ON PERIODIC ALTERNATE BASE EXPANSIONS

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The real base expansions of real numbers were introduced by Rényi [8]. Given a real base $\beta > 1$, a representation of a real number $x \in [0,1)$ is an infinite sequence $(a_n)_{n\in\mathbb{N}}$ of non-negative integer digits such that $x = \sum_{n=0}^{\infty} \frac{a_n}{\beta^{n+1}}$. Choosing at each step the largest possible digit a_n so that the partial sum $\sum_{k=0}^{n} \frac{a_k}{\beta^{k+1}}$ does not exceed x, we obtain one particular β-representation of x called the *β-expansion of x* and denoted by $d_\beta(x)$. Rényi observed that the digits of the β-expansion of x can also obtained by iterating the so-called *β-transformation* $T_\beta\colon [0,1) \to [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor$, where $\lfloor\cdot\rfloor$ denotes the floor function. More precisely, the computation of the n-th digit is given by the formula $a_n = \lfloor \beta T_\beta^n(x) \rfloor$. Then Rényi showed, among other things, that the map T_β defines an ergodic dynamical system. The dynamical properties of the β-expansions were extensively studied since the seminal work of Rényi.

In particular, the *β-shift S_β* received a lot of attention. This set is defined as the topological closure (with respect to the product topology on infinite words) of the set $\{d_\beta(x) : x \in [0,1)\}$. It is shift invariant and it defines a dynamical system that is measure theoretically isomorphic to the dynamical system built on T_β. Parry provided a combinatorial characterization of elements in the β-shift [7] involving one particular infinite word $d^*_\beta(1)$, which is nowadays called the *quasi-greedy β-expansion of* 1 and which is defined as the limit of the sequences $d_\beta(x)$ as x tends to 1^-, that is, $d^*_\beta(1) = \lim_{x\to 1^-} d_\beta(x)$. Ito and Takahashi then showed that the β-shift S_β is of finite type (which property they call *markovian*) if and only if $d^*_\beta(1)$ is purely periodic [6]. Further, Bertrand-Mathis showed that the β-shift S_β is sofic if and only it $d^*_\beta(1)$ is ultimately periodic [1]. From these results, we see the importance of the particular infinite word $d^*_\beta(1)$ in the study of β-expansions of Rényi. Nowadays, real bases β such that $d^*_\beta(1)$ is ultimately periodic are called *Parry numbers*.

In [9], Schmidt studied the set $\mathrm{Per}(\beta)$ of ultimately periodic points of the β-transformation T_β. In particular, his results imply that all Pisot numbers, i.e., algebraic integers $\beta > 1$ whose Galois conjugates (that is, the roots of the minimal polynomial of β) distinct from β all have modulus less than 1, are Parry numbers. The aim of the present paper is to understand the set of real numbers $x \in [0,1)$ having an ultimately periodic alternate base expansion.

Alternate expansions of real numbers are a generalization of Rényi β-expansions [2]. We give here the necessary background in order to state the generalization of Schmidt's result that we seek. An *alternate base* $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$ is a p-tuple of real bases, that is, $\beta_i > 1$ for every $i \in [\![0, p-1]\!]$ (throughout this text, an interval of integers $\{i, \ldots, j\}$ with $i \le j$ is denoted $[\![i, j]\!]$). A *$\boldsymbol{\beta}$-representation* of a real number x is an infinite sequence $a = (a_n)_{n\in\mathbb{N}}$

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of integers such that

$$x=\sum_{m=0}^{\infty}\sum_{i=0}^{p-1}\frac{a_{mp+i}}{(\beta_0\cdots\beta_{p-1})^m\beta_0\cdots\beta_i}. \tag{0.1}$$

We use the convention that for all $n\in\mathbb{N}$, $\beta_n=\beta_{n \bmod p}$ and

$$\boldsymbol{\beta}^{(n)}=(\beta_n,\dots,\beta_{n+p-1}).$$

For $x\in[0,1)$, a distinguished $\boldsymbol{\beta}$-representation $(\varepsilon_n)_{n\in\mathbb{N}}$, called the *$\boldsymbol{\beta}$-expansion* of x, is obtained from the *greedy algorithm*: set $r_0=x$ and, for $n\in\mathbb{N}$, $\varepsilon_n=\lfloor\beta_n r_n\rfloor$ and $r_{n+1}=\beta_n r_n-\varepsilon_n$. The $\boldsymbol{\beta}$-expansion of x is denoted $d_{\boldsymbol{\beta}}(x)$. The n-th digit ε_n belongs to $[\![0,\lceil\beta_n\rceil-1]\!]$. The number r_n is called the n-th *remainder* computed by the greedy algorithm. Note that the remainders all belong to $[0,1)$.

We let $\mathrm{Per}(\boldsymbol{\beta})$ denote the set of real numbers in $[0,1)$ having an ultimately periodic greedy $\boldsymbol{\beta}$-expansion, that is,

$$\mathrm{Per}(\boldsymbol{\beta})=\{x\in[0,1): d_{\boldsymbol{\beta}}(x) \text{ is ultimately periodic}\}. \tag{0.2}$$

As in the real base case, the digits of the $\boldsymbol{\beta}$-expansion may also be obtained by iterating a well-chosen transformation $T_{\boldsymbol{\beta}}$ [3]. The set $\mathrm{Per}(\boldsymbol{\beta})$ may then be seen, up to some technicalities, as the set of ultimately periodic points of this map $T_{\boldsymbol{\beta}}$.

We obtain the following result generalizing Schmidt's theorems [9, Theorems 2.4 and 3.1]. Recall that a *Salem number* is an algebraic integer $\beta>1$ whose Galois conjugates distinct from β all have modulus less than or equal to 1, with equality for at least one of them.

Theorem 1. *Let $\boldsymbol{\beta}=(\beta_0,\dots,\beta_{p-1})$ be an alternate base and set $\beta=\prod_{i=0}^{p-1}\beta_i$.*

1. *If $\mathbb{Q}\cap[0,1)\subseteq\bigcap_{i=0}^{p-1}\mathrm{Per}(\boldsymbol{\beta}^{(i)})$ then $\beta_0,\dots,\beta_{p-1}\in\mathbb{Q}(\beta)$ and β is either a Pisot number or a Salem number.*
2. *If β is a Pisot number and $\beta_0,\dots,\beta_{p-1}\in\mathbb{Q}(\beta)$ then $\mathrm{Per}(\boldsymbol{\beta})=\mathbb{Q}(\beta)\cap[0,1)$.*

Our proof of Theorem 1 is based on algebraic tools such as the alternate base spectrum defined in [4] as a generalization of the β-spectrum originally introduced by Erdős, Joó and Komornik [5]. In the reduced case of one real base, we obtain a proof that is much shorter than Schmidt's original one from [9].

We call $\boldsymbol{\beta}$ a *Parry alternate base* if $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ is eventually periodic for every $i\in[\![0,p-1]\!]$. As a direct consequence of Theorem 1, we reobtain the above-mentioned result from [4] generalizing the fact that all Pisot numbers are Parry numbers.

Corollary 2. *Let $\boldsymbol{\beta}=(\beta_0,\dots,\beta_{p-1})$ be an alternate base and set $\beta=\prod_{i=0}^{p-1}\beta_i$. If β is a Pisot number and $\beta_0,\dots,\beta_{p-1}\in\mathbb{Q}(\beta)$ then $\boldsymbol{\beta}$ is a* Parry alternate base.

As a second corollary, we obtain the following property of Pisot numbers. This result seems to be new; at least we were not able to find a reference for it.

Corollary 3. *If β is a Pisot number then $\beta\in\mathbb{Q}(\beta^p)$ for all $p\in\mathbb{N}_{\geq 1}$.*

We also prove the following theorem generalizing [9, Theorem 2.5]. This result is a refinement of the item (1) of Theorem 1.

Theorem 4. *Let $\boldsymbol{\beta}=(\beta_0,\dots,\beta_{p-1})$ be an alternate base such that $\beta_0,\dots,\beta_{p-1}\in\mathbb{Q}(\beta)$ and set $\beta=\prod_{i=0}^{p-1}\beta_i$. If β is an algebraic integer that is neither a Pisot number nor a Salem number then* $\mathrm{Per}(\boldsymbol{\beta})\cap\mathbb{Q}$ *is nowhere dense in* $[0,1)$.

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