Permutationally invariant processes in arbitrary multiqudit systems

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We establish the theoretical framework for an exact description of the open system dynamics of permutationally invariant (PI) states in arbitrary N-qudit systems when this dynamics preserves the PI symmetry over time. Thanks to Schur-Weyl duality powerful formalism, we identify an orthonormal operator basis in the PI operator subspace of the Liouville space onto which the master equation can be projected and we provide the exact expansion coefficients in the most general case. Our approach does not require to compute the Schur transform as it operates directly within the restricted operator subspace, whose dimension only scales polynomially with the number of qudits. We introduce the concept of 3ν-symbol matrix that proves to be very useful in this context.

I. INTRODUCTION

In quantum science and technologies, multilevel quantum systems (qudits) have proven to offer several advantages over conventional two-level entities (qubits) [1]. These include higher information capacity [1, 2], increased resistance to noise [3, 4], greater security in quantum key distribution [5], more powerful metrological schemes [6, 7], and an improved ability for error correction [8] or for quantum machine learning tasks [9]. Several physical platforms can be used to obtain multiqudit systems. For example, light, with its multiphoton states, is primarily a multiqubit system where the state of a qubit is encoded in the polarisation of a photon or in two of its spatial modes [10]. It can also embody a multiqudit system by giving photons access to $d > 2$ distinct temporal modes or frequency modes [11], or by structuring light to confer orbital angular momentum to photons [12, 13]. Alternatively, individual neutral atoms, which are now routinely cooled, trapped in optical lattices and tweezers and internally controlled by laser light, are being used as registers of qubits and qudits [14, 15]. We can also make use of trapped ions [16, 17], ultracold atomic mixtures (where qudits are encoded in the collective spin of a few atoms whose number can be varied [18]), superconducting devices [19], nitrogen-vacancy (NV) centers in diamond [20], or even molecules [21, 22]. When the multiqudit system is composed of identical though not necessarily indistinguishable qudits, a rich variety of *collective* dynamical behaviors can emerge, such as superradiance [23, 24], spin-squeezing [25], or also dissipative phase transitions [26] to name just a few. In this context, it is therefore essential to find efficient methods to describe the dynamics of the system. Some authors have developed such methods when dissipation acts only collectively or individually, first for qubits [27, 28] and later also for qudits [29–31]. These methods, which rely on the permutation invariance of quantum states, were mainly developed to be numerically useful and do not exploit the powerful connection with group representation theory. Here we fill this gap and establish the general theoretical framework for an exact description.

The manuscript is organized as follows. In Sec. II, permutationally invariant processes are introduced. In Sec. III, the structure of the subspace of permutationally invariant mixed states is established. Thanks to the powerful Schur-Weyl duality formalism, an orthonormal basis in this operator subspace is identified and all relevant properties are discussed. In Sec. IV, the master equation for permutationally invariant processes is detailed and a general Identity that allows for an exact computation of the master equation expansion coefficients is established. This Identity generalizes to arbitrary multiqudit systems Identity 1 of Ref. [27] that was developed in the specific context of multiqubit systems. We then draw conclusions in Sec. V.

Several appendices close the manuscript. First, for the sake of completeness a reminder of the basics of partitions and irreducible representations of the symmetric, general linear, and unitary groups is given in Appendix A. These basics are at the core of the Schur-Weyl duality formalism. The dimension of the subspace of permutationally invariant states is specifically commented in Appendix B. We then review in Appendix C the basics of Clebsch-Gordan coefficients of the tensor product of irreducible representations of the unitary group for the Gel'fand-Tsetlin bases. These coefficients are one of the fundamental ingredients required to compute the master equation expansion coefficients in the context of permutationally invariant processes. In Appendix D, the proof of our general Identity of Sec. IV is given and we show in Appendix E how this Identity particularizes to Identity 1 of Ref. [27] in the specific case of qubit systems ($d = 2$). We finally give in Appendix F the explicit expression of the master equation Lindbladian matrix elements and show in Appendix G how the formalism generalizes when considering more general p-particle terms ($p \geq 1$) in the Lindbladian.

II. PERMUTATIONALLY INVARIANT PROCESSES

In the Born-Markov and secular approximations [32, 33], the open dynamics of a quantum system can always be cast into a Lindblad master equation $d\hat{\rho}(t)/dt = \mathcal{L}[\hat{\rho}(t)]$, with $\hat{\rho}(t)$ the system density operator and $\mathcal{L}$ the Linbladian linear superoperator on the Liouville space $\mathscr{L}(\mathcal{H})$ (the space of linear operators on the system Hilbert space $\mathcal{H}$),

$$\mathcal{L}[\hat{\rho}] = \mathcal{V}[\hat{\rho}] + \mathcal{D}[\hat{\rho}], \tag{1}$$

where $\mathcal{V}[\hat{\rho}] = (i/\hbar)\big[\hat{\rho}, \hat{H}(t)\big]$, with $\hat{H}(t)$ the Hamiltonian (including Lamb-shifts) that drives the unitary evolution of the system, and $\mathcal{D}[\hat{\rho}] \equiv \sum_k \mathcal{D}_{\hat{L}_k}[\hat{\rho}]$ is the dissipator, with $\hat{L}_k$ Lindblad operators that model the system dissipation and decoherence processes and, for all operator $\hat{L}$, $\mathcal{D}_{\hat{L}}$ the superoperator

$$\mathcal{D}_{\hat{L}}[\hat{\rho}] = \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^\dagger\hat{L}. \tag{2}$$

The Lindbladian action (1) is fully determined given the Hamiltonian $\hat{H}$ and the set of Lindblad operators $\hat{L}_k$. It is denoted accordingly $\mathcal{L}_{\hat{H},\{\hat{L}_k\}} \equiv \mathcal{V}_{\hat{H}} + \mathcal{D}_{\{\hat{L}_k\}}$ if explicit notation is required. The sum over k in the dissipator contains at most $\dim(\mathcal{H})^2 - 1$ terms [32, 33].

For an N-qudit system the state space $\mathcal{H}$ identifies to $\mathcal{H}_d^{\otimes N}$, with $\mathcal{H}_d \simeq \mathbb{C}^d$ the individual qudit state space [34]. It has dimension d^N scaling exponentially with N. Endowed with the standard Hilbert-Schmidt scalar product, the Liouville space $\mathscr{L}(\mathcal{H})$ is itself a Hilbert space of dimension d^{2N}. This renders the curse of dimensionality already severe for moderate numbers of qudits. This severity can be significantly downgraded if the system exhibits large symmetries that constraint its dynamics in a much smaller dimensional subspace of the Liouville space. This is in particular the case for so-called *permutationally-invariant* (PI) states $\hat{\rho}$ [35] as long as the Lindbladian $\mathcal{L}$ preserves the PI symmetry over time. A PI operator $\hat{A}_{\rm PI}$ is an operator that satisfies $\hat{P}_\sigma\hat{A}_{\rm PI}\hat{P}_\sigma^\dagger = \hat{A}_{\rm PI} \Leftrightarrow [\hat{P}_\sigma, \hat{A}_{\rm PI}] = 0$ for all permutations σ of 1, ..., N, where $\hat{P}_\sigma$ denotes the standard unitary permutation operator associated to σ in $\mathcal{H}$ [36]. The vector subspace of PI operators in $\mathscr{L}(\mathcal{H})$ is the so-called *commutant* $\mathscr{L}_{S_N}(\mathcal{H})$ of the (unitary) representation $\sigma \mapsto \hat{P}_\sigma$ on $\mathcal{H}$ of the symmetric group S_N [37]. The commutant contains the identity operator and is closed under multiplication of operators and hermitian conjugation [38]. A superoperator $\mathcal{L}$ preserves the PI symmetry if the commutant $\mathscr{L}_{S_N}(\mathcal{H})$ is $\mathcal{L}$-invariant, i.e., if $\mathcal{L}[\hat{A}_{\rm PI}]$ is a PI operator whatever the PI operator $\hat{A}_{\rm PI}$. It is also important that such superoperators avoid contaminating the commutant from any non-PI components, i.e., that the orthogonal complement of the commutant be itself $\mathcal{L}$-invariant, which is equivalent to having the commutant $\mathscr{L}_{S_N}(\mathcal{H})$ both $\mathcal{L}$- and $\mathcal{L}^\dagger$-invariant [39].

The natural class of superoperators that preserve the PI symmetry and avoid contamination from any non-PI components is given by superoperators $\mathcal{L}$ that are themselves PI in the sense that $[\mathcal{P}_\sigma, \mathcal{L}] = 0$ for all permutations σ, with $\mathcal{P}_\sigma$ the (unitary) superoperator of permutation $\mathcal{P}_\sigma[\hat{A}] = \hat{P}_\sigma\hat{A}\hat{P}_\sigma^\dagger, \forall\hat{A} \in \mathscr{L}(\mathcal{H})$. Indeed, in this case for all PI operators $\hat{A}_{\rm PI}$, $\mathcal{P}_\sigma\mathcal{L}[\hat{A}_{\rm PI}] = \mathcal{L}\mathcal{P}_\sigma[\hat{A}_{\rm PI}] = \mathcal{L}[\hat{A}_{\rm PI}], \forall\sigma$, i.e., $\mathscr{L}[\hat{A}_{\rm PI}]$ is PI [41] and the commutant $\mathscr{L}_{S_N}(\mathcal{H})$ is $\mathcal{L}$-invariant. It is also $\mathcal{L}^\dagger$-invariant since the space of PI superoperators is closed under hermitian conjugation.

The superoperators of permutation satisfy $\mathcal{P}_\sigma[\hat{A}\hat{B}] = \mathcal{P}_\sigma[\hat{A}]\mathcal{P}_\sigma[\hat{B}]$ and $\mathcal{P}_\sigma[\hat{A}^\dagger] = \mathcal{P}_\sigma[\hat{A}]^\dagger$ [42]. This implies interestingly that Lindbladians obey $\mathcal{P}_\sigma\mathcal{L}_{\hat{H},\{\hat{L}_k\}} = \mathcal{L}_{\mathcal{P}_\sigma[\hat{H}],\mathcal{P}_\sigma[\{\hat{L}_k\}]}\mathcal{P}_\sigma$, where $\mathcal{P}_\sigma[\{\hat{L}_k\}] \equiv \{\mathcal{P}_\sigma[\hat{L}_k]\}$. If the Hamiltonian $\hat{H}$ is PI as well as the set $\{\hat{L}_k\}$ of Lindblad operators *as a whole*, i.e., $\mathcal{P}_\sigma[\{\hat{L}_k\}] = \{\hat{L}_k\}, \forall\sigma$ (which does not require to have individually $\mathcal{P}_\sigma[\hat{L}_k] = \hat{L}_k, \forall k, \sigma$), the Lindbladian $\mathcal{L}_{\hat{H},\{\hat{L}_k\}}$ is a PI superoperator. This is typically the case for a Lindblad operator set composed of identical *local* operators $\hat{\ell}^{(n)}$, $\forall n = 1, \ldots, N$, and/or a *collective* operator $\hat{L}_c = \sum_{n=1}^N \hat{L}^{(n)}$, where the superscript (n) denotes the specific qudit the local operator acts on [43]. If both contributions are present, the dissipator contains a local and a collective part: $\mathcal{D} = \mathcal{D}_{\hat{\ell}}^{(\rm loc)} + \mathcal{D}_{\hat{L}}^{(\rm col)}$, with $\mathcal{D}_{\hat{\ell}}^{(\rm loc)} = \sum_{n=1}^N \mathcal{D}_{\hat{\ell}^{(n)}}$ and $\mathcal{D}_{\hat{L}}^{(\rm col)} = \mathcal{D}_{\hat{L}_c}$.

More general PI dissipators can also be envisaged with for instance identical two-particle Lindblad operators $\hat{\ell}_2^{(n,m)}$, $\forall n < m = 1, \ldots, N$ and/or a collective two-particle Lindblad operator $\hat{L}_{2,c} = \sum_{n<m} \hat{L}_2^{(n,m)}$, where (n, m) denotes the particle pair the two-particle operators $\hat{\ell}_2$ and $\hat{L}_2$ act on. Strictly generally we can even consider identical p-particle ($p \le N$) Lindblad operators $\hat{\ell}_p^{(n_1,\ldots,n_p)}$, $\forall n_1 < \cdots < n_p$, and also a collective p-particle Lindblad operator $\hat{L}_{p,c} = \sum_{n_1<\cdots<n_p} \hat{L}_p^{(n_1,\ldots,n_p)}$, where $(n_1, \ldots, n_p)$ denotes the particle p-uple the p-particle operators $\hat{\ell}_p$ and $\hat{L}_p$ act on.

The commutant $\mathscr{L}_{S_N}(\mathcal{H})$ is nothing but the symmetric subspace of the Liouville space $\mathscr{L}(\mathcal{H}) \cong (\mathscr{L}(\mathcal{H}_d))^{\otimes N}$, i.e., the space of operators that are invariant under the action of all superoperators of permutation $\mathcal{P}_\sigma$. Its dimension is thus equal to $\binom{N+d^2-1}{N}$ [37] and only scales polynomially with N in $\mathcal{O}(N^{d^2-1})$ instead of exponentially as for the global Liouville space $\mathscr{L}(\mathcal{H})$. This changes drastically the complexity class of PI systems for which large N studies should remain more accessible within classical computational ressources. In this context, it is therefore highly desirable to develop tools that allows one to restrict the master equation treatment in the sole commutant subspace. This requires to identify a natural orthonormal basis of operators in $\mathscr{L}_{S_N}(\mathcal{H})$ onto which the master equation can be projected and to have explicit expressions of the matrix elements. This was specifically done in [27] for qubit systems ($d = 2$). For $d > 2$, nothing similar is identified and we fill this gap in this work with the help of the powerful formalism of Schur-Weyl duality (see, e.g., Refs. [37, 40]). The theory is established for arbitrary d and the results for $d = 2$ are recovered as a special case.

III. STRUCTURE OF THE COMMUTANT $\mathscr{L}_{S_N}(\mathcal{H})$

The N-qudit state space $\mathcal{H} = \mathcal{H}_d^{\otimes N}$ is a natural representation space for both the symmetric group S_N and the general linear group $GL(d) \equiv GL(d, \mathbb{C})$ of $d \times d$ invertible complex matrices (including its subgroup $U(d)$ of $d \times d$ unitary matrices). The standard representation operator for $\sigma \in S_N$ is the unitary permutation operator $\hat{P}_\sigma$ and for $A \in GL(d)$ the tensor product operator $\hat{A}^{\otimes N}$, with $\hat{A}$ the invertible local operator of representation matrix A in the single-qudit basis. Both operators $\hat{P}_\sigma$ and $\hat{A}^{\otimes N}$ commute, so that the product operators $\hat{P}_\sigma \hat{A}^{\otimes N}$ define a representation of the direct product group $S_N \times GL(d)$ on $\mathcal{H}$. For $N, d > 1$, $\mathcal{H}$ is a reducible representation space for both the symmetric and the general linear group, as well as for the direct product group. As a consequence of the Schur-Weyl duality, the state space $\mathcal{H}$ can be decomposed into irreducible subrepresentations of both S_N and $GL(d)$, according to the multiplicity-free decomposition of the direct product group representation $\mathcal{H} \simeq \oplus_{\nu \vdash (N,d)} \mathcal{S}^\nu \otimes \mathcal{U}^\nu(d)$, where the direct sum runs over all partitions ν of N of at most d parts, and $\mathcal{S}^\nu$ and $\mathcal{U}^\nu(d)$ denote irreducible representations (irreps) of S_N and $GL(d)$, respectively (see Appendix A). The restriction of $\mathcal{U}^\nu(d)$ onto the subgroup $U(d)$ is also irreducible and can be denoted similarly. We have on the one side $\mathcal{H} \simeq \oplus_{\nu \vdash (N,d)} \mathcal{S}^{\nu \oplus \dim \mathcal{U}^\nu(d)}$, and on the other side $\mathcal{H} \simeq \oplus_{\nu \vdash (N,d)} \mathcal{U}^\nu(d)^{\oplus \dim \mathcal{S}^\nu}$. In this context, a natural basis in the state space $\mathcal{H}$ is the orthonormal so-called *Schur* basis [9] $\{|\nu, T_\nu, W_\nu\rangle, \forall \nu \vdash (N,d), T_\nu \in \mathcal{T}_\nu, W_\nu \in \mathcal{W}_\nu\}$, with $\mathcal{T}_\nu$ the set of all standard Young Tableaux (SYT) T_ν of shape ν and $\mathcal{W}_\nu$ the set of all standard Weyl Tableaux (SWT) W_ν of shape ν and of content among $0, \ldots, d-1$ (see Appendix A). The cardinalities of the sets $\mathcal{T}_\nu$ and $\mathcal{W}_\nu$ are $f^\nu = \dim \mathcal{S}^\nu$ and $f^\nu(d) = \dim \mathcal{U}^\nu(d)$, respectively. The Schur basis vectors $|\nu, T_\nu, W_\nu\rangle$ belong each to a well defined chain of irreps of both subgroup chains $S_N, S_{N-1}, \ldots, S_1$, and $U(d), U(d-1), \ldots, U(1)$. The two chains of irreps are encoded in the SYT T_ν for the symmetric group and in the SWT W_ν for the unitary group. For all $\nu \vdash (N,d)$ and $W_\nu \in \mathcal{W}_\nu$, $\mathcal{H}_\nu(W_\nu) \equiv \mathrm{span}\{|\nu, T_\nu, W_\nu\rangle, \forall T_\nu \in \mathcal{T}_\nu\}$ is an $\mathcal{S}^\nu$-equivalent irrep subspace of the symmetric group S_N. For all $\nu \vdash (N,d)$ and $T_\nu \in \mathcal{T}_\nu$, $\mathcal{H}_\nu(T_\nu) \equiv \mathrm{span}\{|\nu, T_\nu, W_\nu\rangle, \forall W_\nu \in \mathcal{W}_\nu\}$ is an $\mathcal{U}^\nu(d)$-equivalent irrep subspace of $U(d)$. In each of these irrep subspaces, the orthonormal vectors $|\nu, T_\nu, W_\nu\rangle$ identify to the unique (up to global phases) so-called Gel'fand-Tsetlin (GT) basis vectors of the irrep with respect to either of the abovecited subgroup chains [37, 44].

An operator basis in the commutant $\mathscr{L}_{S_N}(\mathcal{H})$ is nicely given by the set of PI operators

$$\hat{F}_\nu^{(W_\nu, W'_\nu)} = \overline{|\nu, W_\nu\rangle\langle\nu, W'_\nu|} \equiv \frac{1}{\sqrt{f^\nu}} \sum_{T_\nu \in \mathcal{T}_\nu} |\nu, T_\nu, W_\nu\rangle\langle\nu, T_\nu, W'_\nu|, \qquad (3)$$

$\forall \nu \vdash (N,d), W_\nu, W'_\nu \in \mathcal{W}_\nu$. Indeed, these operators are easily seen to be permutationally invariant [45] and their action on the Schur basis states reads $\hat{F}_\nu^{(W_\nu, W'_\nu)}|\nu', T_{\nu'}, \tilde{W}_{\nu'}\rangle = 0$ if $\nu' \neq \nu$ and

$$\hat{F}_\nu^{(W_\nu, W'_\nu)}|\nu, T_\nu, \tilde{W}_\nu\rangle = \frac{1}{\sqrt{f^\nu}}|\nu, T_\nu, W_\nu\rangle \delta_{\tilde{W}_\nu, W'_\nu}. \qquad (4)$$

Hence, their range and kernel are given by $\mathrm{ran}\, \hat{F}_\nu(W_\nu, W'_\nu) = \mathcal{H}_\nu(W_\nu)$ and $\ker \hat{F}_\nu^{(W_\nu, W'_\nu)} = \mathcal{H} \ominus \mathcal{H}_\nu(W'_\nu)$, respectively. Each operator $\hat{F}_\nu^{(W_\nu, W'_\nu)}$ maps a specific $\mathcal{S}^\nu$-equivalent irrep subspace onto an equivalent one: $\hat{F}_\nu^{(W_\nu, W'_\nu)} \mathcal{H}_\nu(W'_\nu) = \mathcal{H}_\nu(W_\nu)$. All this makes the set of operators (3) an operator basis in the commutant $\mathscr{L}_{S_N}(\mathcal{H})$ [37] (as a corollary, $\dim \mathscr{L}_{S_N}(\mathcal{H})$ can also be written $\sum_{\nu \vdash (N,d)} f^\nu(d)^2$ [see Appendix B]). In addition, with respect to the standard Hilbert-Schmidt scalar product between any two linear operators, this basis is orthonormal:

$$\mathrm{Tr}\left(\hat{F}_\nu^{(W_\nu, W'_\nu)\dagger} \hat{F}_{\nu'}^{(\tilde{W}_{\nu'}, \tilde{W}'_{\nu'})}\right) = \delta_{\nu,\nu'} \delta_{W_\nu, \tilde{W}_{\nu'}} \delta_{W'_\nu, \tilde{W}'_{\nu'}}. \qquad (5)$$

It follows that any PI operator $\hat{A}_{\mathrm{PI}}$ admits the expansion

$$\hat{A}_{\mathrm{PI}} = \sum_{\nu \vdash (N,d)} \sum_{W_\nu, W'_\nu \in \mathcal{W}_\nu} A_{\nu, W_\nu, W'_\nu} \hat{F}_\nu^{(W_\nu, W'_\nu)}, \qquad (6)$$

with components $A_{\nu, W_\nu, W'_\nu} \equiv (\hat{A}_{\mathrm{PI}})_{\nu, W_\nu, W'_\nu}$ given by

$$A_{\nu, W_\nu, W'_\nu} = \mathrm{Tr}(\hat{F}_\nu^{(W_\nu, W'_\nu)\dagger} \hat{A}_{\mathrm{PI}}) = \frac{1}{\sqrt{f^\nu}} \sum_{T_\nu \in \mathcal{T}_\nu} \langle \nu, T_\nu, W_\nu | \hat{A}_{\mathrm{PI}} | \nu, T_\nu, W'_\nu \rangle. \qquad (7)$$

The matrix representation of such operators is block diagonal in the Schur basis $\{|\nu, T_\nu, W_\nu\rangle\}$ if the basis vectors are sorted first by ν, then by SYT T_ν, and finally by SWT W_ν, i.e., by vector subspaces $\mathcal{H}_\nu(T_\nu), \forall \nu, T_\nu$. Blocks are of dimension $f^\nu(d) \times f^\nu(d)$ and only depend on ν, but not on T_ν, so that the representation matrix A_{PI} exhibits a double block-diagonal structure, with large "ν-blocks", themselves composed of f^ν identical blocks $A(\nu)$: $A_{\mathrm{PI}} = \oplus_\nu A(\nu)^{\oplus f^\nu}$. The elements of a block $A(\nu)$ read $A(\nu)_{W_\nu, W'_\nu} = A_{\nu, W_\nu, W'_\nu} / \sqrt{f^\nu}$.

The commutant can be decomposed into the direct sum of orthogonal operator subspaces $\mathscr{L}_\nu(\mathcal{H}) \equiv \mathrm{span}\{\hat{F}_\nu^{(W_\nu, W'_\nu)}, \forall W_\nu, W'_\nu \in \mathcal{W}_\nu\}$:

$$\mathscr{L}_{S_N}(\mathcal{H}) = \bigoplus_{\nu \vdash (N,d)} \mathscr{L}_\nu(\mathcal{H}) \qquad (8)$$

and a PI operator that specifically belongs to a subspace $\mathscr{L}_\nu(\mathcal{H})$ is hereafter referenced as a ν-type operator.

The PI orthonormal basis operators $\hat{F}_\nu^{(W_\nu, W'_\nu)}$ are mutually Hermitian conjugate: $\hat{F}_\nu^{(W_\nu, W'_\nu)\dagger} = \hat{F}_\nu^{(W'_\nu, W_\nu)}$ [46]. They fulfill the *multiplication rule* [47]

$$\hat{F}_\nu^{(W_\nu, W'_\nu)} \hat{F}_{\nu'}^{(\tilde{W}_{\nu'}, \tilde{W}'_{\nu'})} = \frac{1}{\sqrt{f^\nu}} \delta_{\nu,\nu'} \delta_{W'_\nu, \tilde{W}_{\nu'}} \hat{F}_\nu^{(W_\nu, \tilde{W}'_{\nu'})}. \qquad (9)$$

As a result the components of a PI operator Hermitian conjugate are given by $(\hat{A}^\dagger_{\rm PI})_{\nu,W_\nu,W'_\nu} = A^*_{\nu,W'_\nu,W_\nu}$ and those of a PI operator product read

$$(\hat{A}_{\rm PI}\hat{B}_{\rm PI})_{\nu,W_\nu,W'_\nu} = \frac{1}{\sqrt{f^\nu}} \sum_{\tilde{W}_\nu \in \mathcal{W}_\nu} A_{\nu,W_\nu,\tilde{W}_\nu} B_{\nu,\tilde{W}_\nu,W'_\nu}. \tag{10}$$

Hence, not only the operator subspaces $\mathscr{L}_\nu(\mathcal{H})$ are closed under multiplication of operators and Hermitian conjugation [48], but also left- or right-multiplying a PI operator with a ν-type operator again yields a ν-type operator. More generally, the product of any number of PI operators is of ν-type as soon as so is one of the operator. Finally, we have the closure relation

$$\sum_{\nu\vdash(N,d)} \sum_{W_\nu\in\mathcal{W}_\nu} \sqrt{f^\nu}\, \overline{|\nu,W_\nu\rangle\langle\nu,W_\nu|} = \mathbb{1}. \tag{11}$$

IV. MASTER EQUATION AND GENERAL IDENTITY

We now consider the system initially in a PI state $\hat{\rho}_{\rm PI}(0)$ with a time evolution governed by a PI Lindbladian of the form $\mathcal{L} = \mathcal{V}_{\hat{H}_c} + \mathcal{D}^{(\rm loc)}_{\hat{\ell}} + \mathcal{D}^{(\rm col)}_{\hat{L}}$, with $\hat{H}_c = \sum_n \hat{H}^{(n)}$, where $\hat{H}$ is a local (single particle) Hamiltonian and $\hat{\ell}$ and $\hat{L}$ are single-particle Lindblad operators (more general PI Lindbladians with p-particle terms in either coherent or dissipative parts are discussed in the Appendices). In this case, the system state is constrained within the commutant and the PI operators $\hat{F}^{(W_\nu,W'_\nu)}_\nu$ provide us with a natural orthonormal operator basis onto which the master equation can be projected: $\forall\lambda\vdash(N,d), W_\lambda, W'_\lambda \in \mathcal{W}_\lambda$,

$$\dot{\rho}_{\lambda,W_\lambda,W'_\lambda} = \sum_{\nu\vdash(N,d)} \sum_{W_\nu,W'_\nu\in\mathcal{W}_\nu} \mathcal{L}_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu} \rho_{\nu,W_\nu,W'_\nu}, \tag{12}$$

where, for every superoperator $\mathcal{O}$,

$$\mathcal{O}_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu} \equiv {\rm Tr}\left(\hat{F}^{(W_\lambda,W'_\lambda)\dagger}_\lambda \mathcal{O}[\hat{F}^{(W_\nu,W'_\nu)}_\nu]\right) \tag{13}$$

is the component of operator $\mathcal{O}[\hat{F}^{(W_\nu,W'_\nu)}_\nu]$ over the commutant basis operator $\hat{F}^{(W_\lambda,W'_\lambda)}_\lambda$.

Both operators $\mathcal{V}_{\hat{H}_c}[\hat{F}^{(W_\nu,W'_\nu)}_\nu]$ and $\mathcal{D}^{(\rm col)}_{\hat{L}}[\hat{F}^{(W_\nu,W'_\nu)}_\nu]$ are of ν-type because they are composed of products of PI operators with the ν-type operator $\hat{F}^{(W_\nu,W'_\nu)}_\nu$. Their expansion in the commutant operator basis follows straightforwardly provided this expansion is explicitly known for each of the involved PI operators. More generally all Linbladian terms can be expressed with the help of the superoperators $\mathcal{K}_{\hat{X},\hat{Y}}$ ($\hat{X},\hat{Y}$ are any two local operators) that act according to

$$\mathcal{K}_{\hat{X},\hat{Y}}[\hat{A}] = \sum_{n=1}^N \hat{X}^{(n)}\hat{A}\hat{Y}^{(n)\dagger}, \quad \forall\hat{A}\in\mathscr{L}(\mathcal{H}). \tag{14}$$

Indeed, $\mathcal{V}_{\hat{H}_c} = (i/\hbar)(\mathcal{K}_{\mathbb{1},\hat{H}} - \mathcal{K}_{\hat{H},\mathbb{1}})$,

$$\mathcal{D}^{(\rm loc)}_{\hat{\ell}} = \mathcal{K}_{\hat{\ell},\hat{\ell}} - \frac{1}{2}\mathcal{K}_{\hat{\ell}^\dagger\hat{\ell},\mathbb{1}} - \frac{1}{2}\mathcal{K}_{\mathbb{1},\hat{\ell}^\dagger\hat{\ell}}, \tag{15}$$

and

$$\mathcal{D}^{(\rm col)}_{\hat{L}}[\hat{\rho}] = \hat{L}_c\mathcal{K}_{\mathbb{1},\hat{L}}[\hat{\rho}] - \frac{1}{2}\hat{L}_c^\dagger\mathcal{K}_{\hat{L},\mathbb{1}}[\hat{\rho}] - \frac{1}{2}\mathcal{K}_{\mathbb{1},\hat{L}}[\hat{\rho}]\hat{L}_c, \tag{16}$$

where $\hat{L}_c$ can similarly be written as $\mathcal{K}_{\hat{L},\mathbb{1}}[\mathbb{1}]$. The superoperators $\mathcal{K}_{\hat{X},\hat{Y}}$ are PI, so that $\mathcal{K}_{\hat{X},\hat{Y}}[\hat{A}_{\rm PI}]$ is itself a PI operator for any PI operator $\hat{A}_{\rm PI}$. With respect to Hermitian conjugation, we have $\mathcal{K}_{\hat{X},\hat{Y}}[\hat{A}]^\dagger = \mathcal{K}_{\hat{Y},\hat{X}}[\hat{A}^\dagger]$ and $\mathcal{K}^\dagger_{\hat{X},\hat{Y}} = \mathcal{K}_{\hat{X}^\dagger,\hat{Y}^\dagger}$.

To get explicit expressions of the matrix elements $\mathcal{L}_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu}$, it is therefore enough to have the expansion in the commutant operator basis of the PI operators $\mathcal{K}_{\hat{X},\hat{Y}}[\hat{F}^{(W_\nu,W'_\nu)}_\nu]$, $\forall\hat{X},\hat{Y},\nu,W_\nu,W'_\nu$. Schur-Weyl duality formalism, trace invariance under cyclic permutations, and Clebsch-Gordan decomposition of tensorial products of unitary irreducible representations of the unitary group $U(d)$ allow one to obtain these expansions.

To this aim, we denote for all $\nu\in\mathscr{P}_d$ (the set of partitions of at most d parts) by ν^- [ν^+] any partition $\in\mathscr{P}_d$ obtained by the removal [addition] of an inner [outer] corner of ν (see Appendix A). The actions of removing [adding] an inner [outer] corner of a partition ν can be combined, so that ν^{-+} denotes any partition $\in\mathscr{P}_d$ obtained first by the removal of an inner corner of ν, then by the addition of an outer corner of the resulting partition at first step.

For every $\nu_L,\nu,\nu_R\in\mathscr{P}_d$, $W_\mu\in\mathcal{W}_\mu$ ($\mu=\nu_L,\nu,\nu_R$), we also introduce the 3ν symbol $\left(\begin{smallmatrix}\nu_L & \nu & \nu_R\\ W_{\nu_L} & W_\nu & W_{\nu_R}\end{smallmatrix}\right)$ as being the square $d\times d$ matrix with entries

$$\begin{pmatrix}\nu_L & \nu & \nu_R\\ W_{\nu_L} & W_\nu & W_{\nu_R}\end{pmatrix}_{i,j} = \langle W_\nu,i|W_{\nu_L}\rangle\langle W_\nu,j|W_{\nu_R}\rangle, \quad \forall i,j=0,\dots,d-1, \tag{17}$$

where $\langle W_\nu,i|W_{\nu_L}\rangle$ and $\langle W_\nu,j|W_{\nu_R}\rangle$ denote Clebsch-Gordan coefficients (CGC's) of the tensorial product $\mathcal{U}^\nu(d)\otimes\mathcal{U}^{(1)}(d)$ for the Gel'fand-Tsetlin bases (see Appendix C). For all $\mu,\nu\in\mathscr{P}_d$, $W_\mu\in\mathcal{W}_\mu$, $W_\nu\in\mathcal{W}_\nu$, $k=0,\dots,d-1$, a CGC $\langle W_\mu,k|W_\nu\rangle$ is zero iff the following two conditions are not simultaneously satisfied (CGC selection rules): $\mu\in\{\nu^-\}$ and $W_\mu\in\mathcal{W}^{(-k)}_\mu(W_\nu)$, where $\mathcal{W}^{(\pm k)}_\mu(W_\nu)$ denotes the set of all SWT's W_μ of shape μ, same content as W_ν $\pm$ one box k, and same Gel'fand-Tsetlin's pattern as that of W_ν $\pm$ one triangular shift pattern (see Appendix C). The set $\mathcal{W}^{(\pm k)}_\mu(W_\nu)$ is a subset of the set $\tilde{\mathcal{W}}^{(\pm k)}_\mu(W_\nu)$ of *all* SWT's of shape μ and same content as W_ν $\pm$ one box k. Its cardinality is at most $(d-1)!/k!$ (in particular, 1 if $d=2$ or $k=d-1$).

It follows from the CGC selection rules that the 3ν-symbol matrix $\left(\begin{smallmatrix}\nu_L & \nu & \nu_R\\ W_{\nu_L} & W_\nu & W_{\nu_R}\end{smallmatrix}\right)$ is necessarily zero if the condition $\nu\in\{\nu_L^-\}\cap\{\nu_R^-\}$ (*partition triangle selection* *rule*) is not satisfied. This condition can only be met if $\nu_L \in \{\nu_R^{-+}\}$ or equivalently $\nu_R \in \{\nu_L^{-+}\}$. We define the *partition triangular delta* $\{\nu_L, \nu, \nu_R\}$ to be 1 if the partition triangle selection rule is satisfied and 0 otherwise. If $\{\nu_L, \nu, \nu_R\} = 1$, an individual element j, l of the 3ν-symbol matrix is zero iff $W_\nu \notin \mathcal{W}_\nu^{(-i)}(W_{\nu_L}) \cap \mathcal{W}_\nu^{(-j)}(W_{\nu_R})$. The CGC's are real and so are the 3ν-symbol matrices. We thus have

$$\begin{pmatrix} \nu_L & \nu & \nu_R \\ W_{\nu_L} & W_\nu & W_{\nu_R} \end{pmatrix} = \begin{pmatrix} \nu_R & \nu & \nu_L \\ W_{\nu_R} & W_\nu & W_{\nu_L} \end{pmatrix}^T. \quad (18)$$

The 3ν-symbol matrices obey the orthogonality relation (see Appendix C)

$$\sum_{W_\nu \in \mathcal{W}_\nu} \mathrm{Tr}\left[\begin{pmatrix} \nu_L & \nu & \nu_R \\ W_{\nu_L} & W_\nu & W_{\nu_R} \end{pmatrix}\right] = \delta_{\nu_L,\nu_R} \delta_{W_{\nu_L}, W_{\nu_R}} \{\nu_L, \nu, \nu_R\} \quad (19)$$

and they represent in the computational basis the single qudit operators

$$\hat{g}_{\nu,W_\nu}^{(\nu_L,W_{\nu_L};\nu_R,W_{\nu_R})} = |\phi_{\nu,W_\nu}^{(\nu_L,W_{\nu_L})}\rangle\langle\phi_{\nu,W_\nu}^{(\nu_R,W_{\nu_R})}|, \quad (20)$$

where we defined $\forall \mu, \nu \in \mathscr{P}_d$, $W_\mu \in \mathcal{W}_\mu$, and $W_\nu \in \mathcal{W}_\nu$, the *unnormalized* single-qudit states $|\phi_{\mu,W_\mu}^{(\nu,W_\nu)}\rangle = \sum_{i=0}^{d-1} \langle W_\mu, i|W_\nu\rangle |i\rangle$.

If $W_\mu \notin \mathcal{W}_\mu$ for $\mu = \nu_L$, ν_R, and/or ν, the 3ν-symbol matrix $\begin{pmatrix} \nu_L & \nu & \nu_R \\ W_{\nu_L} & W_\nu & W_{\nu_R} \end{pmatrix}$ is not defined. However, it may be convenient to adopt the convention that it nevertheless exists and just identifies to the null matrix.

With this stated, we obtain the *general Identity* (proof in Appendix D)

$$\mathcal{K}_{\hat{X},\hat{Y}}[\hat{F}_\nu^{(W_\nu,W'_\nu)}] = \sum_{\lambda \in \{\nu^{-+}\}} \sum_{W_\lambda, W'_\lambda \in \mathcal{W}_\lambda} C_{\hat{X},\hat{Y}}^{(\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu)} \hat{F}_\lambda^{(W_\lambda,W'_\lambda)}, \quad (21)$$

where

$$C_{\hat{X},\hat{Y}}^{(\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu)} = \sum_{\mu \in \{\nu^-\} \cap \{\lambda^-\}} \sqrt{r_\nu^\mu r_\lambda^\mu} \mathrm{Tr}[\hat{g}_\mu^{(\lambda,W_\lambda;\nu,W_\nu)\dagger} \hat{X}] \mathrm{Tr}[\hat{g}_\mu^{(\lambda,W'_\lambda;\nu,W'_\nu)\dagger} \hat{Y}]^*, \quad (22)$$

with $r_\nu^\mu \equiv N f^\mu / f^\nu$, $\forall \nu \vdash N, \mu \in \{\nu^-\}$, and $\hat{g}_\mu^{(\lambda,W_\lambda;\nu,W_\nu)}$ the single qudit operator

$$\hat{g}_\mu^{(\lambda,W_\lambda;\nu,W_\nu)} = \sum_{W_\mu \in \mathcal{W}_\mu} \hat{g}_{\mu,W_\mu}^{(\lambda,W_\lambda;\nu,W_\nu)}. \quad (23)$$

This operator vanishes if $\{\lambda, \mu, \nu\} = 0$ and satisfies $\hat{g}_\mu^{(\lambda,W_\lambda;\nu,W_\nu)\dagger} = \hat{g}_\mu^{(\nu,W_\nu;\lambda,W_\lambda)}$ and

$$\mathrm{Tr}[\hat{g}_\mu^{(\lambda,W_\lambda;\nu,W_\nu)}] = \delta_{\lambda,\nu} \delta_{W_\lambda,W_\nu} \{\lambda, \mu, \nu\}. \quad (24)$$

As a result, $\hat{\rho}_\mu^{(\nu,W_\nu)} \equiv \hat{g}_\mu^{(\nu,W_\nu;\nu,W_\nu)}$ is a trace 1 sum of projection operators, hence positive semidefinite, and represents a single qudit mixed state for every $\mu \in \{\nu^-\}$.

Equation (21) generalizes to arbitrary multiqudit systems and local operators Identity 1 of Ref. [27] that was developed in the specific context of multiqubit systems. We explicitly show in Appendix E how the latter is recovered from Eq. (21) in the case $d = 2$.

If $\hat{Y} = \hat{\mathbb{1}}$, we get thanks to Eq. (24)

$$C_{\hat{X},\hat{\mathbb{1}}}^{(\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu)} = C_{\hat{X}}^{(\nu,W_\lambda,W_\nu)} \delta_{\lambda,\nu} \delta_{W'_\lambda,W'_\nu}, \quad (25)$$

with

$$C_{\hat{X}}^{(\nu,W_\nu,W'_\nu)} = \sum_{\nu^-} r_\nu^{\nu^-} \mathrm{Tr}[\hat{g}_{\nu^-}^{(\nu,W_\nu;\nu,W'_\nu)\dagger} \hat{X}], \quad (26)$$

so that [49]

$$\mathcal{K}_{\hat{X},\hat{\mathbb{1}}}[\hat{F}_\nu^{(W_\nu,W'_\nu)}] = \sum_{\tilde{W}_\nu} C_{\hat{X}}^{(\nu,\tilde{W}_\nu,W_\nu)} \hat{F}_\nu^{(\tilde{W}_\nu,W'_\nu)}. \quad (27)$$

A similar expression is straightforwardly obtained for $\mathcal{K}_{\hat{\mathbb{1}},\hat{Y}}[\hat{F}_\nu^{(W_\nu,W'_\nu)}] = \mathcal{K}_{\hat{Y},\hat{\mathbb{1}}}[\hat{F}_\nu^{(W'_\nu,W_\nu)}]^\dagger$. Finally, thanks to the closure relation (11) any collective operator $\hat{X}_c = \mathcal{K}_{\hat{X},\hat{\mathbb{1}}}[\hat{\mathbb{1}}]$ can be written

$$\hat{X}_c = \sum_{\nu \vdash (N,d)} \sum_{W_\nu, W'_\nu \in \mathcal{W}_\nu} \sqrt{f^\nu} C_{\hat{X}}^{(\nu,W_\nu,W'_\nu)} \hat{F}_\nu^{(W_\nu,W'_\nu)}. \quad (28)$$

The hereabove formalism naturally generalizes when considering more general p-particle Lindbladian terms and the matrix elements $\mathcal{L}_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu}$ immediately follow from this formalism (see Appendices F and G).

V. CONCLUSION

In this paper we established the general theoretical framework that allows for an exact description of the open system dynamics of permutationally invariant states in arbitrary N-qudit systems ($d \geq 2$) when constrained within the commutant $\mathscr{L}_{S_N}(\mathcal{H})$ (the subspace of permutationally invariant operators in the global Liouville space of operators acting in the system Hilbert space). Thanks to Schur-Weyl duality powerful results, we identified a natural orthonormal basis of operators in the commutant onto which the master equation can be projected and we provided the exact expansion coefficients in the most general case (arbitrary dynamics, N and d). The formalism does not require to compute the Schur transform and allows for keeping completely restricted within the commutant subspace, whose dimension only scales polynomially with the number N of qudits instead of exponentially as it is the case for the whole Liouville space of operators. We introduced the concept of 3ν-symbol matrix that proves to be particularly useful in this context. We finally showed how our theoretical framework particularizes for qubit systems ($d = 2$) and how previously known results are recovered for this specific case. Being exact, our formalism should be helpful for studying the dynamics of open many-body quantum systems with large number of constituents.

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Appendix A: Symmetric, general linear, and unitary group irreducible representations

For self-consistency, we briefly recall in this Appendix the basics about the irreducible representations (irreps) of the symmetric group S_N ($N > 0$), the general linear group $GL(d) \equiv GL(d, \mathbb{C})$ ($d > 0$) of invertible $d \times d$ complex matrices, and its subgroup $U(d)$ of $d \times d$ unitary matrices (see, e.g., Refs. [37, 44, 52]). In this context, integer partitions are central.

A partition ν of an integer $N \geq 0$ is a sequence of integers $(\nu_1, \ldots, \nu_l)$, with $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_l > 0$ and $\sum_{i=1}^{l} \nu_i = N$. We write in this case $\nu \vdash N$. The numbers ν_i are called the parts of ν and the number l of parts is the length of ν, denoted by $l(\nu)$. The weight of ν is the sum of its parts, N, also denoted by $|\nu|$. A partition of an integer N of at most d parts ($d > 0$) is a partition ν with $l(\nu) \leq d$. We write in this case $\nu \vdash (N, d)$. We denote by $\mathscr{P}(N)$, $\mathscr{P}_d(N)$, and $\mathscr{P}_d$ the sets of all partitions of N, of N of at most d parts, and of at most d parts, respectively. The sets $\mathscr{P}(N)$ and $\mathscr{P}_d(N)$ are finite, while $\mathscr{P}_d$ is countably infinite. The cardinalities of $\mathscr{P}(N)$ and $\mathscr{P}_d(N)$ are the so-called numbers of partitions $p(N)$ and $p(N, d)$, respectively. Obviously, $\mathscr{P}_d(N) \subseteq \mathscr{P}(N)$, with equality only for $N \leq d$. The case $N = 0$ is particular and only counts the so-called *empty* partition (.) of length 0.

The diagram or shape of a partition $\nu \equiv (\nu_1, \ldots, \nu_l) \vdash N$ is an array of N boxes arranged on l left-justified rows, with row i ($1 \leq i \leq l$) containing ν_i boxes. The shape of a partition ν is usually denoted by the same symbol ν. An inner corner of a shape ν is a box $\in \nu$ whose removal leaves us with a valid partition shape. An outer corner of ν is a box $\notin \nu$ whose addition produces a valid partition shape. Any partition obtained by the removal [addition] of an inner [outer] corner of ν is denoted by ν^- [ν^+] and if the row τ of the removed inner [added outer] corner is specified, the partition is specifically denoted by ν^{-_τ} [ν^{+_τ}]. Conversely, the row at which the removal [addition] occurs for a partition ν^- [ν^+] is denoted by $\tau_{\nu^-/\nu}$ [$\tau_{\nu^+/\nu}$]. We have $\tau_{\nu^\pm/\nu} = \tau_{\nu/\nu^\pm}$ and $\mu = \nu^{\pm_\tau} \Leftrightarrow \nu = \mu^{\mp_\tau}$, so that $\mu \in \{\nu^\pm\} \Leftrightarrow \nu \in \{\mu^\mp\}$.

To each partition $\nu \vdash N$ ($N > 0$), it is possible to associate a unitary irreducible representation $\mathcal{S}^\nu$ of the symmetric group S_N and vice-versa (the number of inequivalent irreducible representations of S_N is the number of partitions $p(N)$). Within each of these representations, there exists a unique orthonormal basis (up to global phases), the so-called Gel'fand-Tsetlin (GT) basis of $\mathcal{S}^\nu$ with respect to the subgroup chain $S_{N-1}, \ldots, S_1$ (also called the Young-Yamanouchi (YY) basis), where the basis vectors each belong to a unitary irrep of each of the subgroup of the chain [37]. These vectors are labeled accordingly by a sequence of partitions $(\nu, \nu(N-1), \ldots, \nu(1))$, with $\nu(i) \vdash i$ ($1 \leq i < N$). A much more compact notation can be used instead, consisting in a so-called Standard Young tableau (SYT) T_ν. Such a tableau is a diagram ν where the N boxes are filled with the numbers 1 to N, each number in exactly one box, strictly increasing both along the rows (from left ro right) and along the columns (from top to bottom). The partitions $\nu(i)$ of the YY-basis vectors are obtained from the shapes of the SYT T_ν restricted to the only boxes 1 to i. The number f^ν of distinct SYT's T_ν of shape ν yields the dimension of the irreducible representation $\mathcal{S}^\nu$ and is given by the elegant hook length formula: $f^\nu = N!/\prod_{(i,j)\in\nu} h_{(i,j)}$, where the product runs over the hook length $h_{(i,j)}$ of each box (i,j) of the diagram ν [51]. The historical Frobenius-Young determinantal formula [53, 54] can also be used instead: $f^\nu = N!\prod_{i<j}(l_i - l_j)/\prod_i l_i!$ with $l_i = \nu_i + l(\nu) - i$. In particular, for $\nu = (N)$, $f^\nu = 1$, and for $\nu \equiv (\nu_1, \nu_2) \vdash (N, 2)$, $f^\nu = (\nu_1 - \nu_2 + 1)\binom{N}{\nu_2}/(\nu_1 + 1)$. The set of distinct SYT's T_ν of shape ν is denoted by $\mathcal{T}_\nu$ and without further specifications the condensed notation $\sum_{T_\nu}$ in sum expressions merely means $\sum_{T_\nu \in \mathcal{T}_\nu}$.

To each partition ν of at most d parts ($d > 0$ and including the empty partition), it is possible to associate an irreducible representation $\mathcal{U}^\nu(d)$ on an inner product space of the general linear group $GL(d)$ and to have the restriction of this representation onto the subgroup $U(d)$ unitary [55]. This restriction of $\mathcal{U}^\nu(d)$ onto the subgroup $U(d)$ is also irreducible and can be denoted similarly. Within each of these representations, there exists a unique orthonormal basis (up to global phases), the so-called Gel'fand-Tsetlin basis with respect to the subgroup chain $U(d-1), \ldots, U(1)$, where the basis vectors each belong to a unitary irrep of each of the subgroup of the chain (at the same time the basis vectors belong to an irrep of each of the $GL(d)$ subgroups $GL(d-1), \ldots, GL(1)$). These $GL(k)$ and $U(k)$ subgroup irreps ($1 \leq k < d$) can be similarly labeled by a partition $\nu(k)$ of at most k parts and the basis vectors are labeled accordingly by a sequence of partitions $(\nu, \nu(d-1), \ldots, \nu(1))$. A more compact notation can be used instead, consisting in a so-called semistandard Young tableau (SSYT), also called standard Weyl tableau (SWT) W_ν. Such a tableau is a diagram ν where the $|\nu|$ boxes are filled with the numbers 0 to $d-1$, each number possibly in more than one box, weakly increasing along the rows (from left ro right) and strictly along the columns (from top to bottom). Again, the partitions $\nu(k)$ of the GT-basis vectors are obtained from the shapes of the SWT W_ν restricted to the only boxes 0 to $k-1$. The number $f^\nu(d)$ of distinct SWT's W_ν of shape ν yields the dimension of the irreducible representation $\mathcal{U}^\nu(d)$, $f^\nu(d) = \prod_{1\leq i<j\leq d}(\nu_i-\nu_j+j-i)/(j-i)$, with $\nu_i \equiv 0, \forall i > l(\nu_i)$ (Weyl dimension formula). In particular, $f^{(\cdot)}(d) = 1$ ($\mathcal{U}^{(\cdot)}(d)$ is the trivial representation), $f^{(1)}(d) = d$, and $f^{(N)}(d) = \binom{N+d-1}{N}, \forall N > 0$. The number $f^\nu(d)$ can also be expressed in the form $f^\nu(d) = s_\nu(1,\ldots,1)$, where $(1,\ldots,1)$ is a d-uple and $s_\nu(x_1,\ldots,x_d)$ is the Schur's polynomial in the d variables $x_1,\ldots,x_d$ associated to partition ν (an homogenous symmetric polynomial of degree $|\nu|$) [37]. The set of distinct SWT's W_ν of shape ν is denoted by $\mathcal{W}_\nu$ and without further specifications the condensed notation $\sum_{W_\nu}$ in sum expressions merely means $\sum_{W_\nu\in\mathcal{W}_\nu}$.

Appendix B: Dimension of the commutant $\mathscr{L}_{S_N}(\mathcal{H})$

The commutant $\mathscr{L}_{S_N}(\mathcal{H})$ coincides with the symmetric subspace of the Liouville space $\mathscr{L}(\mathcal{H}) \cong (\mathscr{L}(\mathcal{H}_d))^{\otimes N}$ and its dimension thus reads $f^{(N)}(d^2) = \binom{N+d^2-1}{N}$. On the other hand, as a result of the Schur-Weyl duality, this dimension also identifies to $\sum_{\nu\vdash(N,d)} f^\nu(d)^2$, with $f^\nu(d)$ the dimension of the $\mathcal{U}^\nu(d)$ irrep of the general linear group $GL(d)$ (and of the unitary group $U(d)$). Both expressions must of course coincide and this can be easily seen as follows.

We first observe that $\sum_{\nu\vdash(N,d)} f^\nu(d)^2$ can also be written (see Appendix A) $\sum_{\nu\vdash(N,d)} s_\nu(1,\ldots,1)^2$, where $(1,\ldots,1)$ is a d-uple and $s_\nu(x_1,\ldots,x_d)$ is the Schur's polynomial in the d variables $x_1,\ldots,x_d$ associated to partition ν (this is an homogeneous symmetric polynomial of degree N in the variables $x_1,\ldots,x_d$ - see, e.g., Ref. [37]). For all $\mathbf{x} \equiv (x_1,\ldots,x_d)$ and $\mathbf{y} \equiv (y_1,\ldots,y_d)$, Cauchy's identity states that

$$\sum_{N=0}^{\infty}\sum_{\nu\vdash(N,d)} s_\nu(\mathbf{x})s_\nu(\mathbf{y}) = \prod_{i,j=1}^{d}(1-x_iy_j)^{-1}, \qquad \text{(B1)}$$

which for $\mathbf{x} = \mathbf{y} = (t,\ldots,t)$ particularizes to

$$\sum_{N=0}^{\infty}\sum_{\nu\vdash(N,d)} s_\nu(t,\ldots,t)^2 = (1-t^2)^{-d^2}. \qquad \text{(B2)}$$

On the one hand, $s_\nu(t,\ldots,t)$ is a multiple of t^N for all $\nu \vdash (N,d)$, so that

$$\sum_{\nu\vdash(N,d)} s_\nu(t,\ldots,t)^2 = \alpha_{N,d}t^{2N}, \quad \forall t, \qquad \text{(B3)}$$

with $\alpha_{N,d}$ a real constant that depends on N and d. On the other hand, $(1-t^2)^{-d^2}$ admits for $-1 < t < 1$ the series expansion $\sum_{N=0}^{\infty}\binom{N+d^2-1}{N}t^{2N}$. As a result, Eq. (B2) can be written for $-1 < t < 1$

$$\sum_{N=0}^{\infty}\alpha_{N,d}t^{2N} = \sum_{N=0}^{\infty}\binom{N+d^2-1}{N}t^{2N} \qquad \text{(B4)}$$

and $\alpha_{N,d}$ must identify to $\binom{N+d^2-1}{N}, \forall N, d$. It then follows from Eq. (B3) that

$$\sum_{\nu\vdash(N,d)} s_\nu(1,\ldots,1)^2 = \binom{N+d^2-1}{N}. \qquad \text{(B5)}$$

Appendix C: Clebsch-Gordan coefficients of tensor products of unitary group irreps

The tensor product of two (unitary) irreps $\mathcal{U}^\mu(d)$ and $\mathcal{U}^\nu(d)$ of the unitary group $U(d)$ ($\mu,\nu \in \mathscr{P}_d$) decomposes into a direct sum of irreducible components according to [52]

$$\mathcal{U}^\mu(d)\otimes\mathcal{U}^\nu(d) = \bigoplus_{\lambda\in\mathscr{P}_d}\bigoplus_{r=1}^{c_{\mu\nu}^\lambda}\mathcal{U}^\lambda(d)_r, \qquad \text{(C1)}$$

with $c_{\mu\nu}^\lambda$ the so-called Littlewood-Richardson coefficients and $\mathcal{U}^\lambda(d)_r$ ($\lambda \in \mathscr{P}_d, r = 1,\ldots,c_{\mu\nu}^\lambda$) equivalent $\mathcal{U}^\lambda(d)$-irreps of $U(d)$. The index r is omitted if it only takes value 1. The Clebsch-Gordan coefficients (CGC's) of the tensor product $\mathcal{U}^\mu(d)\otimes\mathcal{U}^\nu(d)$ for the Gel'fand-Tsetlin (GT) basis are the expansion coefficients $\langle W_\mu, W_\nu|W_\lambda\rangle_r$ of the GT-basis vectors $|W_\lambda\rangle_r$ of $\mathcal{U}^\lambda(d)_r$ in the basis formed by the tensor products of the GT-basis vectors $|W_\mu\rangle$ of $\mathcal{U}^\mu(d)$ with the GT-basis vectors $|W_\nu\rangle$ of $\mathcal{U}^\nu(d)$:

$$|W_\lambda\rangle_r = \sum_{W_\mu,W_\nu}\langle W_\mu,W_\nu|W_\lambda\rangle_r|W_\mu\rangle\otimes|W_\nu\rangle,$$
$$\forall\lambda\in\mathscr{P}_d, W_\lambda\in\mathcal{W}_\lambda, r. \qquad \text{(C2)}$$

The decomposition (C1) is not unique as soon as any of the Littlewood-Richardson coefficients $c_{\mu\nu}^\lambda$ exceeds 1, in which case the CGC's are neither univocally defined.

For $\nu = (1)$, Eq. (C1) specifically reads

$$\mathcal{U}^\mu(d)\otimes\mathcal{U}^{(1)}(d) = \bigoplus_{\lambda\in\{\mu^+\}}\mathcal{U}^\lambda(d) \qquad \text{(C3)}$$

and the CGC's are univocally determined. The d-dimensional representation $\mathcal{U}^{(1)}(d)$ can always be chosen to have each unitary matrix $U \in U(d)$ be represented by a unitary operator having its representation matrix in a given orthonormal basis $|0\rangle,\ldots,|d-1\rangle$ of $\mathcal{U}^{(1)}(d)$ given by U (standard representation of $U(d)$ on the single qudit state space $\mathcal{H}_d \cong \mathcal{U}^{(1)}(d)$). If we view the $U(d)$ subgroups $U(k)$ ($1 \leq k < d$) as the groups of $d\times d$ unitary matrices $U_k \oplus (1)^{\oplus(d-k)}$ (U_k $k\times k$ unitary subblocks) that leave fixed the basis vectors $|k\rangle,\ldots,|d-1\rangle$ in the representation $\mathcal{U}^{(1)}(d)$, the GT-basis vectors $|W_{(1)}\rangle$

of $\mathcal{U}^{(1)}(d)$ merely identify to the orthonormal basis vectors $|j\rangle$ $(j = 0, \ldots, d-1)$ and the CGC's $\langle W_\mu, W_{(1)}|W_\lambda\rangle$ can be accordingly denoted by $\langle W_\mu, j|W_\lambda\rangle$:

$$|W_\lambda\rangle = \sum_{j=0}^{d-1} \sum_{W_\mu \in \mathcal{W}_\mu} \langle W_\mu, j|W_\lambda\rangle |W_\mu\rangle \otimes |j\rangle, \quad \forall\lambda \in \{\mu^+\}, W_\lambda \in \mathcal{W}_\lambda. \quad \text{(C4)}$$

The CGC's $\langle W_\mu, j|W_\lambda\rangle$ are expressed in terms of the Gel'fand-Tsetlin (GT) patterns of the SWT's W_μ [44, 56]. For all $\nu \in \mathscr{P}_d$, the GT pattern $G(W_\nu)$ of a SWT W_ν is the d-row tableau listing in a one row - one partition format all partitions $\nu(k)$ $(k = 1, \ldots, d)$ of shapes of W_ν where only boxes $0, \ldots, k-1$ are kept. The partitions are listed in the reversed order $k = d, \ldots, 1$. Each $\nu(k)$ is a partition of at most k parts, denoted standardly by the numbers $m_{i,k}$, $i = 1, \ldots, k$, set to 0 for all i that exceeds the length of partition $\nu(k)$. The form of a GT pattern only depends on d and is an inverse triangular pattern of d numbers on the first line, $d-1$ on the second, ..., and finally 1 on the last dth line, whatever the partition $\nu \in \mathscr{P}_d$ and the SWT W_ν:

$$G(W_\nu) \equiv \begin{pmatrix} m_{1,d} & & m_{2,d} & & \cdots & & m_{d-1,d} & & m_{d,d} \\ & m_{1,d-1} & & \cdots & & \cdots & & m_{d-1,d-1} & \\ & & & & \cdots & & & & \\ & & & m_{1,2} & & m_{2,2} & & & \\ & & & & m_{1,1} & & & & \end{pmatrix}. \quad \text{(C5)}$$

The numbers $m_{i,d}$ $(i = 1, \ldots, d)$ on the top line merely identify to the parts ν_i of the partition ν of the SWT W_ν, possibly completed with zeroes if the partition length is lower than d. The GT pattern $G(W_\nu)$ is either denoted accordingly by $\binom{\nu}{m}$, or also merely by (m). For all $k = 1, \ldots, d$, the number $\alpha_k = |m_k| - |m_{k-1}|$, with $|m_k| \equiv \sum_{i=1}^{k} m_{i,k}$ and $m_0 \equiv 0$, yields the number n_{k-1} of boxes $(k-1)$ in the SWT W_ν. In particular $n_0 = m_{1,1}$. The numbers $m_{i,k}$ satisfy the betweenness condition: $m_{i,k-1} \in [m_{i+1,k}, m_{i,k}]$, $\forall k = 2, \ldots, d; i = 1, \ldots, k-1$. Any triangular pattern satisfying the betweenness condition is by definition a GT pattern. For all partition ν, there is a one to one correspondence between the set of all GT patterns $\binom{\nu}{m}$ and the set of all SWT's W_ν, so that the knowledge of all numbers $m_{i,k}$ uniquely identify a SWT. In particular, for $d = 2$, the GT pattern for any SWT $W_{\nu\equiv(\nu_1,\nu_2)}$ merely reads

$$G(W_\nu) = \begin{pmatrix} \nu_1 & & \nu_2 \\ & n_0 & \end{pmatrix}. \quad \text{(C6)}$$

For all $i, \tau = 1, \ldots, d$, a triangular shift pattern (i, τ), $\Delta_i(\tau, \tau_{d-1}, \ldots, \tau_i)$, is a pattern of d rows containing only 0's and 1's according to

$$\Delta(\tau, \tau_{d-1}, \ldots, \tau_i) = \begin{pmatrix} e_\tau \\ e_{\tau_{d-1}} \\ \vdots \\ e_{\tau_i} \\ (0)_{i-1} \end{pmatrix}, \quad \text{(C7)}$$

where, $\forall k = i, \ldots, d-1$, $1 \le \tau_k \le k$, e_{τ_k} is a unit row vector of length k with 1 at position τ_k and 0 elsewhere, and $(0)_{i-1}$ denotes a triangular array of zeroes with $i-1$ rows. The set of all triangular shift patterns (i, τ) for all possible values of $\tau_{d-1}, \ldots, \tau_i$ is denoted by $\Delta_i(\tau)$. It is composed of $(d-1)!/(i-1)!$ elements [57]. For $d = 2$ or also $i = d$, this is therefore a singleton.

We define the difference of two GT patterns (m) and (m') by the triangular pattern of elements $m_{i,k} - m'_{i,k}$. If W_μ and W_ν are two SWT's such that

$$G(W_\nu) - G(W_\mu) \in \Delta_i(\tau), \quad \text{(C8)}$$

then $\nu = \mu^{+\tau}$ (equivalently $\mu = \nu^{-\tau}$) and both W_μ and W_ν have globally identical content up to one more box $(i-1)$ in W_ν. Indeed, setting $G(W_\mu) \equiv (m)$ and $G(W_\nu) \equiv (m')$, we then have $|m'_k| = |m_k| + 1, \forall k \ge i$ and $|m'_k| = |m_k|, \forall k < i$, so that $\alpha'_k = \alpha_k$, $\forall k \ne i$ and $\alpha'_i = \alpha_i + 1$. For all $\mu, \nu \in \mathscr{P}_d : \nu \in \{\mu^+\}$ $(\Leftrightarrow \mu \in \{\nu^-\})$, $j = 0, \ldots, d-1$, and SWT W_μ, we denote accordingly by $\mathcal{W}_\nu^{(+j)}(W_\mu)$ the set of all SWT's W_ν of shape ν that fullfill condition (C8) with $i = j+1$ and $\tau = \tau_{\nu/\mu}$. This is a subset of the set $\tilde{\mathcal{W}}_\nu^{(+j)}(W_\mu)$ of *all* SWT's of shape ν and same content as W_μ plus one box j. Similarly, for all SWT W_ν, we denote by $\mathcal{W}_\mu^{(-j)}(W_\nu)$ the set of all SWT's W_μ of shape μ that fullfill condition (C8) with $i = j+1$ and $\tau = \tau_{\mu/\nu}$. This is a subset of the set $\tilde{\mathcal{W}}_\mu^{(-j)}(W_\nu)$ of *all* SWT's of shape ν and same content as W_ν minus one box j. The subset is empty if W_ν does not contain any box j. We have

$$W_\nu \in \mathcal{W}_\nu^{(+j)}(W_\mu) \quad \Leftrightarrow \quad W_\mu \in \mathcal{W}_\mu^{(-j)}(W_\nu). \quad \text{(C9)}$$

The cardinality of the subsets $\mathcal{W}_\nu^{(+j)}(W_\mu)$ and $\mathcal{W}_\mu^{(-j)}(W_\nu)$ is identical and at most $\Delta_{j+1}(\tau_{\nu/\mu})$'s one, i.e., $(d-1)!/j!$ (1 for $d = 2$ or $j = d-1$).

The CGC's $\langle W_\mu, j|W_\lambda\rangle$ in Eq. (C4) are zero if $G(W_\lambda) - G(W_\mu) \notin \Delta_{j+1}(\tau_{\lambda/\mu})$, and otherwise

$$\langle W_\mu, j|W_\lambda\rangle = \left[\frac{\prod_{k=1}^{j}(p_{\tau_{j+1},j+1}-p_{k,j})}{\prod_{\substack{k=1\\k\neq\tau_{j+1}}}^{j+1}(p_{\tau_{j+1},j+1}-p_{k,j+1})}\right]^{1/2}\prod_{l=j+2}^{d}A_{\tau_{l-1},\tau_l}, \tag{C10}$$

where we set $G(W_\lambda) - G(W_\mu) \equiv \Delta(\tau_d, \tau_{d-1}, \ldots, \tau_{j+1})$ (with $\tau_d = \tau_{\lambda/\mu}$), $G(W_\mu) \equiv (m)$, $p_{i,k} = m_{i,k} + k - i$, and where

$$A_{\tau_{l-1},\tau_l} = \mathrm{sgn}(\tau_{l-1}-\tau_l)\left[\prod_{\substack{k=1\\k\neq\tau_l}}^{l}\frac{p_{\tau_{l-1},l-1}-p_{k,l}+1}{p_{\tau_l,l}-p_{k,l}}\prod_{\substack{k=1\\k\neq\tau_{l-1}}}^{l-1}\frac{p_{\tau_l,l}-p_{k,l-1}}{p_{\tau_{l-1},l-1}-p_{k,l-1}+1}\right]^{1/2}, \tag{C11}$$

with sgn the sign function with $\mathrm{sgn}(0) = 1$. The CGC's are real and obey for all $\mu \in \{\lambda^-\} \cap \{\nu^-\}$ the orthogonality relation [44]

$$\sum_{j=0}^{d-1}\sum_{W_\mu\in\mathcal{W}_\mu}\langle W_\mu, j|W_\lambda\rangle\langle W_\mu, j|W_\nu\rangle = \delta_{\lambda,\nu}\delta_{W_\lambda,W_\nu}. \tag{C12}$$

Equation (C4) can be accordingly refined to

$$|W_\lambda\rangle = \sum_{j=0}^{d-1}\sum_{W_\mu\in\mathcal{W}_\mu^{(-j)}(W_\lambda)}\langle W_\mu, j|W_\lambda\rangle|W_\mu\rangle\otimes|j\rangle, \quad \forall\lambda\in\{\mu^+\}, W_\lambda\in\mathcal{W}_\lambda. \tag{C13}$$

Appendix D: Proof of general Identity (21)

For any local operators $\hat{X}$ and $\hat{Y}$, $\mathcal{K}_{\hat{X},\hat{Y}}[\hat{F}_\nu^{(W_\nu,W'_\nu)}]$ is a PI operator and can be expanded according to

$$\mathcal{K}_{\hat{X},\hat{Y}}[\hat{F}_\nu^{(W_\nu,W'_\nu)}] = \sum_{\lambda\vdash(N,d)}\sum_{W_\lambda,W'_\lambda\in\mathcal{W}_\lambda}\mathrm{Tr}(\hat{F}_\lambda^{(W_\lambda,W'_\lambda)\dagger}\mathcal{K}_{\hat{X},\hat{Y}}[\hat{F}_\nu^{(W_\nu,W'_\nu)}])\hat{F}_\lambda^{(W_\lambda,W'_\lambda)}. \tag{D1}$$

We first observe that for any PI operator $\hat{A}_{\mathrm{PI}}$ and any operator $\hat{B}$ we have $\mathrm{Tr}(\hat{A}_{\mathrm{PI}}^\dagger\mathcal{P}_\sigma[\hat{B}]) = \mathrm{Tr}(\hat{A}_{\mathrm{PI}}^\dagger\hat{B}), \forall\sigma\in S_N$ [50]. As a result and since $\mathcal{P}_\sigma[\hat{X}^{(n)}\hat{A}_{\mathrm{PI}}\hat{Y}^{(n)\dagger}] = \hat{X}^{(\sigma(n))}\hat{A}_{\mathrm{PI}}\hat{Y}^{(\sigma(n))\dagger}$, $\forall n,\sigma$, $\mathrm{Tr}(\hat{F}_\lambda^{(W_\lambda,W'_\lambda)\dagger}\hat{X}^{(n)}\hat{A}_{\mathrm{PI}}\hat{Y}^{(n)\dagger})$ is independent of n and we have

$$\begin{aligned}\mathrm{Tr}(\hat{F}_\lambda^{(W_\lambda,W'_\lambda)\dagger}\mathcal{K}_{\hat{X},\hat{Y}}[\hat{F}_\nu^{(W_\nu,W'_\nu)}]) &= N\mathrm{Tr}(\hat{F}_\lambda^{(W_\lambda,W'_\lambda)\dagger}\hat{X}^{(N)}\hat{F}_\nu^{(W_\nu,W'_\nu)}\hat{Y}^{(N)\dagger})\\ &= \frac{N}{\sqrt{f^\lambda f^\nu}}\sum_{T_\lambda,T_\nu}\langle\lambda,T_\lambda,W_\lambda|\hat{X}^{(N)}|\nu,T_\nu,W_\nu\rangle\langle\lambda,T_\lambda,W'_\lambda|\hat{Y}^{(N)}|\nu,T_\nu,W'_\nu\rangle^*.\end{aligned} \tag{D2}$$

The N-qudit system state space can be structured along $\mathcal{H} = (\otimes_i^{N-1}\mathcal{H}_i)\otimes\mathcal{H}_N$. This global structure can further be refined thanks to the specific properties of the Schur-Weyl duality basis states. According to Pieri's rule for the unitary group irreducible representations [Eq. (C3) in Appendix C] and considering the chain of irreps of $S_N, \ldots, S_1$ each Schur-Weyl basis state belongs to (here, S_k ($1 \leq k < N$) denotes the S_N subgroup that fixes each $j \in \{k+1, \ldots, N\}$ and only permutes $\{1, \ldots, k\}$), we get that, for all $\nu \vdash (N,d)$ and $T_\nu \in \mathcal{T}_\nu$, $\mathcal{H}_\nu(T_\nu) \simeq \mathcal{U}^\nu(d)$ is an irreducible component of $\mathcal{H}_{\nu(N-1)}(T_{\nu(N-1)})\otimes\mathcal{H}_N \simeq \mathcal{U}^{\nu(N-1)}(d)\otimes\mathcal{U}^{(1)}(d)$, with $\nu(N-1)$ the shape of T_ν without box N and $T_{\nu(N-1)}$ the SYT T_ν without box N. This implies

$$|\nu,T_\nu,W_\nu\rangle = \sum_{j=0}^{d-1}\sum_{\substack{W_{\nu(N-1)}\in\\ \mathcal{W}_{\nu(N-1)}^{(-j)}(W_\nu)}}\langle W_{\nu(N-1)}, j|W_\nu\rangle|\nu(N-1), T_{\nu(N-1)}, W_{\nu(N-1)}\rangle\otimes|j\rangle_N, \tag{D3}$$

where the coefficients $\langle W_{\nu(N-1)}, j|W_\nu\rangle$ are the (real) Clebsch-Gordan coefficients (CGC's) of the tensor product $\mathcal{U}^{\nu(N-1)}(d)\otimes\mathcal{U}^{(1)}(d)$ for the Gel'fand-Tsetlin bases (see Appendix C). These coefficients are zero iff $W_{\nu(N-1)}\notin\mathcal{W}^{(-j)}_{\nu(N-1)}(W_\nu)$, so that the sum in Eq. (D3) can be formally extended to the whole set $\mathcal{W}_{\nu(N-1)}$. Proceeding similarly for expanding the state $|\lambda, T_\lambda, W_\lambda\rangle$, we get

$$\begin{aligned}\langle\lambda, T_\lambda, W_\lambda|\hat{X}^{(N)}|\nu, T_\nu, W_\nu\rangle &= \sum_{i,j=0}^{d-1}\sum_{\substack{W_{\nu(N-1)}\\ \in\mathcal{W}_{\nu(N-1)}}}\begin{pmatrix}\lambda & \nu(N-1) & \nu\\ W_\lambda & W_{\nu(N-1)} & W_\nu\end{pmatrix}_{i,j}\langle i|\hat{X}|j\rangle\delta_{\lambda(N-1),\nu(N-1)}\delta_{T_{\lambda(N-1)},T_{\nu(N-1)}}\\ &= \mathrm{Tr}[\hat{g}^{(\lambda,W_\lambda;\nu,W_\nu)\dagger}_{\nu(N-1)}\hat{X}]\delta_{\lambda(N-1),\nu(N-1)}\delta_{T_{\lambda(N-1)},T_{\nu(N-1)}}.\end{aligned}\tag{D4}$$

Inserting this result into Eq. (D2) and observing that $\sum_{T_\lambda}=\sum_{\lambda^-}\sum_{T_{\lambda^-}}$ and similarly for the sum over T_ν, the general Identity is directly obtained. Interestingly, Eq. (D4) also shows that

$$\langle\hat{X}^{(N)}\rangle_{|\nu,T_\nu,W_\nu\rangle}=\langle\hat{X}\rangle_{\hat{\rho}^{(\nu,W_\nu)}_{\nu(N-1)}},\quad\forall\hat{X}.\tag{D5}$$

Appendix E: Qubit case

For $d=2$, the commutant basis operators $\hat{F}^{(W_\nu,W'_\nu)}_\nu$ are indexed with partitions $\nu\equiv(\nu_1,\nu_2)\in\mathscr{P}_2$ ($\nu_1>0,\nu_2\geq 0$). In this case, the set $\{\nu^-\}$ is only composed of the valid partitions among the two partitions $\nu^{-1}\equiv(\nu_1-1,\nu_2)$ and $\nu^{-2}\equiv(\nu_1,\nu_2-1)$, so that $\{\nu^{-+}\}$ is in turn only composed of the valid partitions among the three partitions $\nu_a\equiv\nu$, $\nu_b\equiv\nu^{1\to2}=(\nu_1-1,\nu_2+1)$, and $\nu_c\equiv\nu^{2\to1}=(\nu_1+1,\nu_2-1)$ (see Appendix A). The cardinality of the sets $\mathcal{W}^{(\pm j)}_\mu(W_\nu)$ and $\tilde{\mathcal{W}}^{(\pm j)}_\mu(W_\nu)$ is at most 1. Indeed, for $d=2$ the content of the SWT boxes is either a 0 or a 1 and there is a unique SWT $W^{n_0}_\nu$ of shape ν with prescribed admissible content of n_0 boxes 0 and $n_1=|\nu|-n_0$ boxes 1 (the boxes 0 have no other option than being located at the beginning of the first row of the SWT and the boxes 1 only on the rest). For all $j\in\{0,1\}$, $\nu\in\mathscr{P}_2$, $\mu\in\{\nu^\pm\}$, and SWT $W^{n_0}_\nu$, we have $\mathcal{W}^{(\pm j)}_\mu(W^{n_0}_\nu)=\tilde{\mathcal{W}}^{(\pm j)}_\mu(W^{n_0}_\nu)=\{W^{n_0\pm(1-j)}_\mu\}$ or $\emptyset$ (if $W^{n_0\pm(1-j)}_\mu$ is not a valid SWT). As a result, $\forall\lambda,\mu,\nu:\{\lambda,\mu,\nu\}=1$, $i,j\in\{0,1\}$, $\mathcal{W}^{(-i)}_\mu(W^{\tilde{n}_0}_\lambda)\cap\mathcal{W}^{(-j)}_\mu(W^{n_0}_\nu)=\{W^{\tilde{n}_0+i-1}_\mu\}\cap\{W^{n_0+j-1}_\mu\}$ and this set is not empty only if the two singletons coincide with a valid SWT, which at least requires $\tilde{n}_0=n_0+(j-i)$. In addition to the generic vanishing condition $\{\lambda,\mu,\nu\}=0$, the single qubit operator $\hat{g}^{(\lambda,W^{\tilde{n}_0}_\lambda;\nu,W^{n_0}_\nu)}_\mu$ is necessarily zero if $|\tilde{n}_0-n_0|>1$.

Setting $n_0(q)=n_0-q$, the only possibly nonzero $\hat{g}^{(\lambda,W^{n_0(q)}_\lambda;\nu,W^{n_0}_\nu)}_\mu$ operators are obtained for $q=0,\pm1$ (they can vanish within this condition for specific λ, μ, ν, $W^{n_0(q)}_\lambda$, and $W^{n_0}_\nu$). They are listed in Table I, along with their matrix elements, explicit expression, and a relevant trace property they fulfill. To this aim, we defined $\zeta^{\nu,n_0}_{k\tau}\equiv\langle W^{n_0+k-1}_{\nu^{-\tau}},k|W^{n_0}_\nu\rangle$ ($k=0,1$, $\tau=1,2$), $\hat{s}_{+1}=|1\rangle\langle0|$, $\hat{s}_{-1}=|0\rangle\langle1|$, $\hat{s}_0=(|1\rangle\langle1|-|0\rangle\langle0|)/2$, and

$$A^{\nu,n_0}_q=\begin{cases}\sqrt{(\nu_1-n_0+1)(n_0-\nu_2)} & \text{for } q=1\\ (\nu_1+\nu_2-2n_0)/2 & q=0\\ \sqrt{(\nu_1-n_0)(n_0+1-\nu_2)} & q=-1\end{cases},\tag{E1}$$

$$B^{\nu,n_0}_q=\begin{cases}\sqrt{(n_0-\nu_2)(n_0-\nu_2-1)} & \text{for } q=1\\ \sqrt{(\nu_1-n_0)(n_0-\nu_2)} & q=0\\ -\sqrt{(\nu_1-n_0-1)(\nu_1-n_0)} & q=-1\end{cases},\tag{E2}$$

$$D^{\nu,n_0}_q=\begin{cases}-\sqrt{(\nu_1-n_0+1)(\nu_1-n_0+2)} & \text{for } q=1\\ \sqrt{(\nu_1-n_0+1)(n_0-\nu_2+1)} & q=0\\ \sqrt{(n_0-\nu_2+1)(n_0-\nu_2+2)} & q=-1\end{cases}.\tag{E3}$$

The Kronecker delta in the third column of Table I accounts for the necessary condition $n_0(q)=n_0+j-i$ for the matrix elements to be nonzero. The operator expressions in the fourth and fifth columns directly follow from the explicit expressions of the CGC's $\zeta^{\nu,n_0}_{k\tau}$ for all $\nu=(\nu_1,\nu_2)\in\mathscr{P}_2$, i.e. (see Eq. (C10) with $d=2$ in Appendix C),

$$\zeta^{\nu,n_0}_{01}=\sqrt{\frac{n_0-\nu_2}{\Delta\nu}},\quad\zeta^{\nu,n_0}_{11}=\sqrt{\frac{\nu_1-n_0}{\Delta\nu}},\quad\zeta^{\nu,n_0}_{02}=-\sqrt{\frac{\nu_1+1-n_0}{\Delta\nu+2}},\quad\zeta^{\nu,n_0}_{12}=\sqrt{\frac{n_0-\nu_2+1}{\Delta\nu+2}},\tag{E4}$$

λ	μ	$[g_\mu^{(\lambda,W_\lambda^{n_0(q)};\nu,W_\nu^{n_0})}]_{i,j}$	$\hat{g}_\mu^{(\lambda,W_\lambda^{n_0(\pm1)};\nu,W_\nu^{n_0})}$	$\hat{g}_\mu^{(\lambda,W_\lambda^{n_0(0)};\nu,W_\nu^{n_0})}$	$\mathrm{Tr}[\hat{g}_\mu^{(\lambda,W_\lambda^{n_0(q')};\nu,W_\nu^{n_0})\dagger}\hat{s}_q]$
ν	ν^{-1}	$\zeta_{i1}^{\nu,n_0(q)}\zeta_{j1}^{\nu,n_0}\delta_{q,i-j}$	$\frac{A_{\pm1}^{\nu,n_0}}{\Delta\nu}\hat{s}_{\pm1}$	$\frac{1}{\Delta\nu}[(\nu_1-n_0)\lvert 1\rangle\langle 1\rvert+(n_0-\nu_2)\lvert 0\rangle\langle 0\rvert]$	$\frac{A_q^{\nu,n_0}}{\Delta\nu}\delta_{q,q'}$
	ν^{-2}	$\zeta_{i2}^{\nu,n_0(q)}\zeta_{j2}^{\nu,n_0}\delta_{q,i-j}$	$-\frac{A_{\pm1}^{\nu,n_0}}{\Delta\nu+2}\hat{s}_{\pm1}$	$\frac{1}{\Delta\nu+2}[(n_0-\nu_2)\lvert 1\rangle\langle 1\rvert+(\nu_1-n_0)\lvert 0\rangle\langle 0\rvert+\hat{\mathbb{1}}]$	$-\frac{A_q^{\nu,n_0}}{\Delta\nu+2}\delta_{q,q'}$
ν_b	$\nu^{-1}=\nu_b^{-2}$	$\zeta_{i2}^{\nu_b,n_0(q)}\zeta_{j1}^{\nu,n_0}\delta_{q,i-j}$	$\frac{B_{\pm1}^{\nu,n_0}}{\Delta\nu}\hat{s}_{\pm1}$	$2\frac{B_0^{\nu,n_0}}{\Delta\nu}\hat{s}_0$	$\frac{B_q^{\nu,n_0}}{\Delta\nu}\delta_{q,q'}$
ν_c	$\nu^{-2}=\nu_c^{-1}$	$\zeta_{i1}^{\nu_c,n_0(q)}\zeta_{j2}^{\nu,n_0}\delta_{q,i-j}$	$\frac{D_{\pm1}^{\nu,n_0}}{\Delta\nu+2}\hat{s}_{\pm1}$	$2\frac{D_0^{\nu,n_0}}{\Delta\nu+2}\hat{s}_0$	$\frac{D_q^{\nu,n_0}}{\Delta\nu+2}\delta_{q,q'}$

TABLE I. Only possibly nonzero operators $\hat{g}_\mu^{(\lambda,W_\lambda^{n_0(q)};\nu,W_\nu^{n_0})}$ $(q=0,\pm1)$ for a given $\nu\equiv(\nu_1,\nu_2)\in\mathscr{P}_2$ and $W_\nu^{n_0}\in\mathcal{W}_\nu$.

where $\Delta\nu\equiv\nu_1-\nu_2$. The trace property in the sixth column merely stems from the elementary relations $\mathrm{Tr}[\hat{s}_q]=0$, $\mathrm{Tr}[\hat{s}_{\pm1}^\dagger\hat{s}_q]=\delta_{q,\pm1}$, and $\mathrm{Tr}[\hat{s}_0^\dagger\hat{s}_q]=\delta_{q,0}/2$, $\forall q=0,\pm1$.

Finally, we have for all $\nu\equiv(\nu_1,\nu_2)\vdash(N,2)$ (see Appendix A),

$$f^\nu=\frac{\Delta\nu+1}{\nu_1+1}\binom{N}{\nu_2}, \tag{E5}$$

so that

$$r_\nu^{\nu^{-1}}=\frac{\Delta\nu}{\Delta\nu+1}(\nu_1+1),\quad r_\nu^{\nu^{-2}}=\frac{\Delta\nu+2}{\Delta\nu+1}\nu_2,\quad \frac{f^\nu}{f^{\nu_b}}=\frac{(\nu_2+1)(\Delta\nu+1)}{(\nu_1+1)(\Delta\nu-1)},\quad \frac{f^\nu}{f^{\nu_c}}=\frac{(\nu_1+2)(\Delta\nu+1)}{\nu_2(\Delta\nu+3)}. \tag{E6}$$

As a result, for $\hat{X}=\hat{s}_q$ and $\hat{Y}=\hat{s}_r$ $(q,r=0,\pm1)$ the general Identity (21) straightforwardly simplifies to

$$\begin{aligned}\sum_{n=1}^N\hat{s}_q^{(n)}\overline{|\nu,W_\nu^{n_0}\rangle\langle\nu,W_\nu^{n_0'}|}\hat{s}_r^{(n)\dagger}&=\frac{\nu_1+\nu_2+2}{\Delta\nu(\Delta\nu+2)}A_q^{\nu,n_0}A_r^{\nu,n_0'}\overline{|\nu,W_\nu^{n_0(q)}\rangle\langle\nu,W_\nu^{n_0'(r)}|}\\&+\frac{\nu_1+1}{\Delta\nu(\Delta\nu+1)}B_q^{\nu,n_0}B_r^{\nu,n_0'}\sqrt{\frac{f^\nu}{f^{\nu_b}}}\overline{|\nu_b,W_{\nu_b}^{n_0(q)}\rangle\langle\nu_b,W_{\nu_b}^{n_0'(r)}|}+\frac{\nu_2}{(\Delta\nu+1)(\Delta\nu+2)}D_q^{\nu,n_0}D_r^{\nu,n_0'}\sqrt{\frac{f^\nu}{f^{\nu_c}}}\overline{|\nu_c,W_{\nu_c}^{n_0(q)}\rangle\langle\nu_c,W_{\nu_c}^{n_0'(r)}|}.\end{aligned} \tag{E7}$$

For $d=2$, the Schur-Weyl duality basis states $|\nu,T_\nu,W_\nu\rangle$ are nothing but the standard Clebsch-Gordan basis states $|J,M,i\rangle$, according to the correspondance $J=\Delta\nu/2$, $M=N/2-n_0$ (equivalently $\nu_1=N/2+J$, $\nu_2=N/2-J$, and $n_0=N/2-M$), and where i is indexed by the distinct SYT's T_ν, hence from 1 to $f^\nu=\binom{N}{N/2-J}(2J+1)/(J+1+N/2)\equiv d_N^J$. Equation (E7) is just Identity 1 of Ref. [27] expressed in the Schur-Weyl duality language. For $(\nu,W_\nu^{n_0})\equiv(J,M)$, $(\nu,W_\nu^{n_0(q)})=(J,M_q)$, $(\nu_b,W_{\nu_b}^{n_0(q)})=(J-1,M_q)$ and $(\nu_c,W_{\nu_c}^{n_0(q)})=(J+1,M_q)$, with $M_q=M+q$.

Appendix F: Master equation matrix elements $\mathcal{L}_{\lambda,W_\lambda,W_\lambda';\nu,W_\nu,W_\nu'}$

For a Lindbladian $\mathcal{L}=\mathcal{V}_{\hat{H}_c}+\mathcal{D}_{\hat{\ell}}^{(\mathrm{loc})}+\mathcal{D}_{\hat{L}}^{(\mathrm{col})}$, the matrix elements $\mathcal{L}_{\lambda,W_\lambda,W_\lambda';\nu,W_\nu,W_\nu'}$ are composed of the three main contributions

$$\mathcal{L}_{\lambda,W_\lambda,W_\lambda';\nu,W_\nu,W_\nu'}=[\mathcal{V}_{\hat{H}_c}]_{\lambda,W_\lambda,W_\lambda';\nu,W_\nu,W_\nu'}+[\mathcal{D}_{\hat{\ell}}^{(\mathrm{loc})}]_{\lambda,W_\lambda,W_\lambda';\nu,W_\nu,W_\nu'}+[\mathcal{D}_{\hat{L}}^{(\mathrm{col})}]_{\lambda,W_\lambda,W_\lambda';\nu,W_\nu,W_\nu'}. \tag{F1}$$

The commutator between any PI operator $\hat{A}_{\mathrm{PI}}$ and the basis operator $\hat{F}_\nu^{(W_\nu,W_\nu')}$ follows straightforwardly from the commutant algebra multiplication rule. We have

$$[\hat{F}_\nu^{(W_\nu,W_\nu')},\hat{A}_{\mathrm{PI}}]=\frac{1}{\sqrt{f^\nu}}\left(\sum_{\tilde{W}_\nu'}A_{\nu,W_\nu',\tilde{W}_\nu'}\hat{F}_\nu^{(W_\nu,\tilde{W}_\nu')}-\sum_{\tilde{W}_\nu}A_{\nu,\tilde{W}_\nu,W_\nu}\hat{F}_\nu^{(\tilde{W}_\nu,W_\nu')}\right), \tag{F2}$$

so that

$$[\mathcal{V}_{\hat{H}_c}]_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu} = \frac{i}{\hbar}\left(C_{\hat{H}}^{(\nu,W'_\nu,W'_\lambda)}\delta_{W_\lambda,W_\nu} - C_{\hat{H}}^{(\nu,W_\lambda,W_\nu)}\delta_{W'_\lambda,W'_\nu}\right)\delta_{\lambda,\nu}, \tag{F3}$$

$$[\mathcal{D}_{\hat{\ell}}^{(\mathrm{loc})}]_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu} = C_{\hat{\ell},\hat{\ell}}^{(\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu)}\delta_{\lambda,\{\nu-+\}} - \frac{1}{2}\left(C_{\hat{\ell}^\dagger\hat{\ell}}^{(\nu,W'_\nu,W'_\lambda)}\delta_{W_\lambda,W_\nu} + C_{\hat{\ell}^\dagger\hat{\ell}}^{(\nu,W_\lambda,W_\nu)}\delta_{W'_\lambda,W'_\nu}\right)\delta_{\lambda,\nu}, \tag{F4}$$

$$\begin{aligned}[\mathcal{D}_{\hat{L}}^{(\mathrm{col})}]_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu} = {} & C_{\hat{L}}^{(\nu,W_\lambda,W_\nu)}C_{\hat{L}}^{(\nu,W'_\lambda,W'_\nu)*}\delta_{\lambda,\nu} - \frac{1}{2}\left(\sum_{\tilde{W}'_\nu} C_{\hat{L}}^{(\nu,\tilde{W}'_\nu,W'_\lambda)}C_{\hat{L}}^{(\nu,\tilde{W}'_\nu,W'_\nu)*}\right)\delta_{W_\lambda,W_\nu}\delta_{\lambda,\nu} \\ & - \frac{1}{2}\left(\sum_{\tilde{W}_\nu} C_{\hat{L}}^{(\nu,\tilde{W}_\nu,W_\nu)}C_{\hat{L}}^{(\nu,\tilde{W}_\nu,W_\lambda)*}\right)\delta_{W'_\lambda,W'_\nu}\delta_{\lambda,\nu}.\end{aligned} \tag{F5}$$

Appendix G: Generalization to PI Lindbladians with p-particle terms

We consider here the generalized case where the PI Linbladian is of the form

$$\mathcal{L} = \sum_p \left(\mathcal{V}_{\hat{H}_{p,c}} + \mathcal{D}_{\hat{\ell}_p}^{(\mathrm{loc})} + \mathcal{D}_{\hat{L}_p}^{(\mathrm{col})}\right), \tag{G1}$$

where the sum over p only runs for values between 1 and N for which either of the three Lindbladian p-particle terms is nonzero, $\hat{H}_{p,c} = \sum_{n_1<\cdots<n_p=1}^N \hat{H}_p^{(n_1,\ldots,n_p)}$, $\mathcal{D}_{\hat{\ell}_p}^{(\mathrm{loc})} = \sum_{n_1<\cdots<n_p=1}^N \mathcal{D}_{\hat{\ell}_p^{(n_1,\ldots,n_p)}}$, and $\mathcal{D}_{\hat{L}_p}^{(\mathrm{col})} = \mathcal{D}_{\hat{L}_{p,c}}$, with $\hat{L}_{p,c} = \sum_{n_1<\cdots<n_p=1}^N \hat{L}_p^{(n_1,\ldots,n_p)}$. Here, $\hat{H}_p$ is a p-particle Hamiltonian, $\hat{\ell}_p$ and $\hat{L}_p$ are p-particle Lindblad operators, and $(n_1,\ldots,n_p)$ denotes the particle p-uple these p-particle operators act on. These operators satisfy $\hat{X}_p^{(n_1,\ldots,n_p)} = \hat{X}_p^{(n_{\pi(1)},\ldots,n_{\pi(p)})}$, for all $n_1 \neq \ldots \neq n_p$ and permutations $\pi \in S_p$.

All Linbladian terms can again be expressed with the help of generic superoperators, namely $\mathcal{K}_{\hat{X}_p,\hat{Y}_p}$ ($\hat{X}_p,\hat{Y}_p$ any two p-particle operators) that act according to

$$\mathcal{K}_{\hat{X}_p,\hat{Y}_p}[\hat{A}] = \sum_{n_1<\cdots<n_p=1}^N \hat{X}_p^{(n_1,\ldots,n_p)}\hat{A}\hat{Y}_p^{(n_1,\ldots,n_p)\dagger}, \quad \forall \hat{A} \in \mathscr{L}(\mathcal{H}). \tag{G2}$$

Indeed, $\mathcal{V}_{\hat{H}_{p,c}} = (i/\hbar)(\mathcal{K}_{\hat{\mathbb{1}},\hat{H}_p} - \mathcal{K}_{\hat{H}_p,\hat{\mathbb{1}}})$, $\mathcal{D}_{\hat{\ell}_p}^{(\mathrm{loc})} = \mathcal{K}_{\hat{\ell}_p,\hat{\ell}_p} - (\mathcal{K}_{\hat{\ell}_p^\dagger\hat{\ell}_p,\hat{\mathbb{1}}} - \mathcal{K}_{\hat{\mathbb{1}},\hat{\ell}_p^\dagger\hat{\ell}_p})/2$, and $\mathcal{D}_{\hat{L}_p}^{(\mathrm{col})}[\hat{\rho}] = \hat{L}_{p,c}\mathcal{K}_{\hat{\mathbb{1}},\hat{L}_p}[\hat{\rho}] - (\hat{L}_{p,c}^\dagger\mathcal{K}_{\hat{L}_p,\hat{\mathbb{1}}}[\hat{\rho}] - \mathcal{K}_{\hat{\mathbb{1}},\hat{L}_p}[\hat{\rho}]\hat{L}_{p,c})/2$, where $\hat{L}_{p,c}$ can similarly be written as $\mathcal{K}_{\hat{L}_p,\hat{\mathbb{1}}}[\hat{\mathbb{1}}]$. The superoperators $\mathcal{K}_{\hat{X}_p,\hat{Y}_p}$ are PI, so that $\mathcal{K}_{\hat{X}_p,\hat{Y}_p}[\hat{A}_{\mathrm{PI}}]$ is itself a PI operator for any PI operator $\hat{A}_{\mathrm{PI}}$. With respect to Hermitian conjugation, we have $\mathcal{K}_{\hat{X}_p,\hat{Y}_p}[\hat{A}]^\dagger = \mathcal{K}_{\hat{Y}_p,\hat{X}_p}[\hat{A}^\dagger]$.

To get explicit expressions of the matrix elements $\mathcal{L}_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu}$, it is therefore again completely enough to have the expansion in the commutant operator basis of the PI operators $\mathcal{K}_{\hat{X}_p,\hat{Y}_p}[\hat{F}_\nu^{(W_\nu,W'_\nu)}]$, $\forall \hat{X}_p,\hat{Y}_p,\nu,W_\nu,W'_\nu$. This expansion reads

$$\mathcal{K}_{\hat{X}_p,\hat{Y}_p}[\hat{F}_\nu^{(W_\nu,W'_\nu)}] = \sum_{\lambda\vdash(N,d)}\sum_{W_\lambda,W'_\lambda\in\mathcal{W}_\lambda} \mathrm{Tr}(\hat{F}_\lambda^{(W_\lambda,W'_\lambda)\dagger}\mathcal{K}_{\hat{X}_p,\hat{Y}_p}[\hat{F}_\nu^{(W_\nu,W'_\nu)}])\hat{F}_\lambda^{(W_\lambda,W'_\lambda)}. \tag{G3}$$

Since $\mathcal{P}_\sigma[\hat{X}_p^{(n_1,\ldots,n_p)}\hat{A}_{\mathrm{PI}}\hat{Y}_p^{(n_1,\ldots,n_p)\dagger}] = \hat{X}_p^{(\sigma(n_1),\ldots,\sigma(n_p))}\hat{A}_{\mathrm{PI}}\hat{Y}^{(\sigma(n_1),\ldots,\sigma(n_p))\dagger}$, for all $n_1 \neq \cdots \neq n_p$ and permutations $\sigma \in S_N$, $\mathrm{Tr}(\hat{F}_\lambda^{(W_\lambda,W'_\lambda)\dagger}\hat{X}_p^{(n_1,\ldots,n_p)}\hat{A}_{\mathrm{PI}}\hat{Y}_p^{(n_1,\ldots,n_p)\dagger})$ is independent of the p-uple $(n_1,\ldots,n_p)$ and we get

$$\begin{aligned}\mathrm{Tr}(\hat{F}_\lambda^{(W_\lambda,W'_\lambda)\dagger}\mathcal{K}_{\hat{X}_p,\hat{Y}_p}[\hat{F}_\nu^{(W_\nu,W'_\nu)}]) &= \binom{N}{p}\mathrm{Tr}(\hat{F}_\lambda^{(W_\lambda,W'_\lambda)\dagger}\hat{X}_p^{(N-p+1,\ldots,N)}\hat{F}_\nu^{(W_\nu,W'_\nu)}\hat{Y}_p^{(N-p+1,\ldots,N)\dagger}) \\ &= \binom{N}{p}\frac{1}{\sqrt{f^\lambda f^\nu}}\sum_{T_\lambda,T_\nu}\langle\lambda,T_\lambda,W_\lambda|\hat{X}_p^{(N-p+1,\ldots,N)}|\nu,T_\nu,W_\nu\rangle \\ &\qquad \langle\lambda,T_\lambda,W'_\lambda|\hat{Y}_p^{(N-p+1,\ldots,N)}|\nu,T_\nu,W'_\nu\rangle^*.\end{aligned} \tag{G4}$$

Applying successively p times Eq. (D3) first to isolate the N-th qudit, then the $(N-1)$-th, and so on until the $(N-p+1)$-th, we get in similar notations

$$\begin{aligned}
|\nu, T_\nu, W_\nu\rangle &= \sum_{j_p=0}^{d-1} \sum_{\substack{W_{\nu(N-1)} \\ \in \mathcal{W}_{\nu(N-1)}}} \langle W_{\nu(N-1)}, j_p | W_\nu\rangle |\nu(N-1), T_{\nu(N-1)}, W_{\nu(N-1)}\rangle \otimes |j_p\rangle_N \\
&= \sum_{j_{p-1},j_p=0}^{d-1} \sum_{\substack{W_{\nu(N-1)} \\ \in \mathcal{W}_{\nu(N-1)}}} \sum_{\substack{W_{\nu(N-2)} \\ \in \mathcal{W}_{\nu(N-2)}}} \langle W_{\nu(N-1)}, j_p | W_\nu\rangle \langle W_{\nu(N-2)}, j_{p-1} | W_{\nu(N-1)}\rangle \\
&\qquad |\nu(N-2), T_{\nu(N-2)}, W_{\nu(N-2)}\rangle \otimes |j_{p-1}, j_p\rangle_{N-1,N} \\
&\ \ \vdots \\
&= \sum_{j_1,\ldots,j_p=0}^{d-1} \sum_{\substack{W_{\nu(N-1)} \\ \in \mathcal{W}_{\nu(N-1)}}} \cdots \sum_{\substack{W_{\nu(N-p)} \\ \in \mathcal{W}_{\nu(N-p)}}} \langle W_{\nu(N-1)}, j_p | W_\nu\rangle \cdots \langle W_{\nu(N-p)}, j_1 | W_{\nu(N-p+1)}\rangle \\
&\qquad |\nu(N-p), T_{\nu(N-p)}, W_{\nu(N-p)}\rangle \otimes |j_1, \ldots, j_p\rangle_{N-p+1,\ldots,N},
\end{aligned} \tag{G5}$$

with, $\forall k = 1, \ldots, p$, $\nu(N-k)$ the shape of T_ν without boxes $N, \ldots, N-k+1$, and $T_{\nu(N-k)}$ the SYT T_ν without these k boxes.

For every $\nu_L, \nu_R \in \mathscr{P}_d$, $W_\mu \in \mathcal{W}_\mu$ $(\mu = \nu_L, \nu_R)$, $\boldsymbol{\nu} \equiv (\nu_{l,p-1}, \ldots, \nu_{l,1}, \nu_c, \nu_{r,1}, \ldots, \nu_{r,p-1}) \in \mathscr{P}_d^{2p-1}$ (i.e., $\boldsymbol{\nu}$ is a vector of $2p-1$ partitions of at most d parts), $\boldsymbol{W_\nu} \equiv (W_{\nu_{l,p-1}}, \ldots, W_{\nu_{l,1}}, W_{\nu_c}, W_{\nu_{r,1}}, \ldots, W_{\nu_{r,p-1}})$, with $W_\mu \in \mathcal{W}_\mu$ $(\mu = \nu_{l,p-1}, \ldots, \nu_{r,p-1})$, we define the generalized 3ν symbol $\begin{pmatrix} \nu_L & \boldsymbol{\nu} & \nu_R \\ W_{\nu_L} & \boldsymbol{W_\nu} & W_{\nu_R} \end{pmatrix}$ as being the square $d^p \times d^p$ matrix with entries

$$\begin{pmatrix} \nu_L & \boldsymbol{\nu} & \nu_R \\ W_{\nu_L} & \boldsymbol{W_\nu} & W_{\nu_R} \end{pmatrix}_{\mathbf{i},\mathbf{j}} = \prod_{k=1}^{p} \langle W_{\nu_{l,k-1}}, i_k | W_{\nu_{l,k}}\rangle \langle W_{\nu_{r,k-1}}, j_k | W_{\nu_{r,k}}\rangle, \tag{G6}$$

where $\mathbf{i} \equiv (i_1, \ldots, i_p)$, $\mathbf{j} \equiv (j_1, \ldots, j_p)$, $i_k, j_k = 0, \ldots, d-1$ for $k = 1, \ldots, p$, and where we set $\nu_{l,0} = \nu_{r,0} \equiv \nu_c$, $\nu_{l,p} \equiv \nu_L$, and $\nu_{r,p} \equiv \nu_R$.

The generalized 3ν-symbol matrix $\begin{pmatrix} \nu_L & \boldsymbol{\nu} & \nu_R \\ W_{\nu_L} & \boldsymbol{W_\nu} & W_{\nu_R} \end{pmatrix}$ is necessarily zero if the condition $\nu_{l,k-1} \in \{\nu_{l,k}^-\}$ and $\nu_{r,k-1} \in \{\nu_{r,k}^-\}$, $\forall k = 1, \ldots, p$ (*generalized partition triangle selection rule*) is not satisfied. This condition can only be met if $\nu_L \in \{\nu_R^{-^p +^p}\}$ or equivalently $\nu_R \in \{\nu_L^{-^p +^p}\}$, where the superscript $-^p$ $[+^p]$ denotes the action of removing [adding] successively p inner [outer] corners to the partition it applies. We define the *generalized partition triangular delta* $\{\nu_L, \boldsymbol{\nu}, \nu_R\}$ to be 1 if the generalized partition triangle selection rule is satisfied and 0 otherwise.

Since the CGC's are real, we have

$$\begin{pmatrix} \nu_L & \boldsymbol{\nu} & \nu_R \\ W_{\nu_L} & \boldsymbol{W_\nu} & W_{\nu_R} \end{pmatrix} = \begin{pmatrix} \nu_R & \tilde{\boldsymbol{\nu}} & \nu_L \\ W_{\nu_R} & \boldsymbol{W_{\tilde{\nu}}} & W_{\nu_L} \end{pmatrix}^T, \tag{G7}$$

where $\tilde{\boldsymbol{\nu}}$ is the vector of partitions $\boldsymbol{\nu}$ listed in reversed order: $\tilde{\boldsymbol{\nu}} \equiv (\nu_{r,p-1}, \ldots, \nu_{r,1}, \nu_c, \nu_{l,1}, \ldots, \nu_{l,p-1})$. The generalized 3ν-symbol matrices obey the generalized orthogonality relation (see Appendix A)

$$\sum_{\boldsymbol{W_\nu}} \mathrm{Tr}\left[\begin{pmatrix} \nu_L & \boldsymbol{\nu} & \nu_R \\ W_{\nu_L} & \boldsymbol{W_\nu} & W_{\nu_R} \end{pmatrix}\right] = \delta_{\nu_L,\nu_R} \delta_{W_{\nu_L},W_{\nu_R}} \delta_{\boldsymbol{\nu},\tilde{\boldsymbol{\nu}}} \{\nu_L, \boldsymbol{\nu}, \nu_R\}, \tag{G8}$$

with $\sum_{\boldsymbol{W_\nu}} \equiv \sum_{W_{\nu_{l,p-1}}} \cdots \sum_{W_{\nu_{l,1}}} \sum_{W_{\nu_c}} \sum_{W_{\nu_{r,1}}} \cdots \sum_{W_{\nu_{r,p-1}}}$, and they are the representation matrices in the computational basis of the p-qudit product operators

$$\hat{g}_{\boldsymbol{\nu},\boldsymbol{W_\nu}}^{(\nu_L,W_{\nu_L};\nu_R,W_{\nu_R})} = \bigotimes_{k=1}^{p} |\phi_{\nu_{l,k-1},W_{\nu_{l,k-1}}}^{(\nu_{l,k},W_{\nu_{l,k}})}\rangle\langle\phi_{\nu_{r,k-1},W_{\nu_{r,k-1}}}^{(\nu_{r,k},W_{\nu_{r,k}})}|. \tag{G9}$$

We also define the p-qudit operators

$$\hat{g}_{\boldsymbol{\nu}}^{(\nu_L,W_{\nu_L};\nu_R,W_{\nu_R})} = \sum_{\boldsymbol{W_\nu}} \hat{g}_{\boldsymbol{\nu},\boldsymbol{W_\nu}}^{(\nu_L,W_{\nu_L};\nu_R,W_{\nu_R})}. \tag{G10}$$

These operators vanish if $\{\nu_L,\boldsymbol{\nu},\nu_R\}=0$ and they satisfy $\hat{g}_{\boldsymbol{\nu}}^{(\nu_L,W_{\nu_L};\nu_R,W_{\nu_R})\dagger} = \hat{g}_{\tilde{\boldsymbol{\nu}}}^{(\nu_R,W_{\nu_R};\nu_L,W_{\nu_L})}$ and

$$\mathrm{Tr}[\hat{g}_{\boldsymbol{\nu}}^{(\nu_L,W_{\nu_L};\nu_R,W_{\nu_R})}] = \delta_{\nu_L,\nu_R}\delta_{W_{\nu_L},W_{\nu_R}}\delta_{\boldsymbol{\nu},\tilde{\boldsymbol{\nu}}}\{\nu_L,\boldsymbol{\nu},\nu_R\}. \tag{G11}$$

As a result, $\forall \nu\in\mathscr{P}_d$, $W_\nu\in\mathcal{W}_\nu$, $\boldsymbol{\mu}\in\mathscr{P}_d^{2p-1}:\boldsymbol{\mu}=\tilde{\boldsymbol{\mu}}$ and $\{\nu,\boldsymbol{\mu},\nu\}=1$, $\hat{\rho}_{\boldsymbol{\mu}}^{(\nu,W_\nu)}\equiv\hat{g}_{\boldsymbol{\mu}}^{(\nu,W_\nu;\nu,W_\nu)}$ is a trace 1 positive semidefinite operator and represents a separable p-qudit mixed state.

With this stated and expanding $|\lambda,T_\lambda,W_\lambda\rangle$ similarly as $|\nu,T_\nu,W_\nu\rangle$ in Eq. (G5), we directly obtain

$$\begin{aligned}\langle\lambda,T_\lambda,W_\lambda|\hat{X}_p^{(N-p+1,\dots,N)}|\nu,T_\nu,W_\nu\rangle &= \sum_{\mathbf{i},\mathbf{j}}\sum_{\boldsymbol{W}_{\boldsymbol{\mu}_{(T_\lambda,T_\nu)_p}}}\begin{pmatrix}\lambda & \boldsymbol{\mu}_{(T_\lambda,T_\nu)_p} & \nu\\ W_\lambda & \boldsymbol{W}_{\boldsymbol{\mu}_{(T_\lambda,T_\nu)_p}} & W_\nu\end{pmatrix}_{\mathbf{i},\mathbf{j}}\langle\mathbf{i}|\hat{X}_p|\mathbf{j}\rangle\delta_{\lambda(N-p),\nu(N-p)}\delta_{T_{\lambda(N-p)},T_{\nu(N-p)}}\\ &= \mathrm{Tr}[\hat{g}_{\boldsymbol{\mu}_{(T_\lambda,T_\nu)_p}}^{(\lambda,W_\lambda;\nu,W_\nu)\dagger}\hat{X}_p]\delta_{\lambda(N-p),\nu(N-p)}\delta_{T_{\lambda(N-p)},T_{\nu(N-p)}},\end{aligned} \tag{G12}$$

with $\boldsymbol{\mu}_{(T_\lambda,T_\nu)_p}=(\lambda(N-1),\dots,\lambda(N-p+1),\lambda(N-p),\nu(N-p+1),\dots,\nu(N-1))$. Interestingly, this also implies that

$$\langle\hat{X}_p^{(N-p+1,\dots,N)}\rangle_{|\nu,T_\nu,W_\nu\rangle} = \langle\hat{X}_p\rangle_{\hat{\rho}_{\boldsymbol{\mu}_{(T_\nu)_p}}^{(\nu,W_\nu)}}, \quad \forall\hat{X}_p, \tag{G13}$$

with $\boldsymbol{\mu}_{(T_\nu)_p}\equiv\boldsymbol{\mu}_{(T_\nu,T_\nu)_p}$.

Inserting Eq. (G12) into Eq. (G4) and observing that $\sum_{T_\lambda}=\sum_{\lambda^-}\sum_{T_{\lambda^-}}=\sum_{\lambda^-}\sum_{(\lambda^-)^-}\sum_{T_{(\lambda^-)^-}}=\cdots$ (so on p times) and similarly for the sum over T_ν, we immediately get

$$\mathcal{K}_{\hat{X}_p,\hat{Y}_p}[\hat{F}_\nu^{(W_\nu,W'_\nu)}] = \sum_{\lambda\in\{\nu^{-p+p}\}}\sum_{W_\lambda,W'_\lambda\in\mathcal{W}_\lambda} C_{\hat{X}_p,\hat{Y}_p}^{(\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu)}\hat{F}_\lambda^{(W_\lambda,W'_\lambda)}, \tag{G14}$$

with

$$C_{\hat{X}_p,\hat{Y}_p}^{(\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu)} = \sum_{\substack{\boldsymbol{\mu}\in\mathscr{P}_d^{2p-1}:\\ \{\lambda,\boldsymbol{\mu},\nu\}=1}}\sqrt{r_\lambda^{\mu_c}r_\nu^{\mu_c}}\,\mathrm{Tr}[\hat{g}_{\boldsymbol{\mu}}^{(\lambda,W_\lambda;\nu,W_\nu)\dagger}\hat{X}_p]\,\mathrm{Tr}[\hat{g}_{\boldsymbol{\mu}}^{(\lambda,W'_\lambda;\nu,W'_\nu)\dagger}\hat{Y}_p]^*, \tag{G15}$$

where we defined $r_\nu^\mu\equiv\binom{N}{p}f^\mu/f^\nu$, $\forall\nu\vdash N,\mu\in\{\nu^{-p}\}$.

In the particular case where $\hat{Y}_p=\hat{\mathbb{1}}_p$, with $\hat{\mathbb{1}}_p$ the p-particle identity, Eq. (G15) simplifies to

$$C_{\hat{X}_p,\hat{\mathbb{1}}_p}^{(\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu)} = C_{\hat{X}_p}^{(\nu,W_\lambda,W_\nu)}\delta_{\lambda,\nu}\delta_{W'_\lambda,W'_\nu}, \tag{G16}$$

with

$$C_{\hat{X}_p}^{(\nu,W_\nu,W'_\nu)} = \sum_{\substack{\boldsymbol{\mu}\in\mathscr{P}_d^{2p-1}:\\ \{\nu,\boldsymbol{\mu},\nu\}=1\,\&\,\boldsymbol{\mu}=\tilde{\boldsymbol{\mu}}}} r_\nu^{\mu_c}\,\mathrm{Tr}[\hat{g}_{\boldsymbol{\mu}}^{(\nu,W_\nu;\nu,W'_\nu)\dagger}\hat{X}_p]. \tag{G17}$$

It follows that $\mathcal{K}_{\hat{X}_p,\hat{\mathbb{1}}_p}[\hat{F}_\nu^{(W_\nu,W'_\nu)}]=\sum_{\tilde{W}_\nu}C_{\hat{X}_p}^{(\nu,\tilde{W}_\nu,W_\nu)}\hat{F}_\nu^{(\tilde{W}_\nu,W'_\nu)}$ and $\hat{X}_{p,c}=\sum_{\nu,W_\nu,W'_\nu}\sqrt{f^\nu}C_{\hat{X}_p}^{(\nu,W_\nu,W'_\nu)}\hat{F}_\nu^{(W_\nu,W'_\nu)}$, so that the master equation matrix elements (F3) to (F5) merely generalizes in presence of p-particle operators according to

$$[\mathcal{V}_{\hat{H}_{p,c}}]_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu} = \frac{i}{\hbar}\left(C_{\hat{H}_p}^{(\nu,W'_\nu,W'_\lambda)}\delta_{W_\lambda,W_\nu}-C_{\hat{H}_p}^{(\nu,W_\lambda,W_\nu)}\delta_{W'_\lambda,W'_\nu}\right)\delta_{\lambda,\nu}, \tag{G18}$$

$$[\mathcal{D}_{\hat{\ell}_p}^{(\mathrm{loc})}]_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu} = C_{\hat{\ell}_p,\hat{\ell}_p}^{(\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu)}\delta_{\lambda,\{\nu^{-p+p}\}}-\frac{1}{2}\left(C_{\hat{\ell}_p^\dagger\hat{\ell}_p}^{(\nu,W'_\nu,W'_\lambda)}\delta_{W_\lambda,W_\nu}+C_{\hat{\ell}_p^\dagger\hat{\ell}_p}^{(\nu,W_\lambda,W_\nu)}\delta_{W'_\lambda,W'_\nu}\right)\delta_{\lambda,\nu}, \tag{G19}$$

$$\begin{aligned}[\mathcal{D}_{\hat{L}_p}^{(\mathrm{col})}]_{\lambda,W_\lambda,W'_\lambda;\nu,W_\nu,W'_\nu} &= C_{\hat{L}_p}^{(\nu,W_\lambda,W_\nu)}C_{\hat{L}_p}^{(\nu,W'_\lambda,W'_\nu)*}\delta_{\lambda,\nu}-\frac{1}{2}\left(\sum_{\tilde{W}'_\nu}C_{\hat{L}_p}^{(\nu,\tilde{W}'_\nu,W'_\lambda)}C_{\hat{L}_p}^{(\nu,\tilde{W}'_\nu,W'_\nu)*}\right)\delta_{W_\lambda,W_\nu}\delta_{\lambda,\nu}\\ &\quad-\frac{1}{2}\left(\sum_{\tilde{W}_\nu}C_{\hat{L}_p}^{(\nu,\tilde{W}_\nu,W_\nu)}C_{\hat{L}_p}^{(\nu,\tilde{W}_\nu,W_\lambda)*}\right)\delta_{W'_\lambda,W'_\nu}\delta_{\lambda,\nu}.\end{aligned} \tag{G20}$$

For $p = 1$, all results of this Appendix just particularize to the standard formalism presented ahead.

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[42] This means that each superoperator $\mathcal{P}_\sigma$ defines a $*$-isomorphism on the Liouville space $\mathscr{L}(\mathcal{H})$.
[43] Indeed, $\forall\sigma, n$, and local operator $\hat{X}$, we have $\mathcal{P}_\sigma[\hat{X}^{(n)}] = \hat{X}^{(\sigma(n))}$, which implies $\mathcal{P}_\sigma[\{\hat{X}^{(n)}\}] = \{\hat{X}^{(\sigma(n))}\} = \{\hat{X}^{(n)}\}$ and $\mathcal{P}_\sigma[\hat{X}_c] = \hat{X}_c$.
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[45] $\sqrt{f^\nu}\mathcal{P}_\sigma[\hat{F}_\nu^{(W_\nu, W'_\nu)}] = \sum_{\lambda, T_\lambda, W_\lambda} \sum_{\lambda', T'_{\lambda'}, W'_{\lambda'}} \sum_{T_\nu} |\lambda, T_\lambda, W_\lambda\rangle\langle\lambda, T_\lambda, W_\lambda|\hat{P}_\sigma|\nu, T_\nu, W_\nu\rangle \langle\nu, T_\nu, W'_\nu|\hat{P}_\sigma^\dagger|\lambda', T'_{\lambda'}, W'_{\lambda'}\rangle\langle\lambda', T'_{\lambda'}, W'_{\lambda'}| = \sum_{\nu, T_\nu, T'_\nu, T_\nu} |\nu, T_\nu, W_\nu\rangle\langle T_\nu|\hat{\sigma}|T_\nu\rangle\langle T_\nu|\hat{\sigma}^\dagger|T'_\nu\rangle\langle\nu, T'_\nu, W'_\nu| = \sqrt{f^\nu}\hat{F}_\nu^{(W_\nu, W'_\nu)}$, with $\hat{\sigma}$ and $|T_\nu\rangle$ the representation

operators and YY-basis states in the $\mathcal{S}^\nu$-irrep of the symmetric group S_N, respectively (see Appendix A).

[46] Hence, replacing $\hat{F}_\nu^{(W_\nu,W'_\nu)}$ and $\hat{F}_\nu^{(W'_\nu,W_\nu)}$ by $(\hat{F}_\nu^{(W_\nu,W'_\nu)}+\hat{F}_\nu^{(W'_\nu,W_\nu)})/\sqrt{2}$ and $i(\hat{F}_\nu^{(W_\nu,W'_\nu)}-\hat{F}_\nu^{(W'_\nu,W_\nu)})/\sqrt{2}$, $\forall\nu\vdash(N,d)$, $W_\nu,W'_\nu\in\mathcal{W}_\nu:W_\nu\neq W'_\nu$, yields together with the operators $\hat{F}_\nu^{(W_\nu,W_\nu)}$ an orthonormal basis of PI Hermitian operators in the commutant $\mathscr{L}_{S_N}(\mathcal{H})$. In addition, having $\mathrm{Tr}[\hat{F}_\nu^{(W_\nu,W'_\nu)}]=\sqrt{f^\nu}\delta_{W_\nu,W'_\nu}$, an orthogonal basis in the subspace of traceless PI Hermitian operators is straightfowardly obtained by further replacing all but one operators $\hat{F}_\nu^{(W_\nu,W_\nu)}$ by traceless linear combinations of them.

[47] The structure constant of the commutant operator algebra follows immediately: $\hat{F}_\lambda^{(W_\lambda,W'_\lambda)}\hat{F}_\mu^{(W_\mu,W'_\mu)}=\sum_{\nu,W_\nu,W'_\nu}c^{\nu,W_\nu,W'_\nu}_{\lambda,W_\lambda,W'_\lambda;\mu,W_\mu,W'_\mu}\hat{F}_\nu^{(W_\nu,W'_\nu)}$, with $\sqrt{f^\lambda}c^{\nu,W_\nu,W'_\nu}_{\lambda,W_\lambda,W'_\lambda;\mu,W_\mu,W'_\mu}=\delta_{\lambda,\mu}\delta_{\lambda,\nu}\delta_{W'_\lambda,W_\mu}\delta_{W_\lambda,W_\nu}\delta_{W'_\mu,W'_\nu}$.

[48] The product of two ν-type operators is a ν-type operator and so is the Hermitian conjugate of a ν-type operator: each operator subspace $\mathscr{L}_\nu(\mathcal{H})$ is a $*$-algebra of operators on $\mathcal{H}$ and a subalgebra of the commutant $\mathscr{L}_{S_N}(\mathcal{H})$.

[49] Having $\sum_{\nu^-}f^{\nu^-}=f^\nu$, we get in particular $C_{\hat{\mathbb{1}}}^{(\nu,W_\nu,W'_\nu)}=N\delta_{W_\nu,W'_\nu}$, so that Eq. (27) yields as expected from definition $\mathcal{K}_{\hat{\mathbb{1}},\hat{\mathbb{1}}}[\hat{F}_\nu^{(W_\nu,W'_\nu)}]=N\hat{F}_\nu^{(W_\nu,W'_\nu)}$.

[50] The trace is invariant under cyclic permutations, so that $\mathrm{Tr}(\mathcal{P}_\sigma[\hat{A}])=\mathrm{Tr}(\hat{A})$. As a result, we get $\mathrm{Tr}(\hat{A}^\dagger_{\mathrm{PI}}\hat{B})=\mathrm{Tr}(\mathcal{P}_\sigma[\hat{A}^\dagger_{\mathrm{PI}}\hat{B}])=\mathrm{Tr}(\mathcal{P}_\sigma[\hat{A}^\dagger_{\mathrm{PI}}]\mathcal{P}_\sigma[\hat{B}])=\mathrm{Tr}(\hat{A}^\dagger_{\mathrm{PI}}\mathcal{P}_\sigma[\hat{B}]),\forall\sigma$. Alternatively, we can also write $\mathrm{Tr}(\hat{A}^\dagger_{\mathrm{PI}}\mathcal{P}_\sigma[\hat{B}])=\mathrm{Tr}(\mathcal{P}^\dagger_\sigma[\hat{A}_{\mathrm{PI}}]^\dagger\hat{B})=\mathrm{Tr}(\mathcal{P}_{\sigma^{-1}}[\hat{A}_{\mathrm{PI}}]^\dagger\hat{B})=\mathrm{Tr}(\hat{A}^\dagger_{\mathrm{PI}}\hat{B})$.

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[55] Strictly generally, irreducible representations of the general linear group $GL(d)$ are indexed by so-called *highest weights* $\lambda\equiv(\lambda_1,\ldots,\lambda_d)$, with $\lambda_1\geq\cdots\geq\lambda_d$ and λ_i $(i=1,\ldots,d)$ positive *or negative* integers. In the context of the Schur-Weyl duality, only $GL(d)$ irreps of highest weights λ with positive parts play a role, in which case λ identifies to a partition of at most d parts (including the empty partition).

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