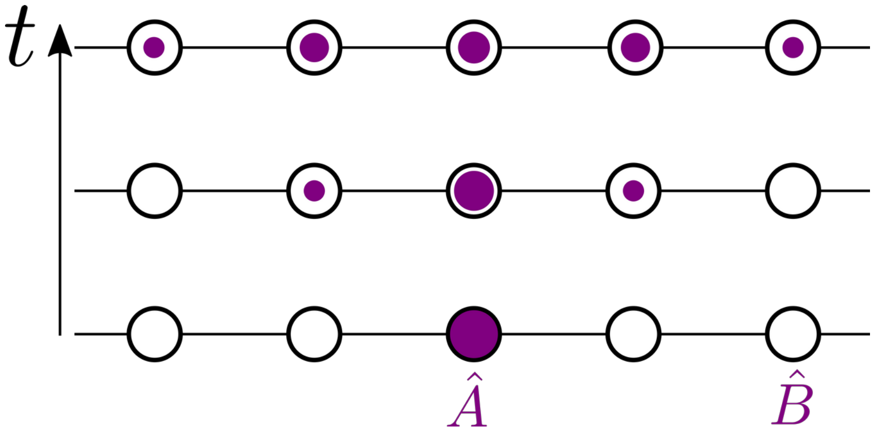
Quasiclassical description of out-of-time-ordered correlators



Thomas MICHEL
Peter SCHLAGHECK
Juan Diego URBINA

LIÈGE université

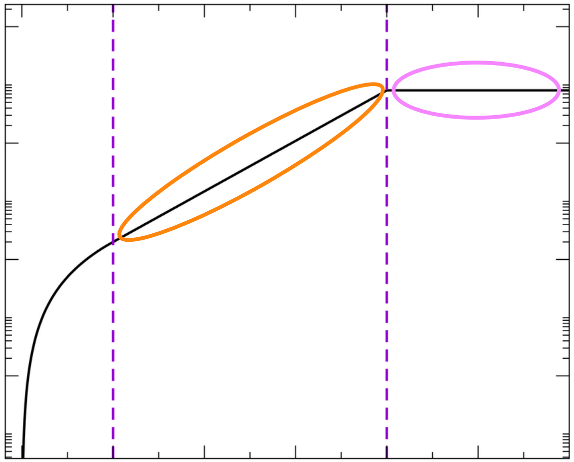
Introduction

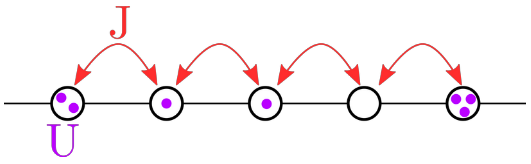
- Study the quantum chaos through the lense of out-of-time-ordered correlators (OTOCs)
  $\rightarrow$ Quasiclassical theory of OTOCs

- Recover the initial exponential growth?

- Derive a saturation value?

- Numerical simulations in Bose-Hubbard systems



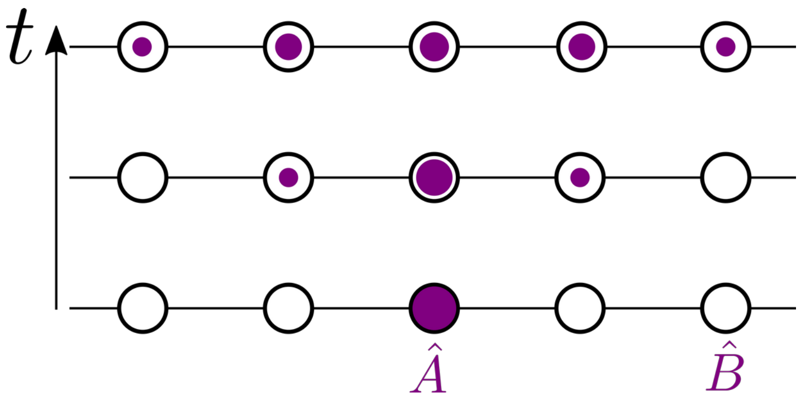


Out-of-time-ordered correlators (OTOCs)

- $C(t) = \langle\psi| \left| \left[\hat{A}(t), \hat{B}(0) \right] \right|^2 |\psi\rangle$

- Characterises the propagation of information

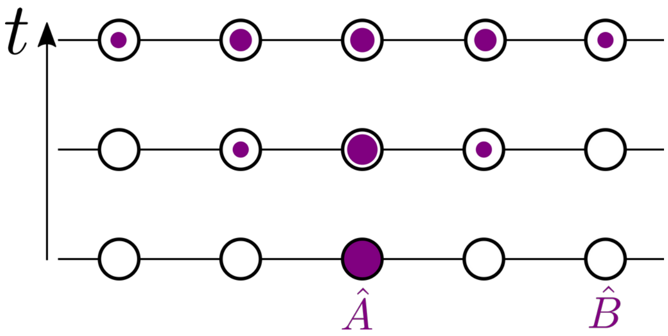
  $\rightarrow$ butterfly speed

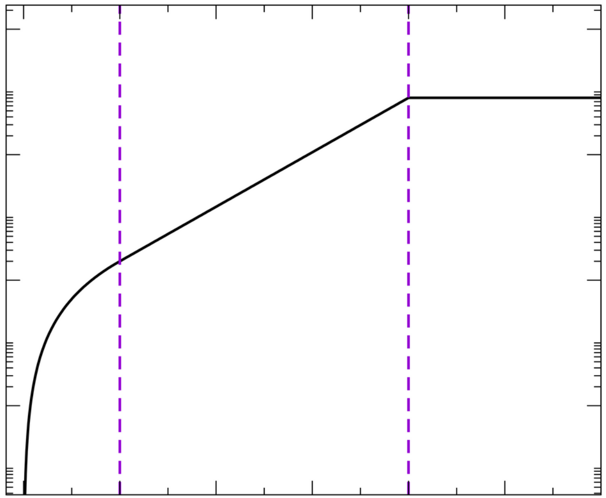


Out-of-time-ordered correlators (OTOCs)

If there is chaos: 3 regimes

- ultra short time $t < \frac{1}{\lambda_L}$: power-law regime, λ_L = Lyapunov exponent
- short time $\frac{1}{\lambda_L} < t < t_E$: exponential regime $\propto e^{2\lambda_L t}$
- after the Ehrenfest time t_E : saturation





Wigner-Moyal formalism

- Well known in the literature
- Crude classical limit: $\hbar \to 0$
- Equivalent to transform commutator into Poisson bracket
- For $\hat{A} = \hat{q}_i,\ \hat{B} = \hat{q}_j$:

$$\langle\psi| \left|\left[\hat{A}(t), \hat{B}\right]\right|^2 |\psi\rangle \to \hbar^2 \int \mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{p}\ \{A_W(\boldsymbol{q},\boldsymbol{p},t), B_W(\boldsymbol{q},\boldsymbol{p},0)\}^2\ W_\psi(\boldsymbol{q},\boldsymbol{p})$$

$$= \hbar^2 \int \mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{p}\ \left(\frac{\partial q_i}{\partial q_j}(t)\right)^2 W_\psi(\boldsymbol{q},\boldsymbol{p})$$

W_ψ = Wigner function of initial state

Wigner-Moyal formalism

- If chaos: exponential growth

$$\{A_W(\boldsymbol{q},\boldsymbol{p},t), B_W(\boldsymbol{q},\boldsymbol{p},0)\}^2 \propto \hbar^2 e^{2\lambda_L t}$$

- Problem: valid only for short time:

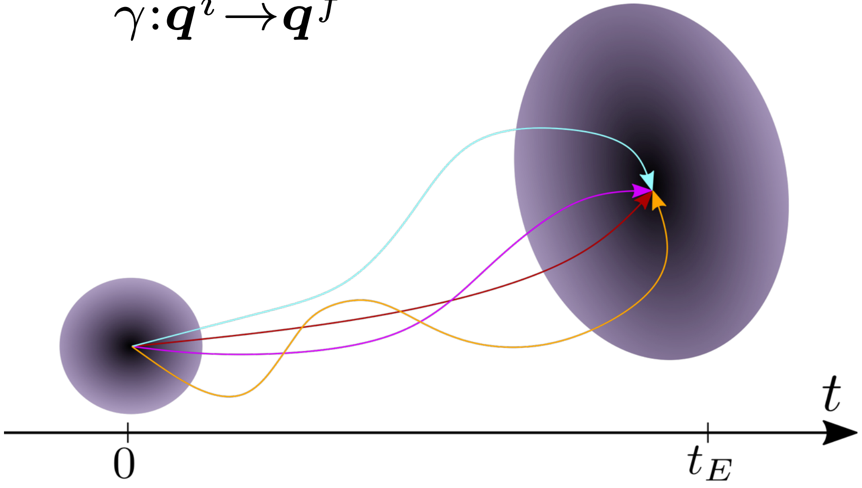
$$t_E \sim \frac{1}{\lambda_L} \log\left(\frac{1}{\hbar}\right) \to \infty$$

$\to$ need for a more elaborate formalism for $t > t_E$

Semiclassical propagator

- Also called the van Vleck-Gutzwiller propagator
- Stationary-phase approximation on the Feynman path integral

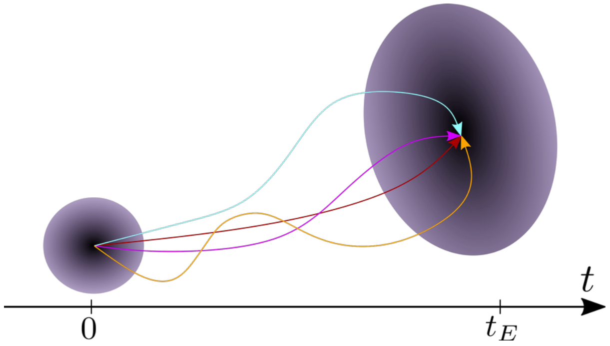
$$\langle \boldsymbol{q}^f | \hat{U}(t) | \boldsymbol{q}^i \rangle \simeq \sum_{\gamma:\boldsymbol{q}^i \to \boldsymbol{q}^f} A_\gamma(\boldsymbol{q}^f, \boldsymbol{q}^i, t)\; e^{\frac{i}{\hbar} R_\gamma(\boldsymbol{q}^f, \boldsymbol{q}^i, t)}$$



Semiclassical propagator

- More granularity towards the classical limit
- Takes interferences into account through $e^{\frac{i}{\hbar}\Delta R_\gamma}$
- No $\hbar$ expansion

$$\langle \boldsymbol{q}^f | \hat{U}(t) | \boldsymbol{q}^i \rangle \simeq \sum_{\gamma:\boldsymbol{q}^i \to \boldsymbol{q}^f} A_\gamma(\boldsymbol{q}^f, \boldsymbol{q}^i, t)\, e^{\frac{i}{\hbar} R_\gamma(\boldsymbol{q}^f, \boldsymbol{q}^i, t)}$$

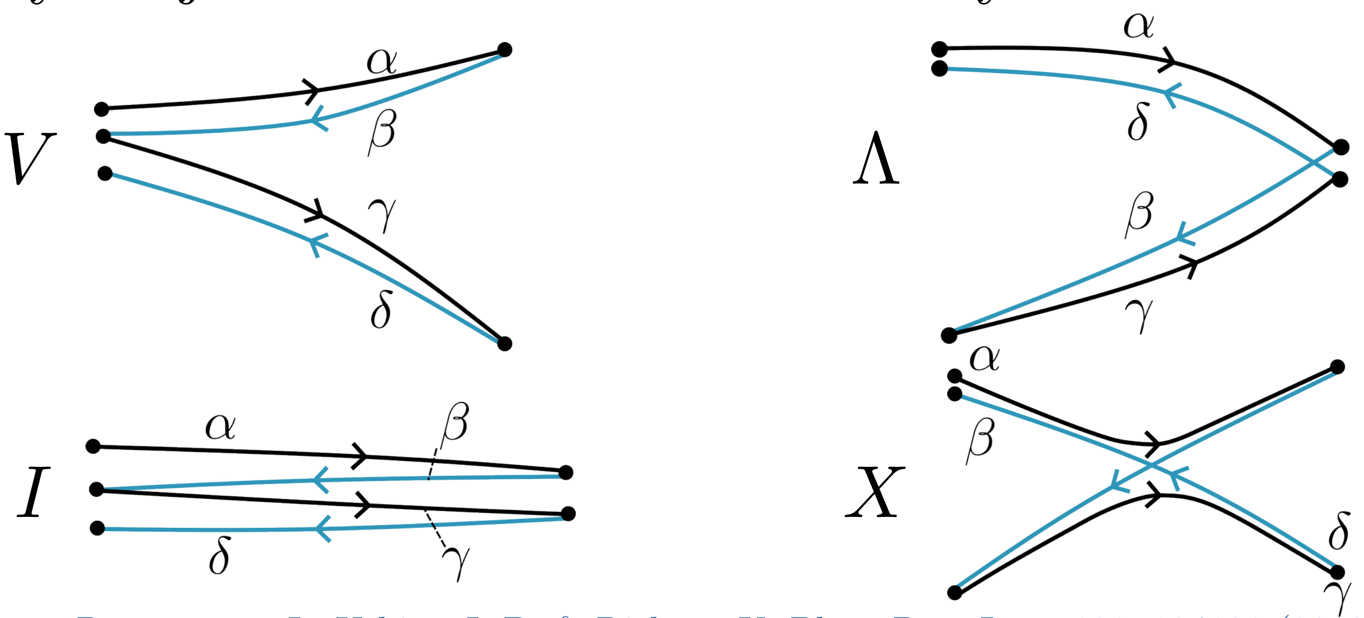


Diagonal (quasiclassical) approximation

- Most of the trajectories average out through $e^{\frac{i}{\hbar}\Delta R_\gamma}$

  $\rightarrow$ ~~interferences~~

- Only trajectories of the same family contribute

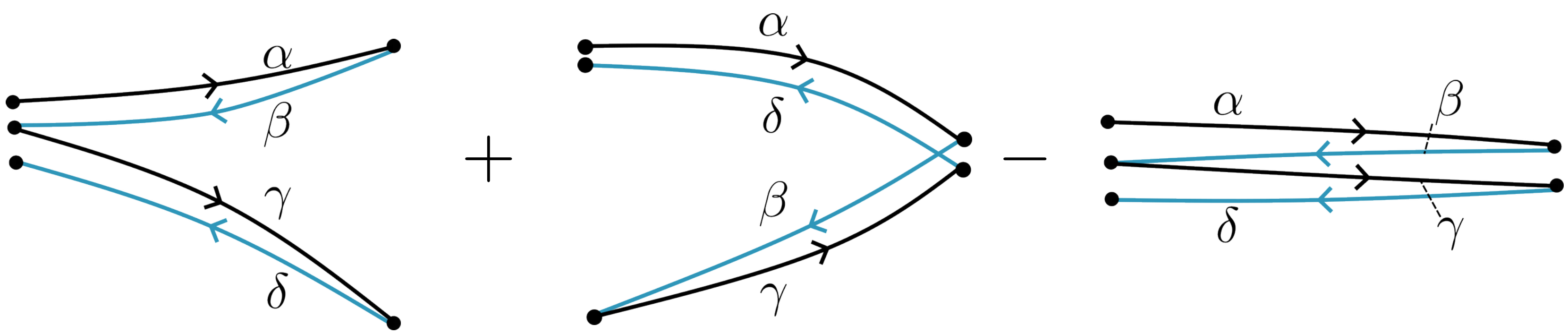


Rammensee J., Urbina J.-D. & Richter, K. Phys. Rev. Lett. 121, 124101 (2018).

Diagonal approximation (quasiclassical)

- X pairing beyond quasiclassical scope

$\rightarrow$ contributions: $C_{qcl}(t) = C_V(t) + C_\Lambda(t) - C_I(t)$



Semiclassical + diagonal approximation

- Quasiclassical OTOC:

$$C_{qcl}(t) = \int \mathrm{d}X_{1,2,3}\mathrm{d}\Delta_{1,2}\; e^{i(\Delta_1 \otimes (X_2 - X_3) + \Delta_2 \otimes (X_1 - 2X_2 + X_3))} A_t\left(X_1 + \frac{\hbar\Delta_2}{4}\right)\; A_t\left(X_1 - \frac{\hbar\Delta_2}{4}\right)$$

$$\left[B\left(X_2 + \frac{\hbar\Delta_1}{4}\right)\; B\left(X_2 - \frac{\hbar\Delta_1}{4}\right) - B\left(2X_1 - 2X_2 + X_3\right)\left(B\left(X_2 + \frac{\hbar\Delta_1}{4}\right) + B\left(X_2 - \frac{\hbar\Delta_1}{4}\right)\right)\right.$$

$$\left. + (\hat{B}^2)_W\left(2X_1 - 2X_2 + X_3\right)\right]\frac{W(X_3)}{(2\pi)^{4L}}$$

- Symplectic formalism: $R = (\boldsymbol{q}, \boldsymbol{p})$

$$R_1 \wedge R_2 = \boldsymbol{q}_1 \cdot \boldsymbol{p}_2 - \boldsymbol{p}_1 \cdot \boldsymbol{q}_2$$

- Cancellation of Λ and I for short time

Rammensee J., Urbina J.-D. & Richter, K. Phys. Rev. Lett. 121, 124101 (2018).
Jalabert R., Garcia-Mata I. & Wisniacki D., Physical Review E 98, (2018).
Kurchan J., J Stat Phys 171, 965-979 (2018).

Semiclassical + diagonal approximation

- First non-vanishing order in $\hbar$:

$$C_{qcl}(t) = \hbar^2 \int \mathrm{d}X \, \{A_t(X), B(X)\}^2 \, W(X) + \mathcal{O}(\hbar^3)$$

- We recover Wigner-Moyal, *i.e.* the Poisson bracket and the exponential growth
- Valid for short time, but we can go further if we don't make $\hbar$ expansions!

Long-time value: $t \to \infty$

- Goal: obtain a finite long-time value C_∞ of C_{qcl}
- Hypotheses: ergodicity + mixing

$$\implies \quad A_t(\boldsymbol{q},\boldsymbol{p}) \to \bar{A}(\vec{c}(\boldsymbol{q},\boldsymbol{p}))$$

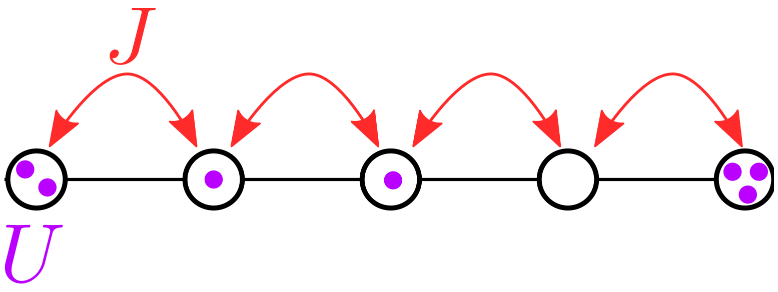
$\vec{c}(\boldsymbol{q},\boldsymbol{p}) = (c_1(\boldsymbol{q},\boldsymbol{p}), c_2(\boldsymbol{q},\boldsymbol{p}), \ldots)$ the constants of motion

- With some $\hbar$ expansions **after** $t \to \infty$ but not inside evolution:

$$C_\infty = \hbar^2 \int \mathrm{d}X \left\{\bar{A}(\vec{c}(X)), B(X)\right\}^2 W(X) + \mathcal{O}(\hbar^3)$$

Numerical simulations: Bose-Hubbard

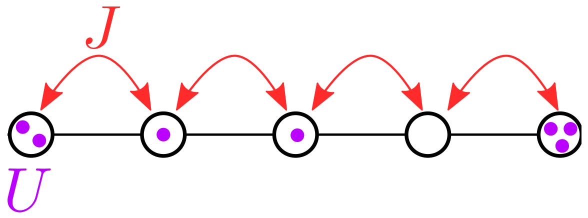
- Model of ultra-cold bosonic atoms in optical lattices with
  - on-site energy E_l
  - 2-body interaction on a site U
  - hopping between adjacent sites J
  - Energy driving δ with frequency ω



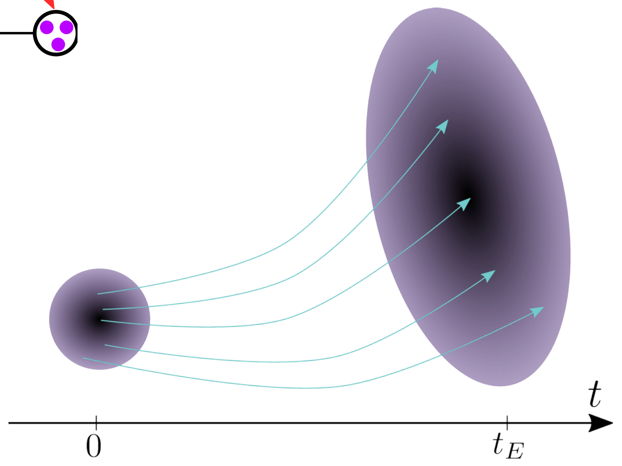
Numerical simulations: Bose-Hubbard

- Model of ultra-cold bosonic atoms in optical lattices

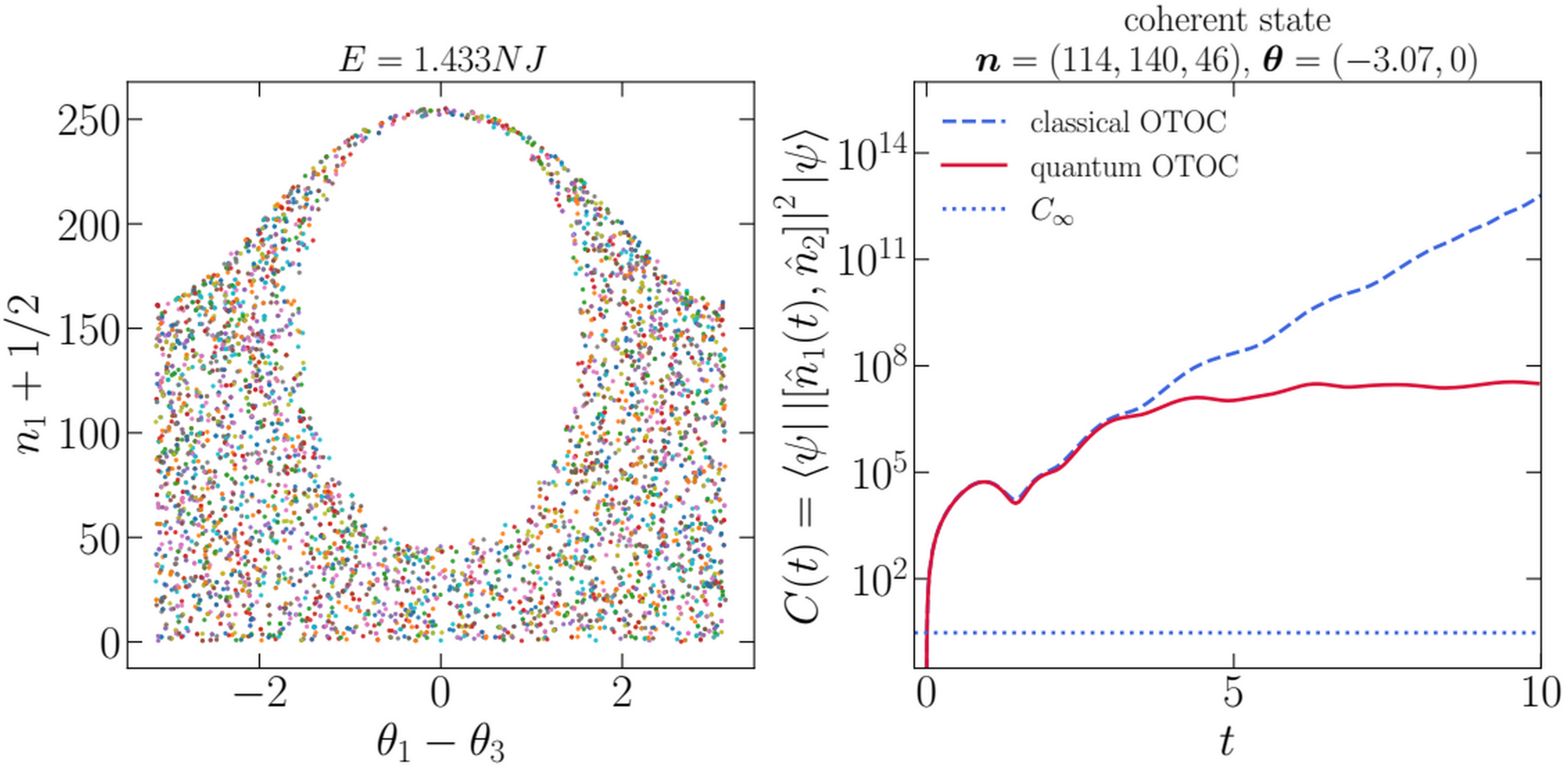
$$\hat{H} = \sum_{l=1}^{L} \left(E_l \hat{n}_l + \frac{U}{2} \hat{n}_l \left(\hat{n}_l - 1 \right) \right) - J \sum_{l=1}^{L-1} \left(\hat{b}_l^\dagger \hat{b}_{l+1} + \hat{b}_{l+1}^\dagger \hat{b}_l \right) + \delta \cos(\omega t) \left(\hat{n}_1 - \hat{n}_2 \right)$$



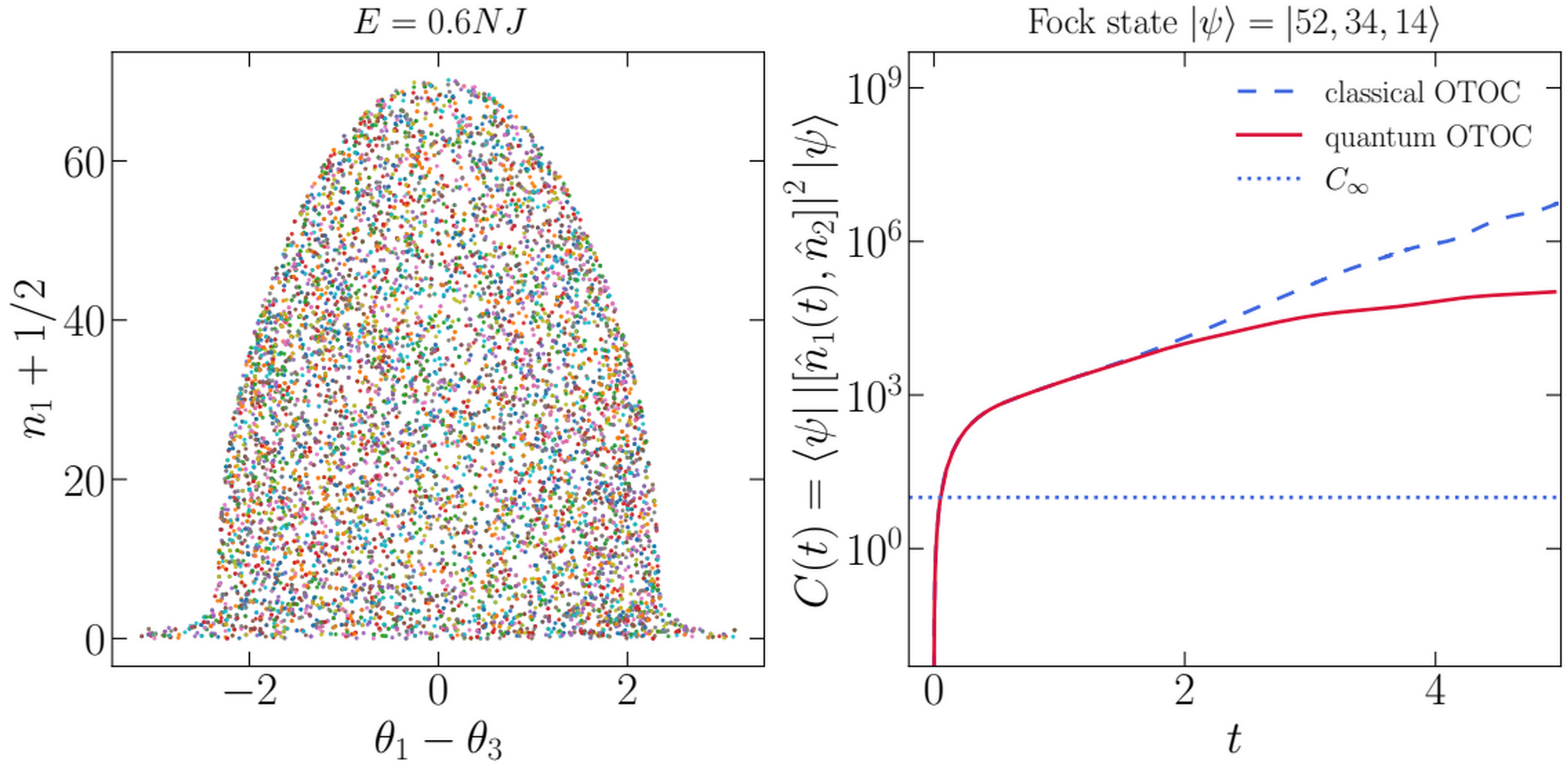
- Quasiclassical simulations using truncated Wigner



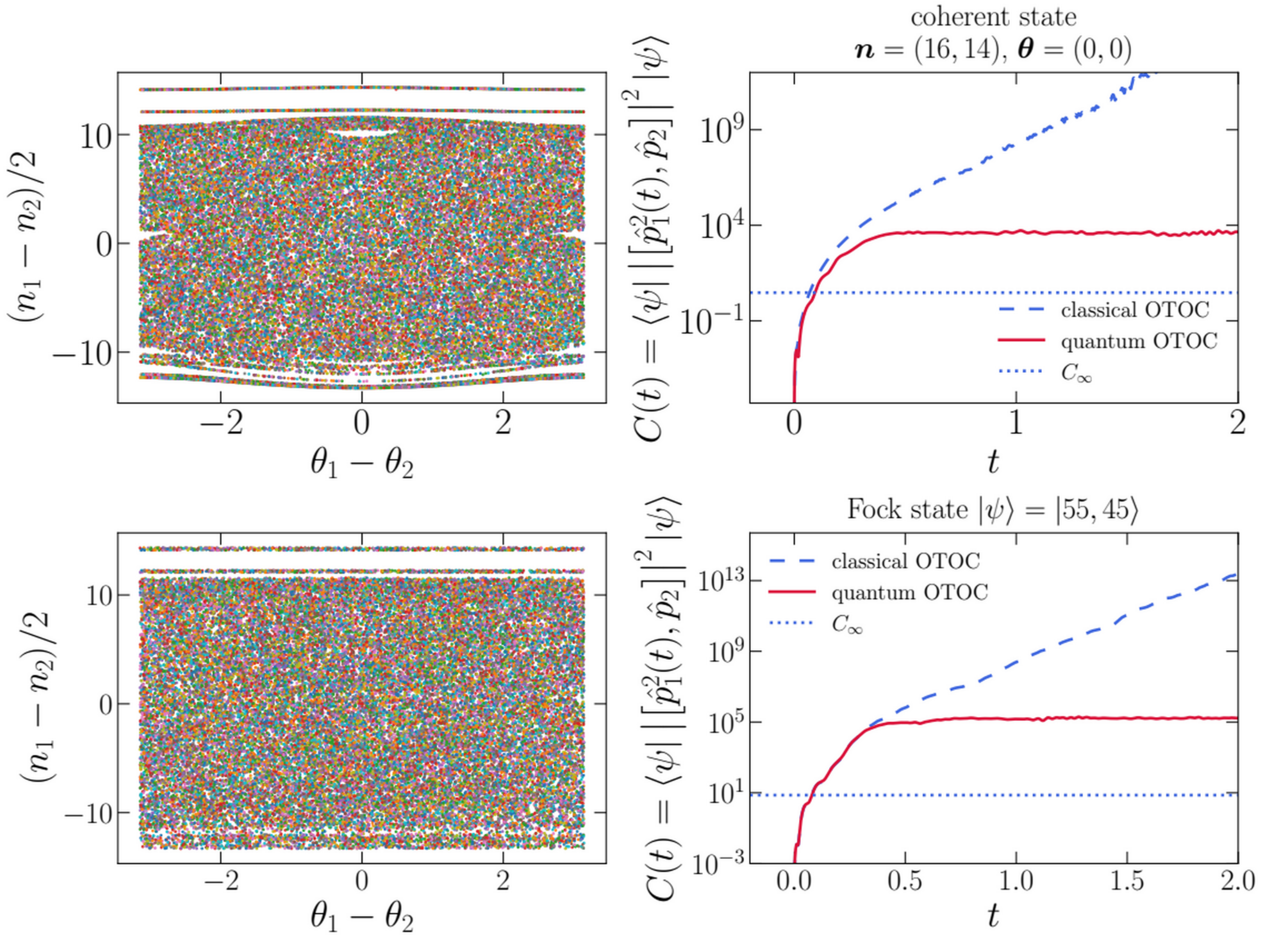
Bose-Hubbard trimer:
coherent states



Bose-Hubbard trimer:
Fock states



Driven dimer

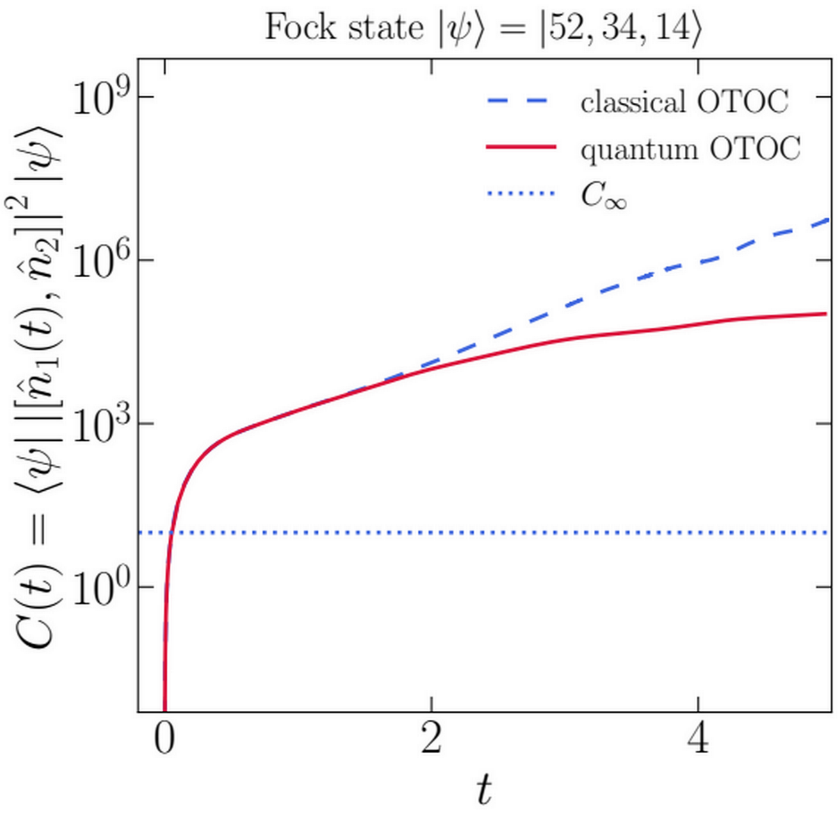


$$N = 30,\ U = 3J$$
$$\delta = 20J,\ \omega = 10J$$

$$N = 100,\ U = J$$
$$\delta = 20J,\ \omega = 10J$$

Observations

- Perfect agreement for short time
- Wigner-Moyal (dashed) never saturates
- Finite long-time value with diagonal approximation (dotted)



Conclusion

- We reproduced the exponential behaviour using the semiclassical propagator and the diagonal approximation

  [Rozenbaum, E. B., Ganeshan, S. & Galitski, V., Phys. Rev. Lett. 118, 086801 (2017)]

- We also reproduced a saturation but not the quantum one $\rightarrow$ quantum nature (interference) of OTOCs

  [Rammensee, J., Urbina, J.-D. & Richter, K., Phys. Rev. Lett. 121, 124101 (2018)]

- Perspective: generalisation to thermal states?

  [Maldacena, J., Shenker, S. H. & Stanford, D. J. High Energ. Phys. 2016, 106 (2016)]

Additional material

Forwards and backwards propagations

$$\langle\psi|\left|[\hat{A}(t),\hat{B}(0)]\right|^2|\psi\rangle = \langle\psi|\,[\hat{A}(t),\hat{B}(0)]^\dagger[\hat{A}(t),\hat{B}(0)]\,|\psi\rangle$$
$$= \langle\psi|\,\hat{B}^\dagger(0)\hat{A}^\dagger(t)\hat{A}(t)\hat{B}(0)+\ldots|\psi\rangle$$