by giving a sufficiently high cost to non-existent edges.

3. Cycle and cycle partition inequalities

In this section, we introduce a second class of valid inequalities for $\mathcal{P}(G, K)$. We describe necessary and sufficient conditions for these inequalities as well as the cycle inequalities to be facet defining. We also discuss separation routines. Further classes of valid inequalities for $\mathcal{P}(G, K)$ will be presented in the next section.

3.1. Cycle inequalities

The next theorem characterizes which cycle inequalities are facet defining.

Theorem 3. *Let $G = (V, E)$ be a complete graph and $K \geq 3$. Let $\pi = (V_0, \dots, V_p)$ be a partition of V such that $p \geq K$. Let $e := uv$ be an edge of $\delta(V_0, V_p)$. Then the cycle inequality $x(T_\pi^e) \geq x_e$ associated with π and e defines a facet of $\mathcal{P}(G, K)$ if and only if*

1. $p = K$,
2. $|V_0| = |V_p| = 1$,
3. $|V_i| + |V_{i+1}| \geq 3$ for $i = 0, \dots, p-1$.

Proof. Suppose that $p > K$. Consider the partition $\pi' = (V'_0, \dots, V'_K)$ defined by $V'_i = V_i$ for $i = 0, \dots, K-1$ and $V'_K = \bigcup_{j=K}^p V_j$. It is easy to see that the cycle inequality associated with π and e is dominated by that associated with π' and e . Hence $x(T_\pi^e) \geq x_e$ does not define a facet.

Now suppose that $p = K$ and assume that $|V_0| > 1$ (the case $|V_p| > 1$ is similar). Let $\pi' = (V'_0, \dots, V'_K)$ be the partition defined by

$$\begin{aligned} V'_0 &= \{u\}, \\ V'_1 &= V_1 \cup (V_0 \setminus \{u\}), \\ V'_i &= V_i \quad \text{for } i = 2, \dots, K. \end{aligned}$$

Again, the cycle inequality associated with π and e is dominated by that associated with π' and e .

Finally, suppose that there exists $i \in \{0, \dots, p-1\}$ such that $|V_i| + |V_{i+1}| < 3$. Since the subsets defining the partition are nonempty, it follows that $|V_i| = |V_{i+1}| = 1$, and that $\delta(V_i, V_{i+1})$ contains a single edge f . If $x(T_\pi^e) \geq x_e$ defines a facet, then there exists a feasible solution F such that $x^F(T_\pi^e) = x_e^F = 0$. Otherwise, the face defined by $x(T_\pi^e) \geq x_e$ is included in the face defined by $x_e \leq 1$. But then,

$$\delta\left(\bigcup_{j=0}^i V_j\right) \cap F = \delta(V_i, V_{i+1}) = \{f\},$$

and F is not 2-edge connected, thus not feasible, which leads to a contradiction.

The three conditions are thus necessary for $x(T_\pi^e) \geq x_e$ to define a facet. Suppose they are all satisfied, and let us denote the cycle inequality by $a^T x \geq 0$. Let $b^T x \geq \beta$ be a facet defining inequality such that the face F_a induced by $a^T x \geq 0$ in $\mathcal{P}(G, K)$ is contained in the face F_b induced by $b^T x \geq \beta$.

We first show that $\bar{E} = E \setminus (T_\pi^e \cup \{e\})$ induces a feasible solution of 2ECSBR, and therefore its incidence vector lies in F_a . To do so, we show that every edge of \bar{E} belongs to a cycle of length 3, and then we show that \bar{E} induces a 2-edge connected subgraph.

Consider an edge $g = v_1 v_2 \in \bar{E}$. If g belongs to $E(V_i)$ for some $i \in \{1, \dots, K-1\}$, then for any $w \in V_{i-1}$, the cycle composed of g , $v_1 w$ and $v_2 w$ is included in \bar{E} . Now consider the case $g \in \delta(V_i, V_{i+1})$ for some $i \in \{0, \dots, K-1\}$. By the third condition, the graph induced by $V_i \cup V_{i+1}$, with edge set $E(V_i) \cup E(V_{i+1}) \cup \delta(V_i, V_{i+1})$, is a complete graph on at least 3 nodes. Therefore g belongs to a cycle of length 3.

As \bar{E} induces a connected graph, each cut contains at least one edge. Since each edge belongs to a cycle, it follows that each cut contains at least two edges, and \bar{E} induces a 2-edge connected subgraph.

Now, consider the edge set $E_f = \bar{E} \cup \{f, e\}$, with $f \in T_\pi^e$. Clearly, E_f is 2-edge connected as \bar{E} is. To show that E_f induces a feasible solution, it is thus sufficient to show that e and f belong to a ring of length $\leq K$. Suppose $f = v_i v_j \in \delta(V_i, V_j)$ with $i < j$. As $f \in T_\pi^e$, $j \geq i+2$. For $k = 0, \dots, i-1, j+1, \dots, K$, select a node $v_k \in V_k$. As $e = v_0 v_K \in E_f$ and $v_k v_{k+1} \in E_f$ for $k = 0, \dots, i-1, j, \dots, K-1$, the edges e , f and $v_k v_{k+1}$ for $k = 0, \dots, i-1, j, \dots, K-1$ form a cycle of length $2+i+K-j \leq K$ since $j \geq i+2$. Therefore, the incidence vector of E_f lies in F_a , thus also in F_b , and $b x^{\bar{E}} = b x^{E_f}$. It follows immediately that $b_f = -b_e$ for all $f \in T_\pi^e$.

It remains to show that $b_g = 0$ for all $g \in \bar{E}$. First suppose $g \in E(V_i)$ for some $i \in \{1, \dots, K-1\}$. Let $g = u_1 u_2$, and consider two nodes $w_1 \in V_{i-1}$ and $w_2 \in V_{i+1}$. Observe that $w_1 w_2 \in T_\pi^e$, and let $\tilde{E} = (\bar{E} \setminus \{g\}) \cup \{w_1 w_2, e\}$. As \tilde{E} induces a connected subgraph, to show that it induces a feasible solution, it is sufficient to show that every

edge of \tilde{E} belongs to a feasible cycle. Let $h = st \in \tilde{E} \setminus \{g\}$. If $h \in E(V_j)$ for some $j \in \{1, \dots, K-1\}$, then clearly, h belongs to a triangle, and hence to a feasible cycle. So suppose that $s \in V_j$ and $t \in V_{j+1}$ for some $j \in \{0, \dots, K-1\}$. If $j \neq i-1$ and $j \neq i$, then by condition 3, we may assume w.l.o.g. that $|V_j| \geq 2$. If $s' \in V_j \setminus \{s\}$, then h belongs to the triangle induced by $\{s, s', t\}$. Now assume that $j = i$ (the case $j = i-1$ is similar). If $|V_{j+1}| \geq 2$, then it is easy to show as before that h belongs to a triangle. If $|V_{j+1}| = 1$, then $w_2 = t$, and h belongs to the triangle induced by $\{s, t, w_1\}$.

Moreover, edges e and w_1w_2 belong to a feasible cycle. Indeed, for $k = 0, \dots, i-2, i+2, \dots, K$, select a node $v_k \in V_k$. Then, $e, w_1w_2, v_{i-2}w_1, w_2v_{i+2}$ and v_kv_{k+1} for $k = 0, \dots, i-3, i+2, \dots, K-1$ form a cycle of length $4 + (i-2) + (K-i-2) = K$. In consequence, the incidence vector of \tilde{E} lies in F_a , thus also in F_b , and $b_x^{\tilde{E}} = b_x^{\tilde{E}}$. It follows immediately that $b_e + b_{w_1w_2} - b_g = 0$. As we have shown that $b_{w_1w_2} = -b_e$, it follows that $b_g = 0$ for all $g \in E(V_i)$, $i = 1, \dots, K-1$.

A similar proof leads to $b_g = 0$ for $g \in \delta(V_i, V_{i+1})$, $i \in \{0, \dots, K-1\}$, and as a direct consequence, $\beta = b_x^{\tilde{E}} = 0$.

We have proved that $b = \gamma a$ with $\gamma = -b_e$. In consequence we have $\beta = 0$ and $\gamma \neq 0$. Since the complete graph defines a feasible solution whose incidence vector does not lie in F_b , $b_x^E = (|T_\pi^e| - 1)\gamma > 0$. As $|T_\pi^e| \geq 2$, it follows that $\gamma > 0$. Thus $a^T x \geq 0$ and $b^T x \geq \beta$ define the same facet $F_a = F_b$. \square

In the following, we discuss the separation problem for the cycle inequalities. In particular, we show that these inequalities can be separated in polynomial time when $K \leq 4$. Note that, based on the proof of Theorem 1, it can be shown that the separation problem for inequalities (2.1) can be easily solved in polynomial time for a 0-1 solution \bar{x} .

Suppose instead that the solution contains fractional values. Let $G = (V, E)$ be a graph and s, t two nodes of V . Given a positive integer B , we define an (s, t) - B -path cut to be any edge set C of E that intersects every (s, t) -path of G with at most B edges. Given a weight vector $w \in \mathbb{R}_+^m$, the minimum (s, t) - B -path cut problem (BPCP) is to find an (s, t) - B -path cut of minimum weight. We now show that BPCP is equivalent to the separation problem for inequalities (2.1).

Lemma 1. *Given a solution \bar{x} of \mathbb{R}_+^m , the separation problem for inequalities (2.1) reduces to solving BPCP for every edge $e = st \in E$ and $B = K-1$ with respect to the weight vector \bar{x} .*

Proof. Suppose that $\bar{x}(C_e) < \bar{x}_e$ where C_e is a minimum (s, t) - B -path cut with respect to \bar{x} . Since there is no cycle of length $\leq K$ in $E \setminus C_e$ containing e , by Theorem 1, there exists a partition $\pi = (V_0, \dots, V_p)$ of V with $e \in \delta(V_0, V_p)$ such that $T_\pi^e \subseteq C_e$. As $\bar{x}_f \geq 0$ for all $f \in E$, we then have

$$\bar{x}(T_\pi) - \bar{x}_e \leq \bar{x}(C_e) - \bar{x}_e < 0,$$

which implies that inequalities (2.1) associated with the partition (V_0, \dots, V_p) and e is violated.

Now suppose that for every edge $e = st \in E$, $\bar{x}(C_e) \geq \bar{x}_e$ where C_e is a minimum (s, t) - $(K-1)$ -path cut with respect to \bar{x} . We claim that no inequality of type (2.1) is violated by \bar{x} . In fact, suppose that for an edge $e = st$, there is a partition $\pi = (V_0, \dots, V_p)$

with $e \in \delta(V_0, V_p)$ and $p \geq K$ such that $\bar{x}(T_\pi^e) - \bar{x}_e < 0$. Since T_π^e is an (s, t) - $(K - 1)$ -path cut, and hence a solution of BPCP, one should have $\bar{x}(C_e) \leq \bar{x}(T_\pi^e)$, a contradiction. \square

If $B = 2$, finding a minimum (s, t) -2-path cut reduces to finding a minimum cut separating s and t in the graph induced by s, t and the nodes adjacent to both s and t .

In what follows, we shall show that, for $B = 3$, the minimum (s, t) - B -path cut problem reduces to a maximum flow problem, and can then be solved in polynomial time. For this, we shall use ideas similar to those developed by Itai, Perl, and Shiloach [18] for solving a closely related problem.

First, note that any node u of V which is not adjacent neither to s nor to t cannot belong to an (s, t) -path of length at most 3 and so can be deleted. So we may assume that G does not contain such nodes.

We will now construct a directed graph $\tilde{G} = (\tilde{N}, \tilde{A})$ from the original one. Let $N = V \setminus \{s, t\}$. Let N' be a disjoint copy of N (where we denote the copy of $u \in N$ that is in N' by u'), and set $\tilde{N} = \{s\} \cup \{t\} \cup N \cup N'$. For each edge $su \in E$ with weight w_{su} make arc $(s, u) \in \tilde{A}$ with capacity w_{su} , for each edge $vt \in E$ make arc $(v', t) \in \tilde{A}$ with capacity w_{vt} , and for each edge $uv \in E$ with $u, v \notin \{s, t\}$, make arcs (u, v') and (v, u') , both with capacity w_{uv} . For each $u \in N$ with $u \neq s, t$ make an arc $(u, u') \in \tilde{A}$ with an infinite capacity (see Figure 2 for an illustration). Note that there is a 1-1 correspondence between the (s, t) -paths of length ≤ 3 in G and the (s, t) -directed paths of length 3 in \tilde{G} .

The following lemmas describe the correspondence between the (s, t) -cuts in \tilde{G} and the (s, t) -3-path cuts in G .

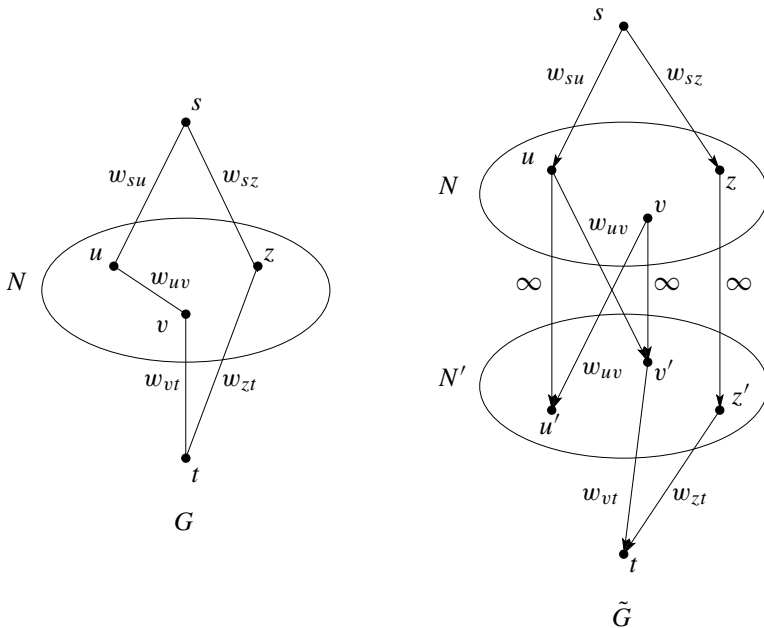


Fig. 2. Construction of \tilde{G}

Lemma 2. *To each finite capacity cut separating s and t in \tilde{G} corresponds a (s, t) -3-path cut of the same weight in G .*

Proof. Let \tilde{C} be a finite capacity cut separating s and t . Then \tilde{C} does not contain any (u, u') arc. Also \tilde{C} cannot contain two arcs of the form (u, v') and (v, u') . For otherwise arcs (u, u') and (v, v') would be in \tilde{C} . Let C be the edge set that contains each edge su with $(s, u) \in \tilde{C}$, each edge vt with $(v', t) \in \tilde{C}$, and each edge uv such that one of the arcs (u, v') and (v, u') is in \tilde{C} . We have that C is a (s, t) -3-path cut with weight $w(\tilde{C})$.
□

If C is a minimal (s, t) -3-path cut in G , then we say that uv is *gone* if either $uv \in C$ or $uv \notin E$. We have the following.

Lemma 3. *Let C be a minimal (s, t) -3-path cut in G . If $uv \in C$ with $u, v \notin \{s, t\}$, then either su and vt are gone or sv and ut are gone but not both.*

Proof. First of all, note that if edges su, vt, sv and ut are all gone then $C \setminus \{uv\}$ is still a (s, t) -3-path cut, contradicting the minimality of C . Now suppose, for instance, that su is not gone. Hence $su \in E \setminus C$, and in consequence, ut is gone. For otherwise, we would have a path of length 2 between s and t going through u and not intersecting C , a contradiction. If sv is not gone, then similarly we obtain that vt is gone. But here, we can remove vt from C , and still have a (s, t) -3-path cut which contradicts again the fact that C is minimal. Consequently, both edges ut and sv are gone. The claim follows by symmetry.
□

Lemma 4. *To each minimal (s, t) -3-path cut in G corresponds a minimal (s, t) -cut of the same weight in \tilde{G} .*

Proof. Let C be a minimal (s, t) -3-path cut in G and $\tilde{C} \subseteq \tilde{A}$ be the arc set constructed as follows: if su (resp. vt) is in C , add arc (s, u) (resp. (v', t)) to \tilde{C} . For $uv \in C$ with $u, v \notin \{s, t\}$, if su and vt are gone, then add (v, u') to \tilde{C} , else by Lemma 3, sv and ut are gone and then add (u, v') to \tilde{C} .

We have that \tilde{C} is a cut in \tilde{G} separating s and t . In fact, if not then there must exist a directed path \tilde{P} from s to t such that $\tilde{P} \cap \tilde{C} = \emptyset$. Hence \tilde{P} is of the form (s, u, v', t) where $u \in N$ and $v' \in N'$. From the definition of \tilde{C} , it follows that edges su and vt do not belong to C and hence are not gone. If $u = v$ then (su, vt) is a path in G that does not intersect C , a contradiction. Therefore $u \neq v$. As $(u, v') \in \tilde{A}$, edge $uv \in E$. If $uv \in C$, as su and vt are not gone, by construction of \tilde{C} we have $(u, v') \in \tilde{C}$, a contradiction. Thus $uv \notin C$ and in consequence, (su, uv, vt) is a path in G not intersecting C . But this is impossible. Hence \tilde{C} is a cut. Clearly \tilde{C} is of capacity $w(C)$.

The cut \tilde{C} is also minimal. In fact, suppose this is not the case, and let $(u, v) \in \tilde{C}$ such that $\tilde{C}' = \tilde{C} \setminus \{(u, v)\}$ is a cut separating s and t . Let C' be the (s, t) -3-path cut obtained from \tilde{C}' using the procedure of Lemma 2. It is easily seen that C' is nothing but the edge set obtained from C by deleting the (unique) edge corresponding to the arc (u, v) . But this contradicts the minimality of C .
□

Theorem 4. *The BPCP can be solved in polynomial time if $B = 3$.*

Proof. We will prove that there is a correspondence between the minimal feasible (s, t) -3-path cuts in G and the minimal finite capacity cuts in \tilde{G} separating s and t , that preserves objective value. Once this fact is established, it will follow that we can solve the BPCP for $B = 3$ in G by solving a maximum flow problem in \tilde{G} .

By Lemma 4, to each minimal (s, t) -3-path cut in G corresponds a minimal cut of the same weight separating s and t in \tilde{G} . In what follows, we will show that the converse also holds, that is, for each finite minimal (s, t) -cut in \tilde{G} , there exists a minimal (s, t) -3-path cut in G with the same weight.

Suppose that \tilde{C} is a minimal finite capacity cut in \tilde{G} , separating s and t . Let C be the (s, t) -3-path cut constructed from \tilde{C} using the procedure of Lemma 2. First note that C has a weight equal to $w(\tilde{C})$. We claim that C is minimal. In fact, suppose not, and let $D \subset C$ be a nonempty edge subset of C such that $C' = C \setminus D$ is a minimal (s, t) -3-path cut. Let \tilde{C}' be the minimal (s, t) -cut in \tilde{G} obtained from C' using the procedure of Lemma 4. We then have that $\tilde{C}' \subseteq \tilde{C}$. If D contains an edge su (vt), then (s, u) ((v', t)) belongs to $\tilde{C} \setminus \tilde{C}'$, but this contradicts the minimality of \tilde{C} . If D contains an edge uv with $u, v \notin \{s, t\}$, then one of the arcs (u, v') and (v, u') belongs to $\tilde{C} \setminus \tilde{C}'$, contradicting again the minimality of \tilde{C} .

In consequence, C is minimal and the proof of the theorem is complete. \square

Unfortunately, McCormick [22] shows that BPCP is NP-hard for $B \geq 12$. Hence we use a straightforward application of the primal-dual method [24] to get an approximation algorithm for BPCP that we use as a heuristic for separating (2.1). This algorithm, described in Algorithm 3.1, works as follows. It considers the linear programming relaxation of the BPCP and its dual. It first constructs an (s, t) - B -path cut C (i.e. a feasible solution of BPCP), and a dual solution $y = (y_P, P \in \mathcal{P}(B))$ where $\mathcal{P}(B)$ is the set of (s, t) -path of length $\leq B$, that satisfy the primal complementary slackness conditions, that is for all $e \in C$, $\sum_{P:e \in P} y_P = c_e$. After that, the algorithm tries to delete the unnecessary edges of C in order to get a solution with lower weight.

It is easy to see that this algorithm returns an (s, t) - B -path cut and runs in $O(|E|^2)$ time. Also from [24] it can be easily shown that this algorithm is a B -approximation algorithm. If \bar{x} is a fractional solution, we can use this algorithm to separate inequalities (2.1) when $K \geq 5$ as follows. Using this algorithm, for an edge $e = st$, compute an (s, t) - $(K - 1)$ -path cut C' in $G - e$. If $\bar{x}(C') < \bar{x}_e$, then it gives a violated cycle inequality. The partition that induces the inequality can be determined by a breadth-first search of the graph $(V, E \setminus (\{e\} \cup C'))$ from either s or t .

Algorithm 3.1 Primal-Dual algorithm for BPCP

Data: a graph $G = (V, E)$, two nodes $s, t \in V$, a weight function w and a bound B .

$l \leftarrow 0$; $y \leftarrow 0$; $C \leftarrow \emptyset$;

while C is not an (s, t) - B -path cut **do**

$l \leftarrow l + 1$;

 Find a path $P \in \mathcal{P}(B)$ such that $P \cap C = \emptyset$;

 Increase y_P until some edge $e_l \in P$ satisfies $\sum_{Q:e \in Q} y_Q = w_{e_l}$;

$C \leftarrow C \cup \{e_l\}$;

$C' \leftarrow C$;

for $j \leftarrow l$ down to 1 **do**

if $C' \setminus \{e_j\}$ is still an (s, t) - B -path cut **then**

$C' \leftarrow C' \setminus \{e_j\}$;

Return C' ;

3.2. Cycle partition inequalities

We present here our second class of valid inequalities for $\mathcal{P}(G, K)$, called cycle partition inequalities.

Theorem 5. *Let $G = (V, E)$ be a graph and $\pi = (V_0, V_1, \dots, V_p)$ be a partition of V with $p \geq K$. Then, the inequality*

$$\left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil\right) x(T_\pi) + x(C_\pi) \geq 2p \quad (3.1)$$

is valid for $\mathcal{P}(G, K)$.

Proof. Let F be an edge set inducing a feasible solution of 2ECSBR. If $F \cap T_\pi = \emptyset$, as $p \geq K$, F must contain at least $2p$ edges of C_π , and hence the incidence vector of F , x^F satisfies (3.3). So let us assume that $F \cap T_\pi \neq \emptyset$. Note that the cyclomatic inequality (4.4) can be rewritten as

$$x(T_\pi) + x(C_\pi) \geq p + \left\lceil \frac{p}{K-1} \right\rceil. \quad (3.2)$$

Using this and the fact that $x^F(T_\pi) \geq 1$, it follows that

$$\begin{aligned} \left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil\right) x^F(T_\pi) + x^F(C_\pi) &= \left(p - \left\lceil \frac{p}{K-1} \right\rceil\right) x^F(T_\pi) \\ &\quad + x^F(T_\pi) + x^F(C_\pi) \\ &\geq p - \left\lceil \frac{p}{K-1} \right\rceil + p + \left\lceil \frac{p}{K-1} \right\rceil = 2p, \end{aligned}$$

and the inequality is valid. \square

Inequalities (3.1) are called *cycle partition inequalities*. The next lemmas show that the only cycle partition inequalities of interest are those for which $p = K$.

Lemma 5. *Let F be a solution to 2ECSBR. If $|F \cap T_\pi| \geq 2$ then x^F , the incidence vector of F , does not satisfy (3.1) with equality.*

Proof. If $|F \cap T_\pi| \geq 2$, then $x^F(T_\pi) \geq 2$. Combining this and (3.2), we get

$$\begin{aligned} \left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil\right) x^F(T_\pi) + x^F(C_\pi) &= \left(p - 1 - \left\lceil \frac{p}{K-1} \right\rceil\right) x^F(T_\pi) \\ &\quad + x^F(T_\pi) + x^F(C_\pi) \\ &\geq 2 \left(p - \left\lceil \frac{p}{K-1} \right\rceil\right) + p + \left\lceil \frac{p}{K-1} \right\rceil \\ &= 3p - \left\lceil \frac{p}{K-1} \right\rceil > 2p, \end{aligned}$$

where the last relation holds since $p \geq K \geq 3$. \square

Lemma 6. *If $p > K$, then (3.1) does not define a facet of $\mathcal{P}(G, K)$.*

Proof. Suppose that $p > K$, and consider an edge $f \in \delta(V_1, V_p)$. If (3.1) defines a facet of $\mathcal{P}(G, K)$, as it is different from a trivial inequality, there must exist a feasible solution F containing f whose incidence vector lies in the face defined by (3.1). By Lemma 5 it follows that $(F \cap T_\pi) \setminus \{f\} = \emptyset$. Suppose first that $F \cap \delta(V_i, V_{i+1}) = \emptyset$ for some $i \in \{1, \dots, p-1\}$. For F to be 2-edge connected, one should then have $|F \cap \delta(V_j, V_{j+1})| \geq 2$ for $j \in \{1, \dots, p-1\} \setminus \{i\}$. This yields to

$$\begin{aligned} \left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil \right) x^F(T_\pi) + x^F(C_\pi) &\geq \left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil \right) + 2 + 2(p-2) \\ &= 3p - 1 - \left\lceil \frac{p}{K-1} \right\rceil > 2p, \end{aligned}$$

since $p > K \geq 3$, which leads to a contradiction.

In consequence, $F \cap \delta(V_i, V_{i+1}) \neq \emptyset$, for $i = 1, \dots, p-1$. Furthermore, it follows from the development above that for some $i \in \{1, \dots, p-1\}$, F contains exactly one edge, say g , from $\delta(V_i, V_{i+1})$. But the shortest cycle in F containing g must go through V_1, \dots, V_p, V_1 and hence, it is of length at least $p > K$, a contradiction. \square

Following this result, we restrict our attention to partitions $\pi = (V_0, \dots, V_K)$ and the cycle partition inequality can be written as

$$(K-1)x(T_\pi) + x(C_\pi) \geq 2K. \quad (3.3)$$

In what follows, we present necessary and sufficient conditions for these inequalities to define facets of $\mathcal{P}(G, K)$.

Theorem 6. *Let $G = (V, E)$ be a complete graph and $\pi = (V_0, V_1, \dots, V_K)$ a partition of V . For notational convenience, let $V_{K+1} := V_0$. Then, the cycle partition inequality*

$$(K-1)x(T_\pi) + x(C_\pi) \geq 2K$$

defines a facet for $\mathcal{P}(G, K)$ if and only if

1. $|V_i| + |V_{i+1}| + |V_j| + |V_{j+1}| \geq 5$ for all $i, j \in \{0, \dots, K\}$, $i \neq j$,
2. $|V_i| \neq 2$ for $i = 0, \dots, K$.

Proof. If the first condition does not hold, then there are two edges $f, g \in C_\pi$ such that $\{f, g\}$ induces a 2-edge cutset of the graph $G - T_\pi$. All the cycles using f in $G - T_\pi$ also use g and are included in C_π , thus are of length $K+1$. It follows that $G - T_\pi$ does not contain any feasible solution, and $x(T_\pi) \geq 1$ is a valid subset inequality. As by Lemma 5, any solution F whose incidence vector x^F lies in the face defined by (3.3) contains at most one edge of T_π , it follows that $x^F(T_\pi) = 1$ and the face defined by (3.3) is included in the face defined by $x(T_\pi) \geq 1$, thus (3.3) cannot define a facet of $\mathcal{P}(G, K)$.

If the second condition is not satisfied, there exists some $i \in \{0, \dots, K\}$ such that $V_i = \{u, v\}$. Let us assume w.l.o.g. that $i = 0$ and let $ax := (K-1)x(T_\pi) + x(C_\pi) \geq 2K$ be the inequality (3.3). Consider first the case $p = K \geq 4$. If $ax \geq 2K$ defines a facet there must exist a solution $F \in \mathcal{F}(G)$ containing an edge of $\delta(V_1, V_3)$ such that $ax^F = 2K$. If there exists $i \in \{0, 3, 4, \dots, K\}$ such that $\delta(V_i, V_{i+1}) \cap F = \emptyset$ then by inequalities (2.2) and Lemma 5 it follows that $|\delta(V_j, V_{j+1}) \cap F| \geq 2$ for all $j \in \{0, 3, 4, \dots, K\} \setminus \{i\}$ and $|\delta(V_2) \cap C_\pi \cap F| \geq 2$. Hence $ax^F \geq K-1 + 2K-2$. As $K > 3$, we have $ax^F > 2K$, a contradiction. Thus $|\delta(V_i, V_{i+1}) \cap F| \geq 1$ for all $i \in \{0, 3, 4, \dots, K\}$. In a similar way, we can show that $|\delta(V_i, V_{i+1}) \cap F| = 1$ for all $i \in \{0, 3, 4, \dots, K\}$. Let $f_0 = F \cap \delta(V_0, V_1)$ and $f_K = F \cap \delta(V_K, V_0)$. As the shortest cycle of F containing f_0 and also containing f_K , goes through the sets $V_1, V_3, V_4, \dots, V_K, V_0$ and must be of length $\leq K$, f_0 and f_K must be incident to the same node of V_0 , say u . This implies that $|F \cap \delta(v)| \leq 1$, contradicting the fact that F induces a 2-edge connected spanning subgraph.

Now let us assume that $p = K = 3$. As $ax \geq 2K$ is not a trivial inequality, there exists a solution $F \in \mathcal{F}(G)$ which does not contains uv and such that $ax^F = 2K$. Consider first the case where $F \cap T_\pi = \emptyset$. Thus any cycle of F of length 3 containing edges of C_π intersects exactly two elements of the partition. By the cut constraints this implies that for $W = \{u\}$ ($W = \{v\}$, $W = V_2$), at least one of the following statements holds: $|F \cap \delta(W, V_1)| \geq 2$ and $|F \cap \delta(W, V_3)| \geq 2$. Therefore $|F \cap C_\pi| \geq 6$. If $|F \cap C_\pi| = 6$ it is not hard to see that there is a node set $W' \subset V$ where $\delta(W') \cap F = \emptyset$, which is impossible. Consequently $|F \cap C_\pi| > 6$. But this implies that x^F does not satisfy $ax \geq 2K$ with equality, a contradiction. If $F \cap T_\pi \neq \emptyset$, then by Lemma 5, $|F \cap T_\pi| = 1$. We can show along the same line that in this case, $|F \cap C_\pi| \geq 5$. But this implies that $ax^F > \alpha$, which yields again a contradiction.

Conversely, suppose that both conditions are satisfied for some inequality $a^T x := (K-1)x(T_\pi) + x(C_\pi) \geq 2K$. Let $b^T x \geq \beta$ be a facet defining inequality such that the face F_a induced by $a^T x \geq 2K$ in $\mathcal{P}(G, K)$ is contained in the face F_b induced by $b^T x \geq \beta$.

We first show that $b_e = b_{e'}$ for every $e, e' \in C_\pi$. Consider an edge set $\delta(V_j, V_{j+1})$ where $|\delta(V_j, V_{j+1})| \geq 3$. By the two conditions, there is at most one $i \in \{0, \dots, K\}$ such that $|\delta(V_i, V_{i+1})| = 1$. Therefore, we may suppose w.l.o.g., that $|\delta(V_i, V_{i+1})| \geq 3$ for $i = 1, \dots, K$ and thus $j \neq 0$. Note that we can have $|V_0| = |V_1| = 1$. Let f_i^1, f_i^2 be two fixed edges of $\delta(V_i, V_{i+1})$ for $i = 1, \dots, K$, and $f_0^1 \in \delta(V_0, V_1)$. As G is complete, we may suppose that f_i^1 and f_i^2 (f_i^1 and f_{i+1}^1) are adjacent for $i = 1, \dots, K$. Let $\bar{E} = \bigcup_{i=0}^K E(V_i)$. By the second condition, we may assume that $|V_{j+1}| \geq 3$. We may also suppose w.l.o.g., that f_j^1 and f_j^2 are incident to a node, say w , of V_j . Consider the edge sets

$$\begin{aligned} E_1 &= \{f_1^1, f_1^2, f_2^1, f_2^2, \dots, f_K^1, f_K^2\} \cup \bar{E}, \\ E_2 &= (E_1 \setminus \{f_j^1\}) \cup \{f\}, \end{aligned}$$

where $f \in \delta(w, V_{j+1}) \setminus \{f_j^1, f_j^2\}$. As both graphs $G(E_1)$ and $G(E_2)$ are 2-edge connected and every edge of E_1 (E_2) belongs to a cycle of length 3, we have that E_1, E_2 are feasible and belong to the face defined by $ax = 2K$. Thus $b x^{E_1} = b x^{E_2}$ and $b_{f_j^1} = b_f$.

Therefore, $b_e = b_{e'}$ for all $e, e' \in \delta(w, V_{j+1})$.

If $|V_j| = 1$, then we are done. If not, then by symmetry, we also obtain that $b_e = b_{e'}$ for all $e, e' \in \delta(V_j, w')$, $w' \in V_{j+1}$. It follows that $b_e = b_{e'}$ for all $e, e' \in \delta(V_j, V_{j+1})$.

Now let $i \in \{1, \dots, K\}$ and g be an edge of $\delta(V_i, V_{i+2})$ adjacent to edges f_{i-1}^1 and f_{i+2}^1 . Note that the edges $g, f_{i+2}^1, \dots, f_{i+K}^1$ form a cycle of length K . Let us examine the sets

$$\begin{aligned} E_3 &= \{f_0^1, f_1^1, \dots, f_K^1, g\} \cup \bar{E}, \\ E_4 &= (E_3 \setminus \{f_{i+1}^1\}) \cup \{f_i^2\}. \end{aligned}$$

Clearly, E_3 and E_4 are feasible and belong to the face defined by $ax = 2K$. Therefore $b_{x^{E_3}} = b_{x^{E_4}}$ and $b_{f_{i+1}^1} = b_{f_i^2}$. This implies that $b_e = b_{e'}$ for all $e \in \delta(V_i, V_{i+1})$, $e' \in \delta(V_{i+1}, V_{i+2})$, $i \in \{1, \dots, K\}$.

It follows that $b_e = \gamma$ for all $e \in C_\pi$.

Since E_1 and E_3 lie in the face defined by $ax = 2K$, $b_{x^{E_1}} = b_{x^{E_3}}$ and $\sum_{i=1}^K b_{f_i^2} = b_g$. It follows that $b_g = (K-1)\gamma$.

As G is complete, for any edge h of T_π , there exist a cycle, say C_h of length $K+1$ going through the sets V_0, \dots, V_K and having h as a chord. Note that $C_h \cap E(V_i) = \emptyset$ for $i = 0, \dots, K$. Let

$$E_h = C_h \cup \{h\} \cup \bar{E}, \text{ for all } h \in T_\pi.$$

Clearly, E_h lies in the face defined by $ax = 2K$ for all $h \in T_\pi$. If h and h' are two edges of T_π , we therefore have $b_{x^{E_h}} = b_{x^{E_{h'}}}$. Since $b_{x^{C_h}} = b_{x^{C_{h'}}}$, it follows that $b_h = b_{h'}$. Thus $b_e = (K-1)\gamma$ for all $e \in T_\pi$.

Next we show that $b_e = 0$ for all $e \in E(V_i)$, $i = 0, \dots, K$. Let $i \in \{0, \dots, K\}$ such that $|V_i| \neq 1$. By 2. it follows that $|V_i| \geq 3$. Let $e := uv \in E(V_i)$. Consider first the case where $|V_i| \geq 4$. W.l.o.g., we may assume that at least one edge among $\{f_{i-1}^1, f_{i-1}^2\}$ ($\{f_i^1, f_i^2\}$) is not incident neither to u nor to v . Let $E'_1 = E_1 \setminus \{e\}$. It is not hard to see that $E'_1 \in \mathcal{F}(G)$. As $ax^{E'_1} = ax^{E_1}$, we have $b_e = 0$. Suppose now that $|V_i| = 3$. Let w be the node in $V_i \setminus \{u, v\}$, $z \in V_{i-1}$ and $z' \in V_{i+1}$. We can assume, w.l.o.g. that f_{i-2}^1, f_{i-2}^2 are incident to z and f_{i+1}^1, f_{i+1}^2 are incident to z' . It is not hard to see that

$$\begin{aligned} E_4 &= \bigcup_{j=0}^K E(V_j) \cup \left(\bigcup_{\substack{j \in \{1, \dots, K\} \\ j \neq i-1, j \neq i}} \{f_j^1, f_j^2\} \right) \cup \{zv, zw, z'u, z'w\}, \\ E'_4 &= E_4 \setminus \{e\}. \end{aligned}$$

define solutions of 2ECSBR that lie in the face defined by $ax \geq 2K$ and thus $b_e = 0$.

Altogether we have shown that

$$b_e = \begin{cases} \gamma & \text{for all } e \in C_\pi, \\ (K-1)\gamma & \text{for all } e \in T_\pi, \\ 0 & \text{for all } e \in \bar{E}, \end{cases}$$

thus $b = \gamma a$. As we did before, we have that $\gamma > 0$, which completes the proof. \square

To separate the cycle partition inequalities, we developed a heuristic which works in two phases. First, we contract edges with high values until a graph on $K + 1$ nodes is obtained. Each node of this graph corresponds to an element of the partition inducing the inequality. In the second phase, we order the elements of the partition in order to get a partition that provides a minimum left hand side in (3.3). For this, we fix a node say v_0 , which will correspond to V_0 , we determine a node v_1 such that $\bar{x}(\delta(v_0, v_1))$ is maximum. Node v_1 corresponds to V_1 . After that, a node v_2 , where $\bar{x}(\delta(v_1, v_2))$ is maximum is computed and so on until a partition V_0, \dots, V_K is obtained. This is repeated $K + 1$ times by changing v_0 . We then consider the partition giving the minimum left hand side. If this is less than $2K$, then a violated cycle partition inequality is found.

4. Further valid inequalities

In this section we present some classes of valid inequalities of $\mathcal{P}(G, K)$ related to the node-case studied by Fortz and Labbé [11]. As we did in the previous section, we also discuss necessary and sufficient conditions for facet defining and devise separation algorithms.

4.1. Cut inequalities

We now provide a characterization of facet defining cut inequalities (2.2).

Theorem 7. *Let $G = (V, E)$ be a complete graph, $K \geq 3$ a given constant, and $W \subseteq V$ a subset of nodes, $\emptyset \neq W \neq V$. The inequality*

$$x(\delta(W)) \geq 2$$

defines a facet of $\mathcal{P}(G, K)$ if and only if

- either $K \geq 4$, $|W| \neq 2$ and $|V \setminus W| \neq 2$,
- or $K = 3$, $|W| \notin \{2, 3\}$ and $|V \setminus W| \notin \{2, 3\}$.

Proof. See [10]. □

The separation of cut constraints can be carried out by computing a minimum cut in the graph, with capacities given by the current LP solution. This can be done in polynomial time, e.g. by the Hao-Orlin algorithm [17] that requires one maximum flow computation. But this algorithm does not output the Gomory-Hu tree [13] that provides the minimum cut between all the pairs of nodes. For this we use Gomory-Hu algorithm [13] for separating these inequalities.

To speed up the computation, we have also developed a simple heuristic to separate the cut inequalities. This works as follow: we repeatedly contract edges with high values until we obtain either a graph of weight less than p , where p is the number of nodes, or a graph on two nodes. If the former case holds, then at least one of the cuts induced by the nodes of the graph is violated. If the latter case holds, we just check whether the cut yielded by the graph is violated.

4.2. Subset inequalities

Fortz et al. have introduced in [9] a general class of inequalities that are valid for the 2-node connected version as follows. Let S be a set of edges such that the graph $G' = (V, E \setminus S)$ does not contain a solution of 2ECSBR. The inequality

$$x(S) \geq 1 \tag{4.1}$$

is called a *subset inequality*. These constraints are also valid for $\mathcal{P}(G, K)$.

This class of inequalities is very generic. Actually, if a feasible solution of a given combinatorial optimization problem has to contain at least one element of a given set S , then inequality (4.1) is valid for the associated polytope.

In what follows, we describe a subclass of inequalities (4.1) which will be used in the framework of a Branch-and-Cut algorithm for the problem that will be introduced in the next section.

Given a partition $\pi = (V_0, \dots, V_K)$ with $|V_0| = |V_K| = 1$, if there exists $i \in \{0, \dots, K - 1\}$ such that $|V_i| = |V_{i+1}| = 1$, then the set T_π induce a subset inequality. We can notice that this inequality dominates the cycle inequality associated to the partition V_0, \dots, V_K and the edge between V_0 and V_K .

To separate these subset inequalities we try to compute for every edge $e := st$ a minimum (s, t) - $(K - 1)$ -path cut C_e . If $\bar{x}(C_e) < 1$, then we compute a partition $\pi = (V_0, \dots, V_K)$ with $V_0 = \{s\}$ by a breadth-first search of the graph $(V, E \setminus (C_e \cup \{e\}))$ from s . Note that $t \in V_K$. If $|V_K| \geq 2$, then we consider the partition $\pi' = (V'_0, \dots, V'_K)$ where $V'_j = V_j$ for $j = 0, \dots, K - 2$, $V'_{K-1} = V_{K-1} \cup (V_K \setminus \{t\})$ and $V'_K = \{t\}$. If there is some $i \in \{0, \dots, K - 1\}$ where $|V'_i| = |V'_{i+1}| = 1$ then $x(T_{\pi'}) \geq 1$ is a violated subset inequality.

4.3. Metric inequalities

Metric inequalities were introduced by Fortz et al. [8] for the 2-node connected version of our problem with general edge lengths. As their result depends only on the fact that any edge in a solution belongs to a feasible cycle, metric inequalities are also valid for $\mathcal{P}(G, K)$. Hence we have the following.

Proposition 1. *Consider an edge $e := ij \in E$ and a set of node potentials $(\alpha_k)_{k \in V}$ satisfying*

$$\alpha_i - \alpha_j > K - 1.$$

Then

$$\sum_{f \in E - e} v_f x_f \geq x_e \tag{4.2}$$

is a valid inequality for $\mathcal{P}(G, K)$ where

$$v_f = \min \left(1, \max \left(0, \frac{|\alpha_l - \alpha_k| - 1}{\alpha_i - \alpha_j + 1 - K} \right) \right) \tag{4.3}$$

for all $f := kl \in E - e$.

Inequalities 4.2 are called *metric inequalities*. The next proposition shows that facet-inducing cycle inequalities form a subset of metric inequalities.

Proposition 2. *Let $G = (V, E)$ be a graph and $K \geq 3$. Let $\pi = (V_0, \dots, V_p)$ be a partition of V such that $p \geq K$ and let $e := ij \in \delta(V_0, V_p)$. If $p = K$ and $T_\pi^e := T_\pi \cup \delta(V_0, V_p) \setminus \{e\}$, then the cycle inequality*

$$x(T_\pi^e) \geq x_e$$

is a metric inequality.

If $p > K$, the cycle inequality is dominated by a metric inequality.

Proof. If $p = K$, it is sufficient to show that there exist node potentials $(\alpha_k)_{k \in V}$ such that v_f defined by (4.3) satisfy $v_f = 1$ for all $f \in T_\pi^e$ and $v_f = 0$ for all $f \in E \setminus (T_\pi^e \cup \{e\})$. Let $\alpha_k = -q$ if and only if $k \in V_q$. Then, $\alpha_i = 0$ and $\alpha_j = -K$, and it is easy to see that (4.3) becomes

$$v_f = \begin{cases} 1 & \text{if } |q - r| > 1, \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in E - e$, $f \in \delta(V_q, V_r)$, $q, r \in \{0, \dots, p\}$. The result follows immediately.

If $p > K$, the same definition of $(\alpha_k)_{k \in V}$ leads to $\alpha_i = 0$, $\alpha_j = -p$, and

$$v_f = \begin{cases} 1 & \text{if } |q - r| > p - K + 1, \\ \frac{|q-r|-1}{p-K+1} & \text{if } 1 < |q - r| \leq p - K + 1, \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in E - e$, $f \in \delta(V_q, V_r)$, $q, r \in \{0, \dots, p\}$. Therefore, the coefficient of an edge $f \in E - e$ is the same in the metric and the cycle inequality if $|q - r| > p - K + 1$ or $|q - r| \leq 1$, and it is smaller in the metric inequality if $1 < |q - r| \leq p - K + 1$. It follows that the metric inequality dominates the cycle inequality. \square

To separate the metric inequalities, we use the heuristic developed by Fortz et al. [8] for the 2-node connected case. As the metric inequalities are independent from the connectivity type (edge or node connectivity), this algorithm is also valid for our problem.

4.4. Cyclomatic inequalities

Cyclomatic inequalities were introduced by Fortz and Labbé [11] for the 2-node connected version of the problem. These inequalities are not valid for $\mathcal{P}(G, K)$, but can easily be adapted to 2-edge connectivity. The following proposition describes these inequalities for $\mathcal{P}(G, K)$. It is stated without proof because the proof is similar to that given in [11] for the node-case.

Proposition 3. *Let $G = (V, E)$ be a graph, $K \geq 3$ a given constant, and V_0, V_1, \dots, V_p a partition of V . Then*

$$x(\delta(V_0, \dots, V_p)) \geq M(p, K) := \left\lceil \frac{Kp}{K-1} \right\rceil \quad (4.4)$$

is a valid inequality for $\mathcal{P}(G, K)$.

The following theorem gives necessary and sufficient conditions for the cyclomatic inequalities to be facet defining.

Theorem 8. *Let $G = (V, E)$ be a complete graph, $K \geq 3$ a given constant and $V_0, V_1, \dots, V_p, p \geq 2$, a partition of V . The inequality*

$$x(\delta(V_0, \dots, V_p)) \geq M(p, K) := \left\lceil \frac{Kp}{K-1} \right\rceil \tag{4.5}$$

defines a facet of $\mathcal{P}(G, K)$ if and only if the following conditions hold:

- $p \geq K$,
- $|V_i| \neq 2$ for $i = 0, \dots, p$,
- either $|V_i| \neq 3$ for $i = 0, \dots, p$, or $(p + 1) \bmod (K - 1) \geq 2$.

Proof. See [10]. □

To separate the cyclomatic inequalities, we developed a heuristic based on Barahona’s algorithm [2] (see also [1]) for separating the so-called partition inequalities

$$x(\delta(V_0, \dots, V_p)) \geq p. \tag{4.6}$$

A first separation algorithm for these inequalities has been devised by Cunningham [5] and requires $|E|$ Minimum-Cut computations. Barahona [2] reduced this computing time to $|V|$ Minimum-Cut computations. Both Cunningham and Barahona’s algorithms give the most violated inequality.

Consider the following inequalities obtained from the cyclomatic inequalities by deleting the upper integral part from the right hand side.

$$x(\delta(V_0, \dots, V_p)) \geq \frac{Kp}{K-1}. \tag{4.7}$$

Clearly inequalities (4.7) are of type (4.6) (it suffices to set $x' = \frac{K-1}{K}x$). Moreover, if (4.7) is violated, then (4.4) is so. However, it may that all the inequalities of type (4.7) are satisfied whereas some cyclomatic inequalities are violated. In order to strengthen inequalities (4.7), we consider the inequalities

$$x(\delta(V_0, \dots, V_p)) \geq \frac{Kp}{K-1} + \epsilon \tag{4.8}$$

and we choose $\epsilon = \frac{p}{100n}$. Note that $\epsilon \leq 0.01$ and the right hand side of (4.8) is linear in p . So inequalities (4.8) can be separated using, for instance, Barahona’s algorithm. Here inequalities (4.8) can be transformed to inequalities of type (4.6) by setting $x' = (\frac{100nK+K-1}{100n(K-1)})x$. As it is pointed out in the next section, the value we considered for ϵ gave the best results.

A second and faster heuristic that we have developed for separating the cyclomatic inequalities consists in contracting edges with high values (in particular edges with value 1) until we get either a graph on $p + 1$ nodes, with $p \geq K$ and whose weight is less than $\lceil \frac{Kp}{K-1} \rceil$ or a graph on less than $K + 1$ nodes. Note that, as given in Theorem 8, cyclomatic inequalities define facets only if $p \geq K$. If the former case holds, then a violated cyclomatic inequality is found. The partition associated to this inequality is given by the resulting graph where each node corresponds to an element of the partition. This heuristic runs in $O(n^3)$ time.

5. A Branch-and-Cut algorithm

In this section, we present a Branch-and-Cut algorithm for the 2-edge connected subgraph problem with bounded rings. Our aim is to address the algorithmic applications of the theoretical results presented in the previous sections and describe some strategic choices made in order to solve that problem.

To start the optimization we consider the following linear program given by the cut inequalities associated with the vertices of the graph together with the cyclomatic inequality induced by the trivial partition (where the elements of the partition correspond to the nodes of the graph) and the trivial inequalities, that is

$$\begin{aligned} \text{Min} \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & x(\delta(v)) \geq 2, \quad \text{for all } v \in V, \\ & x(E) \geq \left\lceil \frac{nK}{K-1} \right\rceil, \\ & 0 \leq x_e \leq 1, \quad \text{for all } e \in E. \end{aligned}$$

An important task in the Branch-and-Cut algorithm is to determine whether or not an optimal solution of the relaxation of the 2ECSBR is feasible. An optimal solution \bar{x} of the relaxation is feasible for the 2ECSBR if it is an integer vector that satisfies the cut inequalities and such that every edge of $G_{\bar{x}}$ is contained in a feasible cycle of $G_{\bar{x}}$. Verifying if \bar{x} is feasible for 2ECSBR can be done in polynomial time. We first check if each edge e with $\bar{x}_e = 1$ belongs to a feasible cycle of $G_{\bar{x}}$ by computing a shortest path between the endnodes of e . And then, by a breadth-first search, we verify if $G_{\bar{x}}$ is connected. If this is the case, as each edge belongs to a cycle, the cut inequalities are also satisfied.

Another important issue in the effectiveness of the Branch-and-Cut algorithm is the computation of a good upper bound. For this, we first try to transform each LP-solution obtained in the Branch-and-Cut to a feasible solution by rounding up to 1 all the variables with fractional value. Then we delete all the edges that do not belong to feasible cycles. And in a final step, we try to reduce the resulting solution F by repeatedly removing edges with high cost e such that $F \setminus \{e\}$ still induces a solution of the 2ECSBR.

If an optimal solution \bar{x} of the linear relaxation of the 2ECSBR is not feasible, the Branch-and-Cut algorithm generates further inequalities that are valid for $\mathcal{P}(G, K)$ and violated by \bar{x} . The separation of valid inequalities is performed in the following order:

- cut inequalities,
- metric inequalities,
- cycle and subset inequalities,
- cyclomatic inequalities,
- cycle partition inequalities.

We remark that all inequalities are global (i.e. valid in all the Branch-and-Cut tree) and several constraints may be added at each iteration. Moreover, we go to the next class of inequalities only if we do not find any violated inequalities in the current class.

To separate the different inequalities, we use the algorithms described in section 3. All our separation algorithms are applied on the graph $G_{\bar{x}} = (V_{\bar{x}}, E_{\bar{x}})$ where \bar{x} is the current LP-solution.

The exact separation of cut constraints can be done using the Gomory-Hu algorithm [13]. This algorithm produces the so-called Gomory-Hu tree with the property that for all pairs of nodes $s, t \in V_{\bar{x}}$ the minimum (s, t) -cut in the tree is also a minimum (s, t) -cut in $G_{\bar{x}}$. Actually, we use the algorithm developed by Gusfield [16] which requires $|V_{\bar{x}}| - 1$ maximum flow computations. The maximum flow computations are handled by the efficient Goldberg and Tarjan algorithm [12] that runs in $O(mn \log \frac{n^2}{m})$ time. The exact algorithm that permits to separate the cut inequalities is then implemented to run in $O(mn^2 \log \frac{n^2}{m})$ time.

The separations of the cycle and subset inequalities are performed simultaneously. We first compute for an edge $e = st$ a (minimum) (s, t) - $(K - 1)$ -path cut C_e using either the exact algorithm if $K \leq 4$ or the primal-dual algorithm if $K \geq 5$. If $\bar{x}(C_e) < 1$, we determine a partition $\pi = (V_0, \dots, V_K)$ by a breadth-first search from s in the graph induced by $E_{\bar{x}} \setminus (C_e \cup \{e\})$. If $|V_K| \geq 2$, then we consider the partition $\pi' = (V'_0, \dots, V'_K)$ where $V'_j = V_j$ for $j = 0, \dots, K - 2$, $V'_{K-1} = V_{K-1} \cup (V_K \setminus \{t\})$ and $V'_K = \{t\}$. The idea behind this is to get a cycle inequality that, by Theorem 3, may define a facet. If there is some $i \in \{0, \dots, K - 1\}$ where $|V'_i| = |V'_{i+1}| = 1$ then $x(T_{\pi'}) \geq 1$ is a violated subset inequality. Note that this inequality dominates the cycle inequality $x(T_{\pi^e}) \geq x_e$. If this is not the case and we have $\bar{x}(T_{\pi^e}) < \bar{x}_e$, then the cycle inequality corresponding to π' and e is violated. Moreover, this inequality is facet defining. We consider this procedure only for edges e with $\bar{x}_e \geq 0.5$. If \bar{x}_e is small, there is little hope to get a violated cycle inequality involving e . This procedure runs in $O(n^3)$ time if $K \leq 4$ and in $O(n^4)$ time if $K \geq 5$.

To separate cyclomatic inequalities, we first use the heuristic based on the contraction of edges. If no violated inequality is found, then we try to generate violated cyclomatic inequalities using the procedure based on Barahona's algorithm [2] for the multicut problem. Both algorithms produce a partition $\pi = (V_0, \dots, V_p)$ with $p \geq K$. For every $i \in \{0, \dots, p\}$ such that $|V_i| = 2$ (resp. $|V_i| = 3$ and either $K = 3$ or $(p + 1) \bmod (K - 1) \leq 1$) then we consider the partition obtained from π by expanding the set V_i . By Theorem 8, The cyclomatic inequality given by this latter partition dominates the one produced by partition π .

When solving instances of the 2ECSBR, we remarked that the separation of cut inequalities using the exact Gusfield algorithm is time consuming. Therefore we adopted the strategy to use that algorithm only if no constraints of any type could be found using the separation routines presented before.

To store the generated inequalities, we created a pool whose size increases dynamically. All the generated inequalities are put in the pool and are dynamic, i.e. they are removed from the current LP when they are not active. We first separate inequalities from the pool. If all the inequalities in the pool are satisfied by the current LP-solution, we separate the classes of inequalities in the order given above.

6. Computational results

The Branch-and-Cut algorithm described in the previous section has been implemented in C++, using BCP [20] to manage the Branch-and-Cut tree and CPLEX 7.1 as LP-solver.

It was tested on a Pentium IV 1,7 GHz with 1 Gb RAM, running under Linux. We fixed the maximum CPU time to 5 hours.

Results are presented here for instances coming from real applications and randomly generated instances. The instances consist in complete graphs with edge costs equal to rounded Euclidean distances. Tests were performed for $K = 3, 4, 5, 6, 7, 10, 13, 16$. Usually in practice, the bound does not exceed 5 as arised from some discussion with researchers from France Telecom [14]. The first set of instances come from the network of the Belgian telecommunications operator Belgacom (52 nodes) and subsets of these nodes. The random problems were generated with 10 to 50 nodes, and five instances of each size were tested. Data on the randomly generated test problems are available at the Web page <http://www.poms.ucl.ac.be/fortz/2cnbm/data.html>.

In the various tables, the entries are:

$ V $:	the number of nodes of the problem,
K	:	the bound on the cycles,
Cu	:	the number of generated cut inequalities,
Cy	:	the number of generated cycle inequalities,
Me	:	the number of generated metric inequalities,
Su	:	the number of generated subset inequalities,
Cc	:	the number of generated cyclomatic inequalities,
Cp	:	the number of generated cycle partition inequalities,
No	:	the number of generated nodes in the Branch-and-Cut tree,
o/p	:	the number of problems solved to optimality over the number of instances tested (only for random instances),
Gap1	:	the gap between the best upper bound (UB) and the lower bound obtained at the root node of the Branch-and-Cut tree without adding cycle and cycle partition inequalities.
Gap2	:	the gap between UB and the lower bound obtained at the root node of the Branch-and-Cut tree.
Gt	:	the gap between UB and the best lower bound found (LB),
CPU	:	the total time in second.

Table 1 reports results obtained for the real instances, while Table 2 presents the average results for the randomly generated problems.

We remark that for 20 nodes or less, all problems could be solved to optimality. Moreover for $K = 3$, all instances have been solved to optimality within the time limit. Comparing these results to those in [11], it appears that the edge connectivity version of the problem is much easier to solve than the node connectivity version for $K = 3$. However for $4 \leq K \leq 7$ and instances with 30 nodes and more, the problem seems to be harder to solve. In fact only a few instances of this type have been solved in less than 5 hours. For the real instances with 52 nodes and $4 \leq K \leq 7$, we got an average gap of 5.26%. For the random instances with 50 nodes, this average gap is 6.93%. A similar increase can also be observed between real and random instances with 30 nodes. Therefore, it seems that real instances are easier to solve.

In both tables, a significant number of cycle and cycle partition inequalities have been generated for most of the instances with 30 nodes and more when $K \leq 6$. In order to evaluate the impact of these inequalities on the performance of the algorithm, Table 3

Table 1. Results for Belgacom instances

$ V $	K	Cu	Cy	Me	Su	Cc	Cp	No	Gap1	Gap2	Gt	CPU
12	3	2	4	6	2	1	1	1	0.38	0.00	0.00	0 : 00 : 00
17	3	13	43	41	15	11	6	7	0.94	0.51	0.00	0 : 00 : 01
30	3	30	91	50	18	49	17	41	0.93	0.58	0.00	0 : 00 : 10
52	3	101	681	426	84	895	124	3015	1.99	1.33	0.00	0 : 42 : 36
12	4	4	24	24	16	5	0	5	1.21	0.52	0.00	0 : 00 : 00
17	4	21	119	88	28	11	0	45	1.90	1.76	0.00	0 : 00 : 02
30	4	111	6676	3437	514	293	81	8723	5.66	3.67	0.00	0 : 55 : 07
52	4	141	6028	2325	251	796	117	4653	6.42	5.30	3.67	5 : 00 : 00
12	5	14	24	24	37	5	0	9	2.19	1.77	0.00	0 : 00 : 00
17	5	28	97	42	103	16	2	29	2.26	2.21	0.00	0 : 00 : 01
30	5	159	20487	5157	5597	525	140	32327	5.13	4.63	0.86	5 : 00 : 00
52	5	134	6151	1302	1144	619	49	4451	7.90	7.17	5.27	5 : 00 : 00
12	6	7	5	8	15	7	0	7	0.72	0.72	0.00	0 : 00 : 00
17	6	9	7	11	26	6	2	1	0.00	0.00	0.00	0 : 00 : 00
30	6	143	17670	3156	11446	331	65	25811	5.98	5.17	1.36	5 : 00 : 00
52	6	121	6712	791	1840	349	12	3035	7.79	7.32	5.69	5 : 00 : 00
12	7	22	23	43	142	9	0	29	1.84	1.84	0.00	0 : 00 : 00
17	7	30	102	78	228	12	0	39	2.86	2.81	0.00	0 : 00 : 02
30	7	108	3627	419	4333	93	0	2545	3.11	3.14	0.00	0 : 17 : 42
52	7	146	7745	644	4121	268	0	3113	8.08	8.08	6.39	5 : 00 : 00
12	10	4	0	20	0	6	0	11	0.83	0.83	0.00	0 : 00 : 00
17	10	2	0	0	0	2	0	1	0.00	0.00	0.00	0 : 00 : 00
30	10	81	606	209	1620	22	0	293	1.45	1.41	0.00	0 : 00 : 57
52	10	126	6058	211	11138	111	1	3701	6.54	6.49	4.99	5 : 00 : 00
17	13	2	0	0	0	3	1	1	0.00	0.00	0.00	0 : 00 : 00
30	13	62	227	143	1452	7	0	127	0.93	0.92	0.00	0 : 00 : 20
52	13	161	6364	228	26030	62	0	3229	5.16	6.66	3.60	5 : 00 : 00
17	16	2	0	0	0	3	0	1	0.00	0.00	0.00	0 : 00 : 00
30	16	89	490	964	4848	20	0	535	2.03	2.03	0.00	0 : 01 : 10
52	16	155	3724	183	20931	67	0	1747	2.22	2.44	1.14	5 : 00 : 00

reports results obtained for real instances with 30 and 52 nodes and $K = 3, 4, 5, 6$ without the use of the cycle and the cycle partition inequalities. The instances with $K = 3, 4$ that were solved to optimality using the cycle and the cycle partition inequalities are also solved to optimality without using these inequalities, but both the computing time and the size of the branch-and-bound tree more than doubled. For the other instances, it can be seen from Tables 1 and 3 that the use of these inequalities allowed to reduce the overall gap by about 13%. This improvement can also be observed for the Belgacom instances and the random instances as well, by comparing the lower bound obtained at the root node with (**Gap2**) and without (**Gap1**) these inequalities. It appears from Table 1 that the use of the cycle and cycle partition inequalities decreases the gap at the root by about 33%. A similar improvement also appears in Table 2 where the decrease is about 27%. However for the instances with $K \geq 7$, the gain is usually not significant. In fact, for higher bounds, these inequalities may be mostly dominated by the subset inequalities. We remark that for these problems, the number of generated subset inequalities is significantly greater than that of the cycle inequalities. Hence our heuristic for separating these inequalities seems to be quite efficient. Also, as the separation for cycle inequalities is not exact for $K \geq 5$, we may not add all the violated inequalities, and hence, these inequalities may be less efficient in this case. However, for small bounds, the cycle inequalities seem to play a central role in the resolution of the 2ECSBR.

Table 2. Results for random instances

$ V $	K	Cu	Cy	Me	Su	Cc	Cp	No	o/p	Gap1	Gap2	Gt	CPU
10	3	3.8	9.0	9.8	6.0	4.2	2.8	3.0	5/5	1.32	0.44	0.00	0 : 00 : 00
20	3	13.0	27.4	19.8	11.0	9.0	7.0	3.4	5/5	1.39	0.15	0.00	0 : 00 : 00
30	3	28.2	98.2	59.8	20.6	33.4	14.8	36.6	5/5	1.80	0.83	0.00	0 : 00 : 08
40	3	53.4	311.0	175.8	40.0	192.4	53.6	1175.8	5/5	3.06	1.31	0.00	0 : 05 : 55
50	3	80.6	776.0	468.4	101.2	710.0	204.6	8053.8	5/5	3.01	1.88	0.00	1 : 28 : 13
10	4	6.6	16.2	18.0	12.2	5.8	2.6	9.4	5/5	1.71	1.26	0.00	0 : 00 : 00
20	4	40.8	1066.0	507.8	120.8	56.0	31.2	551.0	5/5	4.78	3.37	0.00	0 : 01 : 07
30	4	105.0	16707.6	7316.2	914.6	397.8	172.4	22925.4	1/5	7.19	5.72	1.14	4 : 49 : 59
40	4	126.2	10328.8	4098.6	616.2	568.2	168.0	13345.0	0/5	8.55	6.35	2.40	5 : 00 : 00
50	4	123.6	6585.6	2364.2	350.6	668.4	109.0	6466.6	0/5	10.43	7.87	5.06	5 : 00 : 00
10	5	5.0	6.6	9.0	17.2	3.6	0.4	6.2	5/5	1.16	0.89	0.00	0 : 00 : 00
20	5	49.4	2478.6	732.0	1101.8	77.8	28.8	2891.8	5/5	6.37	4.39	0.00	0 : 05 : 49
30	5	108.2	17383.6	3512.8	3949.6	368.6	75.6	18396.2	1/5	7.27	6.09	1.57	4 : 24 : 57
40	5	131.6	11380.0	2111.8	2156.2	457.2	65.4	9775.4	0/5	9.45	7.94	4.71	5 : 00 : 00
50	5	123.2	7240.8	1139.2	1206.8	474.8	55.8	4050.2	0/5	10.38	9.78	7.01	5 : 00 : 00
10	6	9.8	2.8	19.6	51.0	6.2	0.6	9.0	5/5	2.74	2.70	0.00	0 : 00 : 00
20	6	58.0	2413.6	507.2	2105.0	60.8	17.0	2472.6	5/5	5.33	4.46	0.00	0 : 06 : 33
30	6	110.0	17946.4	2536.2	9337.0	232.8	20.6	14776.6	1/5	7.48	6.88	2.37	4 : 43 : 21
40	6	127.0	12085.2	1317.0	4560.8	281.2	35.8	7021.0	0/5	9.99	9.15	5.85	5 : 00 : 00
50	6	121.8	7495.8	741.2	2395.8	270.6	19.8	2711.8	0/5	10.91	10.53	8.19	5 : 00 : 00
10	7	6.2	0.4	14.2	37.6	4.0	0.2	9.0	5/5	1.36	1.25	0.00	0 : 00 : 00
20	7	45.8	910.8	166.0	1049.0	34.2	8.4	625.4	5/5	3.73	3.38	0.00	0 : 01 : 04
30	7	119.6	16455.2	1953.2	16711.6	203.4	6.0	16300.6	0/5	6.19	5.64	1.69	5 : 00 : 00
40	7	137.0	13231.8	1035.8	8448.8	239.2	20.4	6652.6	0/5	7.67	7.61	4.24	5 : 00 : 00
50	7	127.6	8159.6	530.8	4238.0	206.8	12.2	2712.6	0/5	9.77	9.49	7.45	5 : 00 : 00
10	10	2.8	0.0	0.0	0.0	1.4	0.0	1.0	5/5	0.00	0.00	0.00	0 : 00 : 00
20	10	40.2	480.8	117.6	1944.4	25.2	1.4	601.0	5/5	2.30	2.50	0.00	0 : 00 : 45
30	10	122.2	9110.8	753.8	28861.8	113.4	0.8	7852.6	4/5	3.79	3.81	0.14	2 : 57 : 21
40	10	154.2	12217.8	673.8	21737.4	152.2	0.8	6181.8	1/5	4.16	6.24	3.11	4 : 41 : 58
50	10	152.8	10722.8	400.0	16278.2	120.0	1.6	3177.8	0/5	6.45	6.35	4.46	5 : 00 : 00
20	13	25.6	86.8	183.6	812.8	14.0	0.2	130.2	5/5	1.72	1.72	0.00	0 : 00 : 06
30	13	80.6	1427.6	220.6	7718.4	38.0	0.4	769.4	5/5	2.50	2.24	0.00	0 : 05 : 36
40	13	155.2	7688.4	515.6	32764.8	143.6	0.8	5373.4	2/5	4.63	4.61	2.19	4 : 00 : 09
50	13	175.0	10648.2	328.0	30746.6	105.4	0.4	3961.8	0/5	4.59	4.55	2.81	5 : 00 : 00
20	16	18.8	1.8	67.0	121.4	8.4	0.2	13.0	5/5	0.61	0.60	0.00	0 : 00 : 01
30	16	88.2	885.4	306.6	7535.8	56.0	0.4	610.2	5/5	2.57	2.41	0.00	0 : 03 : 52
40	16	141.8	3743.8	335.2	24970.8	100.4	0.2	2433.8	4/5	3.51	3.66	1.26	1 : 32 : 27
50	16	184.4	8520.6	266.8	43858.0	92.0	0.4	3611.0	0/5	3.23	3.24	1.57	5 : 00 : 00

Table 3. Results for Belgacom instances without cycle and cycle partition inequalities

$ V $	K	Cu	Cy	Me	Su	Cc	Cp	No	Gap1	Gt	CPU
30	3	32	0	134	22	66	0	93	0.93	0.00	0 : 00 : 20
52	3	103	0	1187	104	1220	0	6505	1.99	0.00	1 : 28 : 12
30	4	119	0	9687	983	435	0	16183	5.66	0.00	1 : 45 : 07
53	4	146	0	5109	572	816	0	8555	7.31	5.48	5 : 00 : 00
30	5	152	0	14900	8797	670	0	37329	5.13	0.80	5 : 00 : 00
52	5	144	0	4156	2923	649	0	5975	9.02	6.73	5 : 00 : 00
30	6	171	0	8361	17601	287	0	33919	5.89	1.50	5 : 00 : 00
52	6	150	0	3163	5844	383	0	4385	7.93	6.13	5 : 00 : 00

When $K \geq 10$, we notice that the problems are easier to solve. Most of the instances on 30 nodes and some on 40 nodes have been solved to optimality. This is due to the fact that for large value of K , the 2ECSBR is closer to the 2-edge connected subgraph

Table 4. The average overall gap

	Size	$K = 3$	$4 \leq K \leq 7$	$K \geq 10$
Belgacom instances	17	0.00	0.00	0.00
	30	0.00	1.20	0.00
	52	0.00	5.25	3.24
Random instances	20	0.00	0.00	0.00
	30	0.00	1.69	0.05
	40	0.00	4.30	2.19
	50	0.00	6.93	2.95

problem which can be solved efficiently — for the graph sizes considered in this paper — using the cut constraints only.

Table 4 summarizes the main results of our computational study. It reports the average overall gap for the two types of instances with respect to the size of the instances and the bound on the cycles. As it appears, if the bound is 3, the problem seems to be easy to solve, either for the real or the random instances. If the bound is high (≥ 10 for instance), the problem remains relatively easy to solve. However, for bounds between 4 and 7 the problem gets harder, in particular for instances with 40 nodes and more, and this for both real and random instances.

7. Concluding remarks

We studied the two-edge connected subgraph problem where every edge must belong to a bounded cycle. We have given an integer programming formulation for this problem. We have identified various classes of valid inequalities and discussed necessary and sufficient conditions for these inequalities to be facet defining. We have provided separation algorithms for these inequalities. In particular, we have shown that the separation problem for the cycle inequalities can be reduced to a maximum flow problem when the cycle bound is ≤ 4 and thus, can be solved in this case in polynomial time. Using these results we have described a Branch-and-Cut algorithm for this problem. Our computational results have shown that the problem could be hard to solve for K between 4 and 7. We could estimate the effect of the cycle and the cycle partition inequalities in the Branch-and-Cut algorithm. We could also measure the performance of our separation techniques.

It would be interesting to extend the results given in this paper to the more general survivable network design problem [15] with bounded rings.

Acknowledgements. The authors would like to thank the anonymous referees for their valuable comments that permitted to improve the presentation of the paper.

References

1. Baiou, M., Barahona, F., Mahjoub, A.R.: Separation of Partition Inequalities. *Mathematics of Operations Research* **25** (2), 243–254, May 2000
2. Barahona, F.: Separating from the dominant of the spanning tree polytope. *Op. Research Letters* **12**, 201–203 (1992)

3. Barahona, F., Mahjoub, A.R.: On two-connected subgraph polytopes. *Discrete Mathematics* **147**, 19–34 (1995)
4. Chopra, S.: The k -edge connected spanning subgraph polyhedron. *SIAM Journal on Discrete Mathematics* **7**, 245–259 (1994)
5. Cunningham W.H.: Optimal attack and reinforcement of a network. *Journal of ACM* **32**, 549–561 (1985)
6. Didi Biha, M., Mahjoub, A.R.: k -edge connected polyhedra on series-parallel graphs. *Operations Research Letters* **19**, 71–78 (1996)
7. Fortz B.: *Design of Survivable Networks with Bounded Rings. Vol. 2 Network Theory and Applications*. Kluwer Academic Publishers, 2000
8. Fortz, B., Labbé, M., Maffioli, F.: Solving the Two-Connected Network with Bounded Meshes Problem. *Operations Research*, **48** (6), 866–877 November-December 2000
9. Fortz B., Labbé M.: Polyhedral results for two-connected networks with bounded rings. *Mathematical Programming* **93** (1), 27–54 2002
10. Fortz, B., Mahjoub, A.R., McCormick, S.T., Pesneau, P.: Two-edge connected subgraphs with bounded rings: Polyhedral results and Branch-and-Cut. Working paper 98/03 Institut d'Administration et de Gestion Université Catholique de Louvain, Belgique, 2003
11. Fortz, B., Labbé, M.: Two-connected networks with rings of bounded cardinality. *Computational Optimization and Applications* **27**, 123–148 2004
12. Goldberg, A.V., Tarjan, R.E.: A New Approach to the Maximum-Flow Problem. *Journal of the Association for Computing Machinery* **35** (4), 921–940, October 1988
13. Gomory, R.E., Hu, T.C.: Multi-Terminal Network Flows. *SIAM Journal on Applied Mathematics* **9** (4), 551–570, December 1961
14. Gourdin, E., Liao, B.: *Personnal Communication* 2003
15. Grötschel, M., Monma, C.L., Stoer, M.: Design of Survivable Networks. chapter 10, pp. 617–672. *Handbooks in Operations Research and Management Science*. Elsevier, North-Holland Amsterdam. 1995
16. Gusfield, D.: Very Simple Methods for All Pairs Network Flow Analysis. *SIAM Journal on Computing* **19** (1), 143–155 February 1990
17. Hao, J., Orlin, J. B.: A faster algorithm for finding the minimum cut in a graph. *Proc. of 3rd ACM-SIAM Symp. on Discrete Algorithms* 1992, pp. 165–174
18. Itai, A., Perl, Y., Shiloach, Y.: The Complexity of Finding Maximum Disjoint Paths with Length Constraints. *Networks* **12**, 277–286 (1982)
19. Kerivin, H., Mahjoub, A.R.: Design of Survivable Networks: A Survey. To appear in *Networks* 2004
20. Ladányi, L., Ralphs, T.K., Trotter, L.E.: Computational Combinatorial Optimization: Optimal or Provably Near-Optimal Solutions. *Branch Cut and Price: Sequential and Parallel* 223–260. *Lecture Notes in Computer Science*. Springer-Verlag, September 2001
21. Mahjoub, A.R.: Two-edge connected spanning subgraphs and polyhedra. *Mathematical Programming* **64**, 199–208 1994
22. McCormick, S.T.: *Personnal Communication* 2001
23. Tsong-Ho Wu.: *Fiber Network Service Survivability*. Artech House Inc. 1992
24. Williamson, D.P.: *Lecture Notes on Approximation Algorithms*. IBM Research Division 1999