

Dissipative Quantum Phase Transition in an Interacting Many-Particle System: from Two-Level to Multilevel Spins

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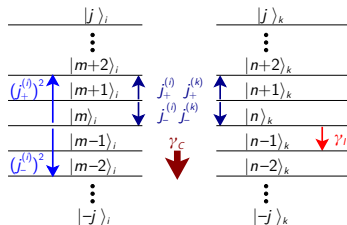
- 1 Theoretical Model
- 2 Dissipative Phase Transition in Mean-Field Limit
- 3 Spectrum of the Liouvillian and Symmetry Breaking
- 4 Purity and Entanglement of the Steady State

Spin- j Dissipative Lipkin-Meshkov-Glick Model

N spin- j particles, $J_\alpha = \sum_{i=1}^N j_\alpha^{(i)}$

Hamiltonian

$$H = \frac{V}{2N} (j_+^2 + j_-^2)$$



Spin- j Dissipative Lipkin-Meshkov-Glick Model

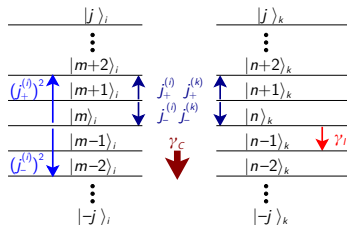
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$$, \quad L_C = \sum_{i=1}^N L_i$$

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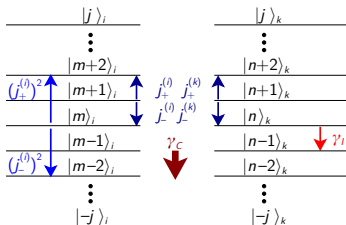
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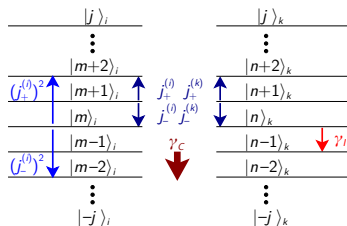


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Lindblad master equation

$$\begin{aligned} \dot{\rho} = & -i[H, \rho] + \gamma_I \sum_{i=1}^N \left(L_i \rho L_i^\dagger - \frac{1}{2} \{L_i^\dagger L_i, \rho\} \right) \\ & + \frac{\gamma_C}{N} \left(L_C \rho L_C^\dagger - \frac{1}{2} \{L_C^\dagger L_C, \rho\} \right) \end{aligned}$$

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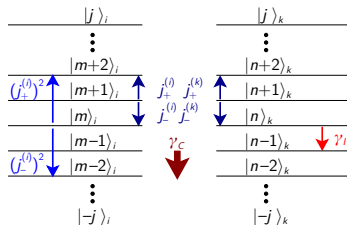
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Dissipators with $\ell_{-j} = \sqrt{2j}$

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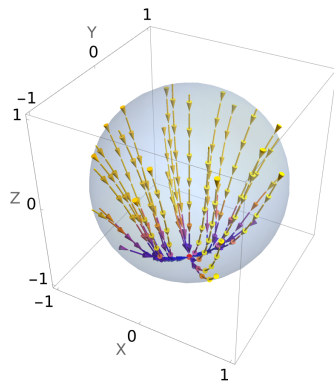


Qubit Model in Mean-Field Limit

Steady states in the limit $N \rightarrow \infty$:

- For $\gamma_I + \gamma_C > 2|V|$:

$$\langle J_x \rangle = \langle J_y \rangle = 0, \frac{\langle J_z \rangle}{Nj} = -1.$$



$$\gamma_I = 2.3V, \gamma_C = 0$$

T. E. Lee et al., PRL **110**, 257204 (2013),

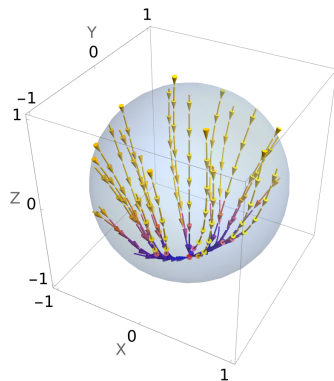
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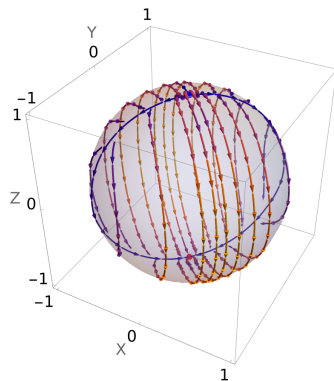
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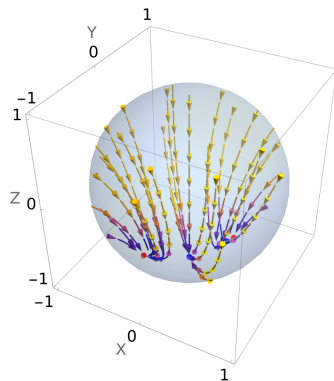
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$$\frac{\langle J_x \rangle}{Nj} = \text{sgn}(V) \frac{\langle J_y \rangle}{Nj} = \pm \frac{\sqrt{\gamma_I(2|V| - \gamma_I - \gamma_C)}}{(2|V| - \gamma_C)}$$
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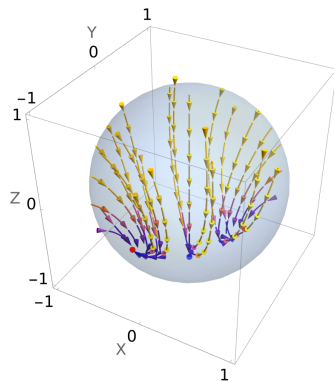
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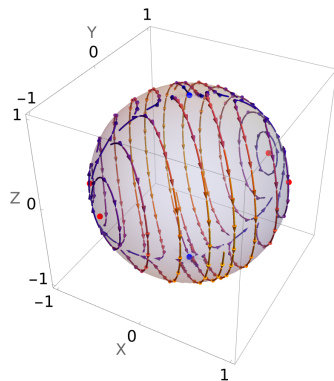
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- For $\gamma_I = 0, \gamma_C < 2|V|$:
4 center fixed points with $\langle J_z \rangle = 0$.

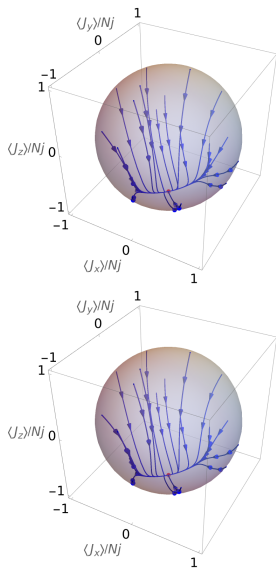
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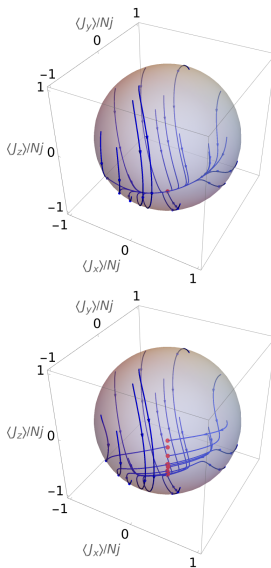


$$\gamma_I = 0, \gamma_C = 1.7V$$

Dynamics for Multilevel Systems, $j = 3/2$, $\gamma_I + \gamma_C > 2|V|$



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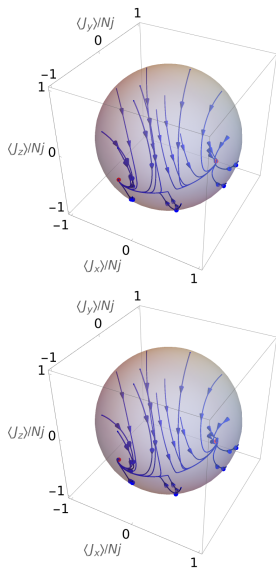


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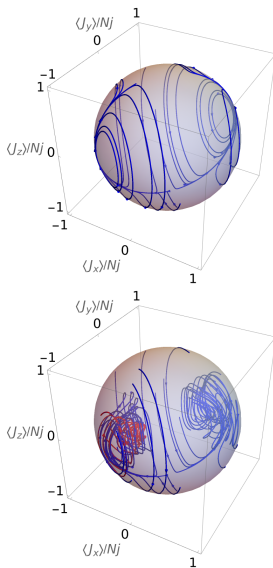
$$L_i^{(1)} = j_-^{(i)}$$

$$L_i^{(2)} = \sqrt{2}j \sum_{m=-j}^{j-1} |m\rangle \langle m+1|$$

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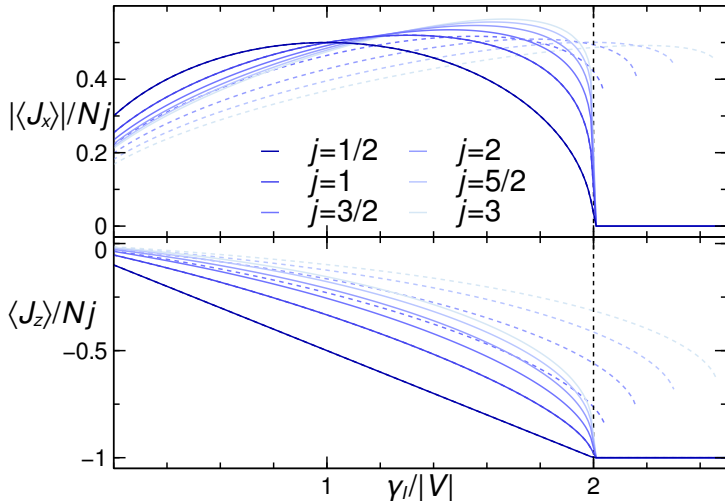


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Steady States for Multilevel Systems



$\gamma_C = 0,$

$L_i^{(1)}$

$L_i^{(2)}$

Spectrum of the Liouvillian and Symmetry Breaking

Master equation: $\dot{\rho} = \mathcal{L}[\rho]$, $\dim \mathcal{L} = \binom{N+4j(j+1)}{N} \times \binom{N+4j(j+1)}{N}$

Steady state: $\mathcal{L}[\rho] = 0$

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Symmetry breaking:

$\Delta = |\lambda_1| \xrightarrow{N \rightarrow \infty} 0$ (eigenvalue $\lambda_1 \neq 0$ closest to 0)

in corresponding symmetry sector

F. Minganti et al., PRA **98**, 042118 (2018).

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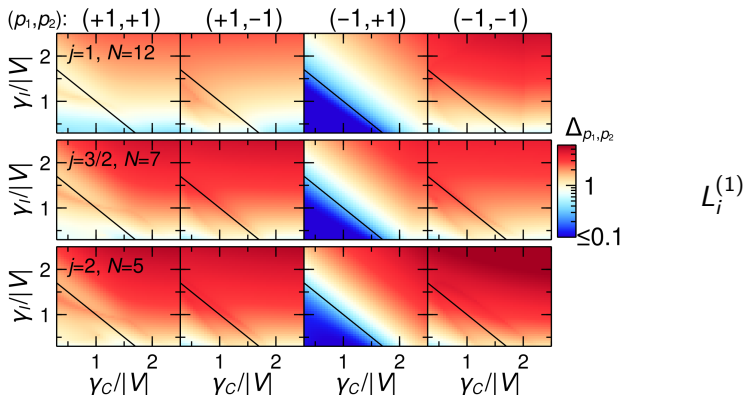
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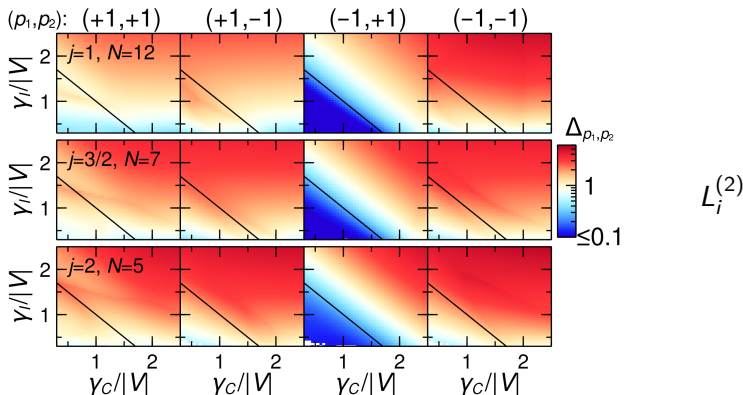
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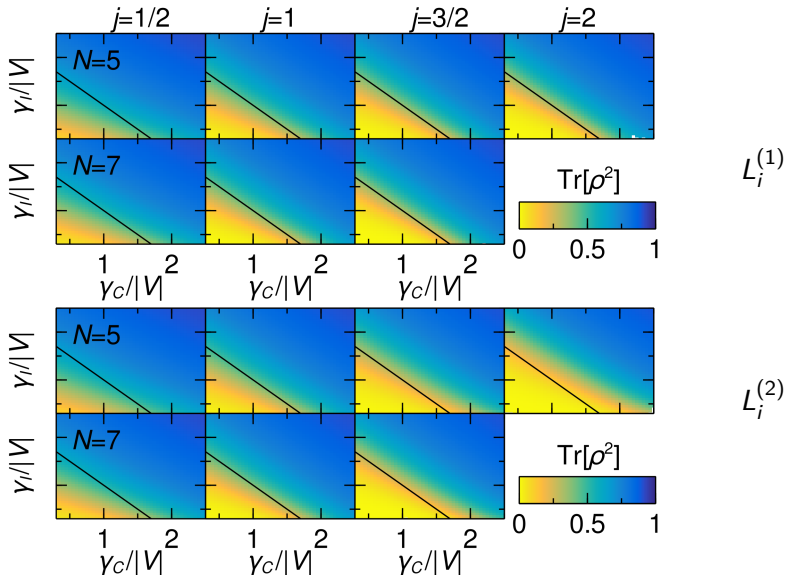
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Steady-State Purity $\text{Tr}[\rho^2]$

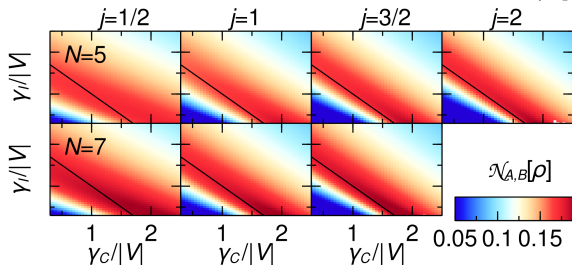


Steady-State Entanglement: Negativity $\mathcal{N}_{A,B}[\rho]$

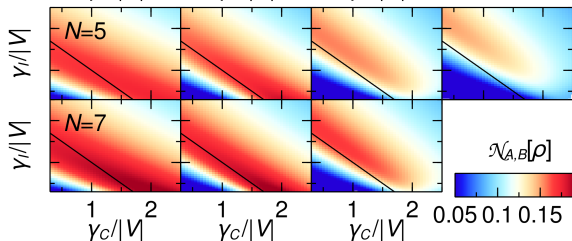
Subsystems A, B , partial transpose ρ^{T_B} with eigenvalues $\lambda_1 \leq \dots \leq \lambda_n < 0 \leq \lambda_{n+1} \leq \dots$ $\mathcal{N}_{A,B}[\rho] = \sum_{i=1}^n |\lambda_i|$

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$L_i^{(1)}$



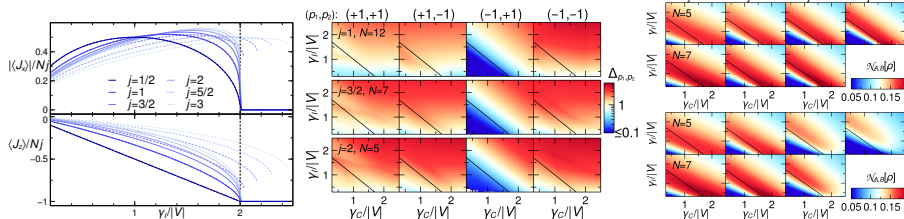
$L_i^{(2)}$

$$A = \left\lceil \frac{N}{2} \right\rceil \text{ particles,}$$

$$B = \left\lfloor \frac{N}{2} \right\rfloor \text{ particles}$$

Conclusions

- 1 Despite same critical point $\gamma_I + \gamma_C = 2|V|$ as for $j = \frac{1}{2}$:
New critical region, change of $\vec{J}$ at transition stronger with j .
- 2 Spectrum of Liouvillian reveals symmetry breaking.
- 3 Sharp change of steady-state purity & maximum of its entanglement negativity at phase transition.



Supplemental Material: Details on Symmetries of $\mathcal{L}$

Symmetry: Superoperator Π such that $[\Pi, \mathcal{L}] = 0$.

1 For all $\pi \in S_N$: $\Pi_\pi : \rho \mapsto \pi \rho \pi^\dagger$.

2 $\Pi_1 : \rho \mapsto e^{i\pi J_z} \rho e^{-i\pi J_z}$, with

$$\langle J_x \rangle_{\Pi_1[\rho]} = \text{Tr}(J_x \Pi_1[\rho]) = -\langle J_x \rangle_\rho,$$

$$\langle J_y \rangle_{\Pi_1[\rho]} = -\langle J_y \rangle_\rho,$$

$$\langle J_z \rangle_{\Pi_1[\rho]} = \langle J_z \rangle_\rho.$$

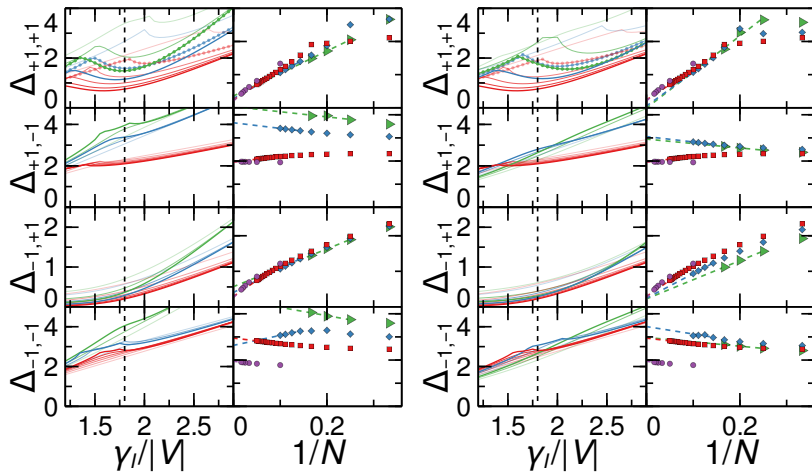
3 If L_i and L_C are real operators, $\Pi_2 : \rho \mapsto e^{i\pi J_z/2} \rho^* e^{-i\pi J_z/2}$, with

$$\langle J_x \rangle_{\Pi_2[\rho]} = \langle J_y \rangle_\rho,$$

$$\langle J_y \rangle_{\Pi_2[\rho]} = \langle J_x \rangle_\rho,$$

$$\langle J_z \rangle_{\Pi_2[\rho]} = \langle J_z \rangle_\rho.$$

Supplemental Material: Details on Spectrum, $\gamma_C = 0.2$



$$L_i = j_-^{(i)}$$

$$j = \frac{1}{2}, j = 1, j = \frac{3}{2}, j = 2$$

$$L_i = \sqrt{2j} \sum_{m=-j}^{j-1} |m\rangle \langle m+1|$$

Supplemental Material: Mean-Field for $j = 1/2$

Mean-field assumptions for $N \rightarrow \infty$:

$$\frac{1}{N^2} \langle \{J_\alpha, J_\beta\} \rangle \approx \frac{2}{N^2} \langle J_\alpha \rangle \langle J_\beta \rangle,$$

justified by

$$\frac{\langle J_\alpha \rangle}{N} \stackrel{N \rightarrow \infty}{\equiv} \mathcal{O}(1), \quad \frac{\langle [J_\alpha, J_\beta] \rangle}{N^2} \stackrel{N \rightarrow \infty}{\equiv} \mathcal{O}(N^{-1}).$$

With $X := 2 \langle J_x \rangle / N$, $Y := 2 \langle J_y \rangle / N$, $Z := 2 \langle J_z \rangle / N$ get

$$\begin{aligned}\dot{X} &= -2VYZ - \gamma_I X + \gamma_C XZ, \\ \dot{Y} &= -2VXZ - \gamma_I Y + \gamma_C YZ, \\ \dot{Z} &= 4VXY - 2\gamma_I(Z + 1) - \gamma_C(X^2 + Y^2),\end{aligned}$$

Supplemental Material: Mean-Field for Arbitrary j

$$S_{x;m,n}^{(i)} = \frac{1}{2N} \sum_{i=1}^N (|m\rangle_i \langle n|_i + |n\rangle_i \langle m|_i),$$
$$S_{y;m,n}^{(i)} = -\frac{i}{2N} \sum_{i=1}^N (|m\rangle_i \langle n|_i - |n\rangle_i \langle m|_i),$$

with

$$\langle \{S_{\alpha;m,n}, S_{\beta;o,p}\} \rangle \approx 2 \langle S_{\alpha;m,n} \rangle \langle S_{\beta;o,p} \rangle \quad (N \rightarrow \infty).$$

Equations of motion:

$$\begin{aligned} \partial_t \langle S_{\alpha;\mu,\nu} \rangle &= i \langle [H, S_{\alpha;\mu,\nu}] \rangle + \frac{\gamma_I}{2j} \sum_{i=1}^N \left\langle [L_i^\dagger, S_{\alpha;\mu,\nu}] L_i + L_i^\dagger [S_{\alpha;\mu,\nu}, L_i] \right\rangle \\ &\quad + \frac{\gamma_C}{2Nj} \left\langle [L_C^\dagger, S_{\alpha;\mu,\nu}] L_C + L_C^\dagger [S_{\alpha;\mu,\nu}, L_C] \right\rangle \\ &= V f_1(\langle S_{\beta;o,p} \rangle) + \gamma_I g(\langle S_{\beta;o,p} \rangle) \\ &\quad + \gamma_C f_2(\langle S_{\beta;o,p} \rangle) \end{aligned}$$

Supplemental Material: Mean-Field Unitary Part

$$\begin{aligned} i[H, S_{x;m,n}] &= \frac{V}{2j} \left(\left\{ \frac{J_x}{N}, \mathcal{A}_{+1,+1}^{y;m,n} \right\} + \left\{ \frac{J_y}{N}, \mathcal{A}_{-1,+1}^{x;m,n} \right\} \right), \\ i[H, S_{y;m,n}] &= \frac{V}{2j} \left(- \left\{ \frac{J_x}{N}, \mathcal{A}_{+1,-1}^{x;m,n} \right\} + \left\{ \frac{J_y}{N}, \mathcal{A}_{-1,-1}^{y;m,n} \right\} \right), \end{aligned}$$

with

$$\begin{aligned} A_{j,m} &:= \sqrt{(j-m)(j+m+1)}, \\ J_x/N &= \sum_{m=-j}^{j-1} A_{j,m} S_{x;m+1,m}, & J_y/N &= \sum_{m=-j}^{j-1} A_{j,m} S_{y;m+1,m} \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_{\sigma_1, \sigma_2}^{\alpha;m,n} &:= A_{j,n-1} S_{\alpha;m,n-1} + \sigma_1 A_{j,n} S_{\alpha;m,n+1} \\ &\quad + \sigma_2 A_{j,m-1} S_{\alpha;n,m-1} + \sigma_1 \sigma_2 A_{j,m} S_{\alpha;n,m+1} \end{aligned}$$

Supplemental Material: Mean-Field Individual Dissipation

$$\frac{1}{2} \sum_{i=1}^N [L_i^\dagger, S_{x;m,n}] L_i + \text{h.c.} = \text{Re} [\ell_m^* \ell_n] S_{x;m+1,n+1} - \text{Im} [\ell_m^* \ell_n] S_{y;m+1,n+1} \\ - \frac{|\ell_{n-1}|^2 + |\ell_{m-1}|^2}{2} S_{x;m,n},$$

$$\frac{1}{2} \sum_{i=1}^N [L_i^\dagger, S_{y;m,n}] L_i + \text{h.c.} = \text{Re} [\ell_m^* \ell_n] S_{y;m+1,n+1} + \text{Im} [\ell_m^* \ell_n] S_{x;m+1,n+1} \\ - \frac{|\ell_{n-1}|^2 + |\ell_{m-1}|^2}{2} S_{y;m,n}$$

Supplemental Material: Mean-Field Collective Dissipation

$$\begin{aligned}
 \frac{1}{2N} [L_C^\dagger, S_{x;m,n}] L_C + \text{h.c.} &= \frac{1}{4} \sum_{o=-j}^{j-1} \left(- \left\{ \mathcal{R}_{+1,+1}^{y;m,n,o} + \mathcal{J}_{-1,+1}^{x;m,n,o}, S_{y,o+1,o} \right\} \right. \\
 &\quad \left. - \left\{ \mathcal{R}_{-1,+1}^{x;m,n,o} - \mathcal{J}_{+1,+1}^{y;m,n,o}, S_{x,o+1,o} \right\} \right) \\
 \frac{1}{2N} [L_C^\dagger, S_{y;m,n}] L_C + \text{h.c.} &= \frac{1}{4} \sum_{o=-j}^{j-1} \left(- \left\{ \mathcal{R}_{-1,-1}^{y;m,n,o} + \mathcal{J}_{+1,-1}^{x;m,n,o}, S_{x,o+1,o} \right\} \right. \\
 &\quad \left. + \left\{ \mathcal{R}_{+1,-1}^{x;m,n,o} - \mathcal{J}_{-1,-1}^{y;m,n,o}, S_{y,o+1,o} \right\} \right)
 \end{aligned}$$

with

$$\begin{aligned}
 \mathcal{R}_{\sigma_1, \sigma_2}^{\alpha;m,n,o} &:= \text{Re}[\ell_{n-1}^* \ell_o] S_{\alpha;m,n-1} + \sigma_1 \text{Re}[\ell_n^* \ell_o] S_{\alpha;m,n+1} \\
 &\quad + \sigma_2 \text{Re}[\ell_{m-1}^* \ell_o] S_{\alpha;n,m-1} + \sigma_1 \sigma_2 \text{Re}[\ell_m^* \ell_o] S_{\alpha;n,m+1}, \\
 \mathcal{J}_{\sigma_1, \sigma_2}^{\alpha;m,n,o} &:= \text{Im}[\ell_{n-1}^* \ell_o] S_{\alpha;m,n-1} + \sigma_1 \text{Im}[\ell_n^* \ell_o] S_{\alpha;m,n+1} \\
 &\quad + \sigma_2 \text{Im}[\ell_{m-1}^* \ell_o] S_{\alpha;n,m-1} + \sigma_1 \sigma_2 \text{Im}[\ell_m^* \ell_o] S_{\alpha;n,m+1}.
 \end{aligned}$$

Supplemental Material: Numerical Method

Permutation invariance: $\pi \mathcal{L}[\rho] \pi^\dagger = \mathcal{L}[\pi \rho \pi^\dagger]$ for all $\pi \in S_N$

- Assume $\rho = \pi \rho \pi^\dagger$ for all $\pi \in S_N$

- Write ρ as vector in basis

$$\begin{aligned} & [n_{-j,-j}, n_{-j,-j+1}, \dots, n_{j,j}] \\ & \sim \sum_{\pi \in S_N} \pi | -j \rangle \langle -j |^{\otimes n_{-j,-j}} \otimes | -j \rangle \langle -j+1 |^{\otimes n_{-j,-j+1}} \otimes \dots \otimes | j \rangle \langle j |^{\otimes n_{j,j}} \pi^\dagger \end{aligned}$$

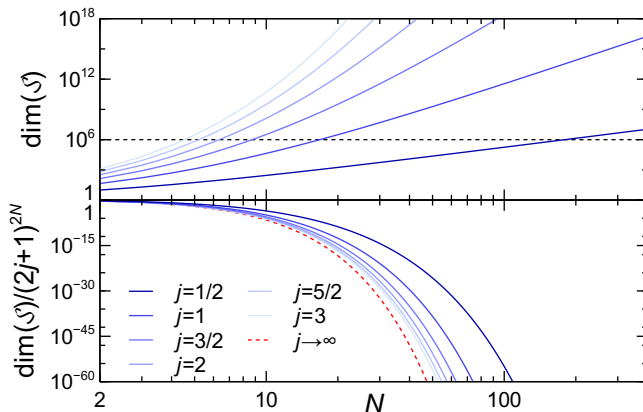
- Dimension of ρ : $\binom{N+4j(j+1)}{N}$ instead of $(2j+1)^N \times (2j+1)^N$

M. Gegg and M. Richter, NJP **18**, 043037 (2016)

D. Huybrechts, PhD Thesis (Universiteit Antwerpen, 2021)

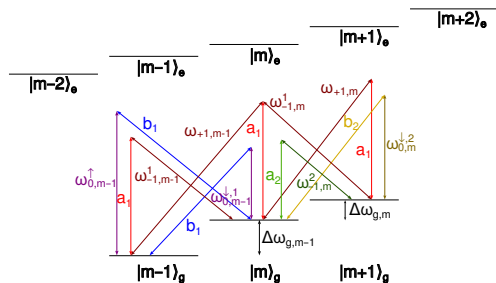
V. Sukharnikov et al., PRA **107**, 053707 (2023)

Supplemental Material: Permutation-Invariant Space



$$\mathcal{S} = \left\{ \rho \in \mathbb{C}^{(2j+1)^N \times (2j+1)^N} \mid \forall \pi \in S_N : \rho = \pi \rho \pi^\dagger \right\}$$

Supplemental Material: Proposal for Experiment



- N atoms with $d = 2j + 1$ ground states, $d + 2$ excited states
- 4 cavity modes $a_{1,2}, b_{1,2}$, $6(d - 1)$ driving fields.
- Off-resonant driving: effective ground-state transitions
- Dissipative cavity dynamics: effective interactions & dissipation of the atomic ground states

Example: Transition from $5S_{1/2}, F = 2$ to $5P_{3/2}, F' = 3$ in ^{87}Rb ,
 E. Suarez et al., PRA **107**, 023714 (2023)