SUBSTITUTIONS AND CANTOR REAL NUMERATION SYSTEMS

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Abstract. We consider Cantor real numeration system as a frame in which every non-negative real number has a positional representation. The system is defined using a bi-infinite sequence $B = (\beta_n)_{n\in\mathbb{Z}}$ of real numbers greater than one. We introduce the set of B-integers and code the sequence of gaps between consecutive B-integers by a symbolic sequence in general over the alphabet $\mathbb{N}$. We show that this sequence is S-adic. We focus on alternate base systems, where the sequence B of bases is periodic, and characterize alternate bases B in which B-integers can be coded by using a symbolic sequence $\mathbf{v}_B$ over a finite alphabet. With these so-called Parry alternate bases we associate some morphisms and show that $\mathbf{v}_B$ is a fixed point of their composition. We then provide two classes of Parry alternate bases B generating sturmian fixed points. The paper generalizes results of Fabre and Burdík et al. obtained for the Rényi numerations systems, i.e., in the case when the Cantor base B is a constant sequence.

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1. Introduction

In the pursuit of a generalization of the famous Cobham's theorem, Fabre [16] has associated substitutions with numeration systems with one real base β which is a Parry number. Such substitutions have for their fixed points infinite sequences which are sometimes called Parry words, or k-bonacci-like words. They include the famous Fibonacci sequence corresponding to the golden ratio numeration system. Although Fabre himself viewed the substitutions in the context of automata recognizing the language of an associated linear numeration system, it can be derived already from Thurston [24] that the substitutions may serve for generating the set of numbers with integer expansion in base β, the so-called β-integers. These numbers were identified as a suitable tool in the description of mathematical models of quasicrystals [9], since their arithmetic and diffractive properties generalize the arithmetic and diffractive properties of ordinary integers which give an underlying structure of periodic crystals. Certain physical properties of materials such as electron conductivity can be derived from the character of the spectrum of a Schrödinger operator associated with the model. As identified in the works of Damanik [14, 15], the spectrum depends on combinatorial properties of infinite sequences coding distances in the β-integers, such as the factor complexity, palindromicity, repetition index, etc. Other physical properties can be derived from their geometric behavior [1, 3, 21].

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First studies of combinatorics of Parry sequences date back to the paper of Burdík et al. [9]. More general results about their factor complexity can be found in [17, 5]. Palindromic complexity is the subject of [2]. Turek in [25] gives a tool to compute abelian complexity and balances of Parry words. The balance properties have been used for determining arithmetical features of numeration systems [4]. Recently, Gheeraert et al. [19, 20] study their string attractors.

Our aim is to generalize the notion of β-integers and corresponding symbolic sequences to positional numeration systems defined using multiple bases. The so-called Cantor real numeration system was introduced recently in [12] for representations of numbers in the interval $[0, 1)$. The system is given by a sequence $B = (\beta_n)_{n\geq 1}$ of real bases $\beta_n > 1$. B-expansion of a number $x \in [0, 1)$ is the lexicographically greatest string $(x_n)_{n\geq 1}$ of non-negative integers such that $x = \sum_{n=1}^{+\infty} \frac{x_n}{\beta_1\beta_2\cdots\beta_n}$. The Rényi numeration system can be viewed as the special case when the sequence B is constant. The adjective "Cantor real" added to the name of numeration system is linked to the fact that the idea of using multiple bases comes already from Cantor [10], who however, required all the bases β_n to be integers. When the Cantor real base B is a periodic sequence, we speak of alternate base systems.

In our paper, we first modify the definition of a Cantor real system so that it represents also positive numbers outside of the unit interval, see Section 3. For that, we consider a bi-infinite sequence $B = (\beta_n)_{n\in\mathbb{Z}}$. Consequently, in Section 4, we introduce the set of B-integers and determine the distances between its consecutive elements. The sequence of distances between consecutive B-integers is coded by an inifnite word. Its properties are studied in Section 5. For the special case of alternate base systems, we study in Section 6 when the distances between consecutive B-integers take finitely many values. It turns out that this happens precisely when the base is a Parry alternate base, as defined in [13]. Then the infinite symbolic sequence coding the distances between B-integers is a fixed point of a primitive substitution. In Section 7, we build an automaton associated with a Parry alternate base accepting the language of the B-shift and which encodes all the notions studied in Section 6, namely the possible distances obtained for each shift of the base as well as every substitution used to construct the primitive substitution mentioned previously. In order to give intuition on our results from Sections 4 to 7, we refer the reader to Section 8, where we use a running example in order to illustrate the concepts under study and the obtained results. This section also contains two smaller examples in order to illustrate phenomena not seen in the running example. Section 9 presents an alternative definition of the set of B-integers based on the dynamical point of view. In Section 10 we describe the bases B which yield a sturmian infinite sequence.

2. Combinatorics on words

An *alphabet* $\mathcal{A}$ is a set of symbols. In this paper we consider both finite and infinite alphabets, usually $\mathcal{A} \subset \mathbb{N}$. The set of all finite words over the alphabet $\mathcal{A}$, equipped with the empty word ε and the operation of concatenation, is a monoid denoted by $\mathcal{A}^*$.

We consider right-sided infinite words $\mathbf{v} = a_0a_1a_2\cdots \in \mathcal{A}^{\mathbb{N}}$ with $a_i \in \mathcal{A}$. Such an infinite word is *purely periodic* if it can be written as an infinite concatenation of a single finite word $w \in \mathcal{A}^*$. We write $\mathbf{v} = www\cdots = w^\omega$. The infinite word $\mathbf{v}$ is *eventually periodic* if there exist words $u, w \in \mathcal{A}^*$ such that $\mathbf{v} = uw^\omega$.

Given alphabets $\mathcal{A}$, $\mathcal{B}$, we consider endomorphisms $\sigma\colon \mathcal{A}^* \to \mathcal{B}^*$ and usually call them simply morphisms. The action of a morphism can be extended to right-sided infinite

sequences by

$$\sigma(a_0a_1a_2\cdots) = \sigma(a_0)\sigma(a_1)\sigma(a_2)\cdots .$$

A *substitution* σ over $\mathcal{A}$ is a morphism $\sigma\colon \mathcal{A}^* \to \mathcal{A}^*$ such that there is a letter $a \in \mathcal{A}$ with $\sigma(a) = aw$ for $w \in \mathcal{A}^*$, $w \neq \varepsilon$. An infinite sequence $\mathbf{v} \in \mathcal{A}^{\mathbb{N}}$ is a *fixed point* of a substitution σ if $\sigma(\mathbf{v}) = \mathbf{v}$. It is easy to see that a substitution σ has always a fixed point, namely $\lim_{i\to\infty} \sigma^i(a)$, where the limit is considered over the product topology.

The incidence matrix of a morphism $\sigma\colon \mathcal{A}^* \to \mathcal{B}^*$ is a matrix M_σ with columns numbered by the letters of the alphabet $\mathcal{A}$ and rows numbered by the letters of $\mathcal{B}$ such that for each pair of letters $a \in \mathcal{A}$, $b \in \mathcal{B}$ the entry $M_{b,a}$ is equal to the number of occurrences of the letter b in the image $\sigma(a)$. For two morphisms $\sigma\colon \mathcal{A}^* \to \mathcal{B}^*, \eta\colon \mathcal{B}^* \to \mathcal{C}^*$ it holds that $M_{\eta\sigma} = M_\eta M_\sigma$. The incidence matrix of a substitution σ is a square matrix and we say that σ is *primitive* if M_σ is primitive, i.e., if M_σ^n is a positive matrix for some $n \geq 1$.

An S-adic expansion of an infinite word $\mathbf{w}$ is a sequence $((\sigma_n, a_n))_{n\in\mathbb{N}}$, where $\sigma_n\colon \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$ are morphisms and a_n are letters belonging to $\mathcal{A}_n$, such that

$$\mathbf{w} = \lim_{n\to+\infty} \sigma_0\sigma_1\cdots\sigma_{n-1}(a_n). \tag{2.1}$$

We then say that $\mathbf{w}$ is an S-adic sequence. For more details about S-adic systems see [7].

3. Two-way Cantor real bases

In this section, we generalize the framework of [12] in order to be able to represent all non-negative real numbers. Let $B = (\beta_n)_{n\in\mathbb{Z}}$ be a bi-infinite sequence of real numbers greater than one and such that $\prod_{n=0}^{+\infty} \beta_n = +\infty$ and $\prod_{n=1}^{+\infty} \beta_{-n} = +\infty$. We call such a sequence a *two-way Cantor real base*. For the positional representation of numbers in base B, we consider two-sided pointed infinite sequences $\mathbf{a} = (a_n)_{n\in\mathbb{Z}} = \cdots a_2a_1a_0 \bullet a_{-1}a_{-2}a_{-3}\cdots$. The associated value map is defined only on those sequences $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ in $\mathbb{N}^{\mathbb{Z}}$ for which there exists some $N \in \mathbb{Z}$ such that $a_n = 0$ for all $n \geq N$. For such a sequence $\mathbf{a}$, we define

$$\mathrm{val}_B(\mathbf{a}) = \sum_{n=0}^{N-1} a_n\beta_{n-1}\cdots\beta_0 + \sum_{n=1}^{\infty} \frac{a_{-n}}{\beta_{-1}\cdots\beta_{-n}}, \tag{3.1}$$

provided that the infinite sum defines a convergent series. In this case, we say that $\mathbf{a}$ is a *B-representation* of the so-obtained non-negative real number $\mathrm{val}_B(\mathbf{a})$. The sequence $\mathbf{a}$ is usually written as

$$\begin{cases} a_{N-1}\cdots a_0 \bullet a_{-1}a_{-2}\cdots & \text{if } N \geq 1; \\ 0 \bullet 0^{-N}a_{N-1}a_{N-2}\cdots & \text{if } N \leq 0, \end{cases}$$

that is, we omit the coefficients a_n for $n \geq \max\{N, 1\}$.

For a two-way real Cantor base $B = (\beta_n)_{n\in Z}$, we write $S(B)$ the two-way real Cantor base $S(B) = (\beta_{n+1})_{n\in\mathbb{Z}}$. Thus, we have

$$\mathrm{val}_B(a_{N-1}\cdots a_0 \bullet a_{-1}a_{-2}\cdots) = \beta_0 \cdot \mathrm{val}_{S(B)}(a_{N-1}\cdots a_1 \bullet a_0a_{-1}a_{-2}\cdots). \tag{3.2}$$

For $N \in \mathbb{N}$, we let S^N be the N-th iterate of S, i.e., $S^N(B) = (\beta_{n+N})_{n\in\mathbb{Z}}$. We extend this definition to all integers $N \in \mathbb{Z}$ in a natural fashion simply by setting $S^N(B) = (\beta_{n+N})_{n\in\mathbb{Z}}$.

In general, a non-negative real number x can have more than one B-representation. Among all of these, we consider the greedy one defined as follows.

First, consider the case where $x \in [0, 1)$. We use the greedy algorithm defined in the one-way Cantor base $(\beta_{-1}, \beta_{-2}, \ldots)$ as introduced in [12]: first, set $r_{-1} = x$, and then, for

$n \leq -1$, iteratively compute $a_n = \lfloor \beta_n r_n \rfloor$ and $r_{n-1} = \beta_n r_n - a_n$. We define

$$d_B(x) = a_{-1}a_{-2}\cdots .$$

In the two-way Cantor real base B, the *B-expansion* of $x \in [0,1)$ is then denoted by

$$\langle x \rangle_B = 0 \bullet d_B(x) = 0 \bullet a_{-1}a_{-2}\cdots .$$

Since we have assumed that $\prod_{n=1}^{+\infty} \beta_{-n} = +\infty$, the series on the right side of (3.1) converges and $\mathrm{val}_B(\langle x \rangle_B) = x$.

Next, we consider the case where $x \geq 1$. In this case, we will need to use the bases β_n for $n \geq 0$ as well. Let $N \geq 1$ be the minimal integer such that $x < \beta_{N-1}\cdots\beta_0$. Note that such an integer exists since we have assumed that $\prod_{n=0}^{+\infty} \beta_n = +\infty$. We compute the $S^N(B)$-expansion of $\frac{x}{\beta_{N-1}\cdots\beta_0}$. If the obtained expansion is $0 \bullet a_{N-1}a_{N-2}\cdots$, then the *$B$-expansion* of x is denoted by

$$\langle x \rangle_B = a_{N-1}\cdots a_0 \bullet a_{-1}a_{-2}\cdots .$$

In particular, the obtained digits a_n belong to the alphabet $\{0, \ldots, \lceil \beta_n \rceil - 1\}$ for all n. Also note that the B-expansion of 1 is always $\langle 1 \rangle_B = 1 \bullet 0^\omega$.

Let us collect some properties of the B-expansions. These properties are straightforward consequences of analogue results in [12] for one-way Cantor real bases.

Lemma 1. *Let $B = (\beta_n)_{n\in\mathbb{Z}}$ be a two-way Cantor real base. A sequence $(a_n)_{n\in\mathbb{Z}}$ over $\mathbb{N}$, with $a_n = 0$ for sufficiently large positive n, is the B-expansion of some non-negative real number if and only if we have*

$$\sum_{n=0}^{k} a_n \beta_{n-1}\cdots\beta_0 + \sum_{n=1}^{\infty} \frac{a_{-n}}{\beta_{-1}\cdots\beta_{-n}} < \beta_k\cdots\beta_0 \quad \textit{for all } k \geq 0$$

and

$$\sum_{n=k}^{\infty} \frac{a_{-n}}{\beta_{-1}\cdots\beta_{-n}} < \frac{1}{\beta_{-1}\cdots\beta_{-k+1}} \quad \textit{for all } k \geq 1.$$

Let us define a total order on the B-representations of non-negative real numbers. Let $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ and $\mathbf{b} = (b_n)_{n\in\mathbb{Z}}$ be sequences in $\mathbb{N}^{\mathbb{Z}}$, for which there exist some $N \in \mathbb{Z}$ such that $a_n = b_n = 0$ for all $n \geq N$. We say that $\mathbf{a}$ is less than or equal to $\mathbf{b}$ in the *radix order* if $a_{N-1}a_{N-2}a_{N-3}\cdots$ is lexicographically less than or equal to $b_{N-1}b_{N-2}b_{N-3}\cdots$. In this case, we write $\mathbf{a} \leq_{\mathrm{rad}} \mathbf{b}$. As usual, we write $\mathbf{a} <_{\mathrm{rad}} \mathbf{b}$ if $\mathbf{a} \leq_{\mathrm{rad}} \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. Note that the radix order is defined on two-way sequences, whereas the lexicographic order compares right-sided infinite sequences.

Lemma 2. *Let $B = (\beta_n)_{n\in\mathbb{Z}}$ be a two-way Cantor real base.*

1. *For all non-negative real numbers x and y, we have $x < y$ if and only if $\langle x \rangle_B <_{\mathrm{rad}} \langle y \rangle_B$.*
2. *The B-expansion of a non-negative real number is maximal in the radix order among all its B-representations.*

4. B-integers

We are interested in the analogue of the set $\mathbb{N}$ in the context of Cantor real bases.

Definition 3. Let $B = (\beta_n)_{n\in\mathbb{Z}}$ be a two-way Cantor real base. A non-negative real number x is called a *B-integer* if its B-expansion is of the form

$$\langle x \rangle_B = a_{N-1}\cdots a_0 \bullet 0^\omega$$

with $N \in \mathbb{N}$. Otherwise stated, its B-expansion has only zeros after the radix point. If a_{N-1} is not zero or $x = 0$, then we say that N is the *length* of the B-expansion of x. The set of all B-integers is denoted by $\mathbb{N}_B$.

In the case where $\beta_n = \beta$ for all $n \in \mathbb{N}$, the set of B-integers coincides with the classical β-integers introduced by Gazeau [18] and widely studied, see for instance [9, 16]. Moreover, note that $\mathbb{N}_B = \mathbb{N}$ if and only if all products $\prod_{i=0}^{n} \beta_i$ are integers.

Clearly, the set of B-integers is unbounded. Let us show that it is also a discrete set.

Proposition 4. *The set $\mathbb{N}_B$ has no accumulation point in $\mathbb{R}$.*

Proof. It suffices to see that for all $n \geq 0$, the set of B-integers that are smaller than $\beta_{n-1} \cdots \beta_0$ is finite. The B-expansion of any such B-integer is of the form $a_m a_{m-1} \cdots a_0 \bullet 0^\omega$ with $m \leq n$. Each digit a_i being bounded by β_i, there are only finitely many B-expansions having this property. □

In view of this result, there exists an increasing sequence $(x_k)_{k\in\mathbb{N}}$ such that

$$\mathbb{N}_B = \{x_k : k \in \mathbb{N}\}. \tag{4.1}$$

In particular, we have $x_0 = 0$ and $x_1 = 1$, and $\lim_{k\to+\infty} x_k = +\infty$.

In the paper [12], the admissible Cantor base expansions were characterized. In order to be able to exploit the results from [12], we extend the definition of quasi-greedy expansions of 1 to two-way Cantor real bases. For $B = (\beta_n)_{n\in\mathbb{Z}}$ a two-way Cantor real base, we define the *quasi-greedy expansion of* 1 as the sequence

$$d_B^*(1) := \lim_{x\to 1^-} d_B(x),$$

where the limit is taken with respect to the product topology. The sequence $d_B^*(1)$ contains infinitely many non-zero digits and it evaluates to 1, meaning that $\mathrm{val}_B(0 \bullet d_B^*(1)) = 1$.

Theorem 5. [12, Theorem 26] *A sequence $0 \bullet a_{-1}a_{-2}\cdots$ of non-negative integers is the B-expansion of some number in $[0, 1)$ if and only if $a_{n-1}a_{n-2}\cdots <_{\mathrm{lex}} d_{S^n(B)}^*(1)$ for all indices $n \leq 0$.*

Note that the inequality in the previous theorem is also valid for positive indices n since for $n > 0$, the digit a_{n-1} is zero and the leading digit of $d_{S^n(B)}^*(1)$ is positive.

For a number $x > 0$, we have

$$\langle x\rangle_B = a_{N-1}\cdots a_0 \bullet a_{-1}a_{-2}\cdots \text{ if and only if } \langle \tfrac{x}{\beta_0}\rangle_{S(B)} = a_{N-1}\cdots a_1 \bullet a_0 a_{-1}\cdots . \tag{4.2}$$

This fact easily follows from the definition of $S(B)$ and the greedy algorithm. Therefore, Theorem 5 generalizes to the context of two-way Cantor real bases.

Corollary 6. *A sequence $a_{N-1}\cdots a_0 \bullet a_{-1}a_{-2}\cdots$ of non-negative integers is the B-expansion of some non-negative real number if and only if $a_{n-1}a_{n-2}\cdots <_{\mathrm{lex}} d_{S^n(B)}^*(1)$ for all indices $n \leq N$.*

In view of these results, we see that we need to compare all shifts of a sequence to the quasi-greedy expansions of 1 with respect to the corresponding shifted bases $S^n(B)$. From now on, for all $n \in \mathbb{Z}$, we write

$$d_{S^n(B)}^*(1) = d_{n,1}d_{n,2}\cdots . \tag{4.3}$$

Observe that unlike previously, we have increasing indices as we read from left to right. By results of [12], we know that

$$d_{n,\ell+1}d_{n,\ell+2}\cdots \leq_{\text{lex}} d^*_{S^{n-\ell}(B)}(1) = d_{n-\ell,1}d_{n-\ell,2}\cdots \tag{4.4}$$

for all $n \in \mathbb{Z}$ and all $\ell \in \mathbb{N}$. Therefore, for all $n \in \mathbb{Z}$, the sequence $d_{n,1}d_{n,2}\cdots d_{n,n} \bullet 0^\omega$ is the B-expansion of a B-integer.

Proposition 7. *Let $k \in \mathbb{N}$, x_k and x_{k+1} be elements of $\mathbb{N}_B$ as introduced in* (4.1)*. Write $\langle x_k\rangle_B = \cdots a_2a_1a_0 \bullet 0^\omega$ and $\langle x_{k+1}\rangle_B = \cdots b_2b_1b_0 \bullet 0^\omega$. Define n to be the maximal index such that $a_n \neq b_n$. Then*

1. $\langle x_k\rangle_B = \cdots a_{n+2}a_{n+1}a_nd_{n,1}d_{n,2}\cdots d_{n,n} \bullet 0^\omega$;
2. $\langle x_{k+1}\rangle_B = \cdots a_{n+2}a_{n+1}(a_n+1)0^n \bullet 0^\omega$;
3. $x_{k+1} - x_k = \mathrm{val}_B(0 \bullet d_{n,n+1}d_{n,n+2}d_{n,n+3}\cdots) \leq 1$.

Proof. Since x_{k+1} is a successor of x_k in $\mathbb{N}_B$, by Lemma 2, there is no integer sequence of the form $c_nc_{n-1}\cdots c_0 \bullet 0^\omega$ greater than $a_na_{n-1}\cdots a_0 \bullet 0^\omega$ and smaller than $b_nb_{n-1}\cdots b_0 \bullet 0^\omega$, with respect to the radix order, that satisfies the inequalities of Corollary 6. Consequently, $b_n\cdots b_0 = (a_n+1)0^n$. On the other hand, $a_{n-1}a_{n-2}\cdots a_0 \bullet 0^\omega$ must be the greatest sequence of this form in the radix order satisfying the inequalities of Corollary 6, which is $d_{n,1}d_{n,2}\cdots d_{n,n} \bullet 0^\omega$ thanks to (4.4).

Using (3.2) and the equality $\mathrm{val}_{S^n(B)}(0 \bullet d_{n,1}d_{n,2}\cdots) = 1$, we get that

$$\begin{aligned} x_{k+1} - x_k &= \mathrm{val}_B(10^n \bullet 0^\omega) - \mathrm{val}_B(d_{n,1}d_{n,2}\cdots d_{n,n} \bullet 0^\omega) \\ &= \beta_{n-1}\cdots\beta_0 - \mathrm{val}_B(d_{n,1}d_{n,2}\cdots d_{n,n} \bullet 0^\omega) \\ &= \mathrm{val}_B(d_{n,1}\cdots d_{n,n} \bullet d_{n,n+1}d_{n,n+2}\cdots) - \mathrm{val}_B(d_{n,1}d_{n,2}\cdots d_{n,n} \bullet 0^\omega) \\ &= \mathrm{val}_B(0 \bullet d_{n,n+1}d_{n,n+2}\cdots). \end{aligned}$$

Finally, note that adding one to the right-most digit $d_{n,n}$ in the greedy representation $d_{n,1}d_{n,2}\cdots d_{n,n} \bullet 0^\omega$ violates the greedy condition, which means that $\mathrm{val}_B(d_{n,1}d_{n,2}\cdots d_{n,n} \bullet 0^\omega) + 1 \geq \beta_{n-1}\cdots\beta_0$, and hence $x_{k+1} - x_k \leq 1$. □

Definition 8. For every $n \in \mathbb{N}$, we define

$$\Delta_n = \mathrm{val}_B(0 \bullet d_{n,n+1}d_{n,n+2}d_{n,n+3}\cdots).$$

In particular, we have $\Delta_0 = 1$ and $\Delta_n \leq 1$ for $n > 0$. The above proposition states that the distances between consecutive B-integers take values Δ_n accordingly to the position n where the B-expansions of these B-integers differ. At specific cases, it may happen that $\Delta_n = \Delta_{n'}$ even though $n \neq n'$.

Lemma 9. *If $n, n' \in \mathbb{N}$ are such that $\Delta_n = \Delta_{n'}$, then the string $d_{n,n+1}d_{n,n+2}d_{n,n+3}\cdots$ coincides with the string $d_{n',n'+1}d_{n',n'+2}d_{n',n'+3}\cdots$.*

Proof. It follows from the equality

$$d_{n,n+1}d_{n,n+2}\cdots = \lim_{x\to(\Delta_n)^-} d_B(x).$$

which is a simple consequence of (4.2). □

In view of Proposition 7, we encode the distances between consecutive B-integers by an infinite sequence.

Definition 10. With a two-way Cantor real base B, we associate a sequence $\mathbf{w}_B = (w_k)_{k\in\mathbb{N}}$ over $\mathbb{N}$ as follows: for all $k \in \mathbb{N}$, we let w_k be the greatest non-negative integer n such that $\langle x_k\rangle_B$ and $\langle x_{k+1}\rangle_B$ differ at index n.

If the set of distances $\{x_{k+1} - x_k : k \in \mathbb{N}\} = \{\Delta_n : n \in \mathbb{N}\}$ between B-integers is finite, then we can project $\mathbf{w}_B$ onto a sequence over a finite alphabet and study its substitutivity. To see the connection of this notion to the Rényi numeration system, we recall that Fabre [16] assigned to a Parry base β a primitive substitution over a finite alphabet. This substitution turned out to be the substitution fixing the β-integers defined by Gazeau [18]. We now present a generalization of these results.

We first show that the infinite sequence $\mathbf{w}_{S(B)}$ can be mapped to the infinite sequence $\mathbf{w}_B$ by using a substitution over the infinite alphabet $\mathbb{N}$. This will then allow us to prove that $\mathbf{w}_B$ is an S-adic sequence.

Definition 11. We define a substitution ψ_B over the alphabet $\mathbb{N}$ as follows:

$$\psi_B \colon n \mapsto 0^{d_{n+1,n+1}}(n+1).$$

Proposition 12. *We have* $\psi_B(\mathbf{w}_{S(B)}) = \mathbf{w}_B$.

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be the strictly increasing sequence for which $\mathbb{N}_B = \{x_k : k \in \mathbb{N}\}$ and let $(\tilde{x}_k)_{k\in\mathbb{N}}$ be the strictly increasing sequence for which $\mathbb{N}_{S(B)} = \{\tilde{x}_k : k \in \mathbb{N}\}$. Write $\mathbf{w}_B = w_0 w_1 \cdots$ and $\mathbf{w}_{S(B)} = \tilde{w}_0 \tilde{w}_1 \cdots$ where the w_k's and the $\tilde{w}_k$'s are letters in $\mathbb{N}$. Let $k \in \mathbb{N}$ and let $n = \tilde{w}_k$. By Proposition 7, we have $\tilde{x}_{k+1} - \tilde{x}_k = \mathrm{val}_{S(B)}(0 \bullet d_{n+1,n+1} d_{n+1,n+2} \cdots)$. Proposition 7 allows us to write

$$\begin{aligned} \langle \tilde{x}_k \rangle_{S(B)} &= a_{N-1} \cdots a_{n+1} \quad a_n \quad d_{n+1,1} \cdots d_{n+1,n} \bullet 0^\omega \\ \langle \tilde{x}_{k+1} \rangle_{S(B)} &= a_{N-1} \cdots a_{n+1} (a_n + 1) \quad 0 \quad \cdots \quad 0 \quad \bullet 0^\omega \end{aligned}$$

for some index $N > n$. By (4.2) the B-expansions of $\beta_0 \tilde{x}_k$ and $\beta_0 \tilde{x}_{k+1}$ are

$$\begin{aligned} \langle \beta_0 \tilde{x}_k \rangle_{S(B)} &= a_{N-1} \cdots \quad a_n \quad d_{n+1,1} \cdots d_{n+1,n}\, 0 \bullet 0^\omega \\ \langle \beta_0 \tilde{x}_{k+1} \rangle_{S(B)} &= a_{N-1} \cdots (a_n + 1) \quad 0 \quad \cdots \quad 0 \quad 0 \bullet 0^\omega. \end{aligned}$$

In particular, the numbers $\beta_0 \tilde{x}_k$ and $\beta_0 \tilde{x}_{k+1}$ belong to $\mathbb{N}_B$. Similarly as in the proof of Proposition 7, Lemma 2 combined with Corollary 6 and (4.4) imply that the B-expansion of every B-integer laying between $\beta_0 \tilde{x}_k$ and $\beta_0 \tilde{x}_{k+1}$ is of the form

$$a_{N-1} \cdots a_n d_{n+1,1} \cdots d_{n+1,n} b \bullet 0^\omega$$

where $b \in \{0, \ldots, d_{n+1,n+1}\}$. We have

$$\mathrm{val}_B(a_{N-1} \cdots a_n d_{n+1,1} \cdots d_{n+1,n} b \bullet 0^\omega) = \beta_0 \tilde{x}_k + b.$$

The letter assigned by Definition 10 to the distance between $\beta_0 \tilde{x}_k + b$ and $\beta_0 \tilde{x}_k + b + 1$ is 0 for each $b \in \{0, \ldots, d_{n+1,n+1} - 1\}$, whereas the letter assigned to the distance between $\beta_0 \tilde{x}_k + d_{n+1,n+1}$ and $\beta_0 \tilde{x}_{k+1}$ is $n+1$. Since k has been chosen arbitrarily, the sequence w_B is the infinite concatenation of the blocks $\psi_B(n) = \psi_B(\tilde{w}_k) = 0^{d_{n+1,n+1}}(n+1)$ for each k. Otherwise stated, we have $\psi_B(\mathbf{w}_{S(B)}) = \mathbf{w}_B$, as announced. □

From the proof of the previous proposition, we derive the following relation between distances Δ_n of consecutive B-integers and distances $\widetilde{\Delta}_n$ of consecutive $S(B)$-integers.

Corollary 13. *For all* $n \in \mathbb{N}$, *we have*

$$\beta_0 \widetilde{\Delta}_n = \Delta_{n+1} + d_{n+1,n+1}.$$

Proof. With the notation of the proof of Proposition 12, the distances between consecutive B-integers occurring in the interval $[\beta_0 \tilde{x}_k, \beta_0 \tilde{x}_{k+1}]$ take once the value Δ_{n+1} and $d_{n+1,n+1}$ times the value $\Delta_0 = 1$. The sum of these gives the length of the interval, namely $\beta_0(\tilde{x}_{k+1} - \tilde{x}_k) = \beta_0 \tilde{\Delta}_n$. □

We are now ready to prove that the infinite sequence associated with the B-integers in Definition 10 is an S-adic sequence. For all $m \in \mathbb{Z}$, we will denote the corresponding substitution $\psi_{S^m(B)} =: \psi_m$, that is,

$$\psi_m \colon n \mapsto 0^{d_{m+n+1,n+1}}(n+1). \tag{4.5}$$

Corollary 14. *The infinite sequence $\mathbf{w}_B$ is the S-adic sequence given by the sequence of substitutions $(\psi_m)_{m\in\mathbb{N}}$ applied on the letter 0, i.e.,*

$$\mathbf{w}_B = \lim_{m\to+\infty} \psi_0\psi_1\cdots\psi_{m-1}(0).$$

In particular if the sequence $(\psi_m)_{m\in\mathbb{N}}$ is purely periodic with period $p \geq 1$, then $\mathbf{w}_B$ is a fixed point of the substitution $\psi_0\cdots\psi_{p-1}$.

Proof. For all $m \in \mathbb{Z}$, we have $d_{m+1,1} \geq 1$ since $\beta_m > 1$. Therefore, the image of 0 under ψ_m has length at least 2 and hence the length of $\psi_0\psi_1\cdots\psi_{m-1}(0)$ increases as m grows to infinity. Moreover, it follows from Proposition 12 that $\psi_{m-1}(\mathbf{w}_{S^m(B)}) = \mathbf{w}_{S^{m-1}(B)}$ for all $m \in \mathbb{Z}$. Since each $\mathbf{w}_{S^m(B)}$ starts with the letter 0, this implies that for all $m \in \mathbb{N}$, the word $\psi_0\psi_1\cdots\psi_{m-1}(0)$ is a prefix of $\mathbf{w}_B$. Hence the sequence of finite words $(\psi_0\psi_1\cdots\psi_{m-1}(0))_{m\in\mathbb{N}}$ converges to the infinite sequence $\mathbf{w}_B$. □

In the next section we will examine the properties of the substitutions ψ_m. For this purpose, the following statement will be of use.

Proposition 15. *For each $m \in \mathbb{N}$, the word $\psi_0\cdots\psi_{m-1}(0)$ contains all letters $0, 1, \ldots, m$. Moreover, if $M \in \mathbb{N}$ is a constant such that for all $m \in \mathbb{N}$, the quasi-greedy expansions $d^*_{S^m(B)}(1)$ do not contain the factor 0^M, then for each $n \in \mathbb{N}$, the word $\psi_0\cdots\psi_{M-1}(n)$ starts with the letter 0.*

Proof. We will show the first statement by induction on m. For $m = 0$, it is obvious. For $m > 0$, we have $\psi_0\cdots\psi_{m-1}(0) = \psi_0\cdots\psi_{m-2}(0^{d_{m,1}}1)$. Since $d_{m,1} \geq 1$, the word $\psi_0\cdots\psi_{m-1}(0)$ has $\psi_0\cdots\psi_{m-2}(0)$ as a prefix, and therefore by induction hypothesis, it contains all letters $0, 1, \ldots, m-1$. But $\psi_0\cdots\psi_{m-1}(0)$ also has $\psi_0\cdots\psi_{m-2}(1)$ as a suffix, and from (4.5), it is clear that it ends with the letter m.

For the second statement, let M be as in the statement and fix some $n \in \mathbb{N}$. Realize that there exists $j \in \{0, 1, \ldots, M-1\}$ such that $d_{M+n,n+j+1} \neq 0$. Take the smallest such j. Then for any $n \in \mathbb{N}$, $\psi_{M-j}\cdots\psi_{M-1}(n) = n + j$. Then

$$\psi_0\cdots\psi_{M-1}(n) = \psi_0\cdots\psi_{M-j-1}(n+j) = \psi_0\cdots\psi_{M-j-2}\big(0^{d_{M+n,n+j+1}}(n+j+1)\big)$$

which has the prefix 0. □

In the following example we illustrate that the assumption concerning boundedness of the lengths of factors formed only from zeros is necessary for the validity of the second statement in Proposition 15.

Example 16. We consider the Cantor real base $(\beta_n)_{n\in\mathbb{Z}}$ with $\beta_n = 2$ for $n < 0$ and for $n \geq 0$, the elements β_n are given recursively by

$$\beta_n = 2 + \frac{1}{\beta_{n-1}\beta_{n-2}\cdots\beta_{-1}}.$$

It is easily seen that $\beta_{n-1}\beta_{n-2}\cdots\beta_{-1} = 3\cdot 2^n - 1$ for $n \geq 0$. The quasi-greedy expansions of 1 are of the form

$$d^*_{S^m(B)}(1) = \begin{cases} 1^\omega, & \text{for } m \leq 0; \\ 20^m1^\omega, & \text{for } m \geq 1. \end{cases}$$

According to (4.5), the substitutions ψ_m for $m \in \mathbb{N}$ are all equal, namely given by

$$\psi_m\colon \begin{cases} 0 \mapsto 001; \\ n \mapsto n+1, \quad \text{for } n \geq 1. \end{cases}$$

The infinite word $\mathbf{w}_B$ is therefore a fixed point of the substitution ψ_0. Obviously, we have $\psi_0 \cdots \psi_{m-1}(n) = \psi_0^m(n) = n + m$ does not contain the digit 0 for any $n \geq 1$ and any $m \geq 0$.

5. Faithful representation of B-integers

In Definition 10 we introduced the infinite word $\mathbf{w}_B = w_0 w_1 w_2 \cdots$ over the alphabet $\mathbb{N}$ which codes the sequence of distances between consecutive elements of $\mathbb{N}_B$. The equality $w_k = w_{k'}$ implies the equality $\Delta_{w_k} = \Delta_{w_{k'}}$, but not necessarily vice versa. Our aim is to find a sequence $\mathbf{v}_B$ that faithfully encodes the sequence of distances.

Definition 17. We say that a sequence $\mathbf{v}_B = v_0 v_1 v_2 \cdots$ over an alphabet $\mathcal{A} \subset \mathbb{N}$ is a faithful coding of the set $\mathbb{N}_B$ of B-integers if for any $k, k' \in \mathbb{N}$, it holds that $v_k = v_{k'}$ if and only if $\Delta_{v_k} = \Delta_{v_{k'}}$.

To identify letters indexed by i, j in $\mathbf{w}_B$ for which $\Delta_{w_i} = \Delta_{w_j}$, we use the projection morphism π_B defined over the alphabet $\mathbb{N}$ as

$$\pi_B(n) = \begin{cases} \pi_B(n'), & \text{if } \Delta_n = \Delta_{n'} \text{ for some } n' < n; \\ n, & \text{otherwise.} \end{cases}$$

Obviously, $\mathbf{v}_B := \pi_B(\mathbf{w}_B)$ is the faithful coding of B-integers. The letters of $\mathbf{v}_B$ take values in the alphabet

$$\mathcal{A}_B := \pi_B(\mathbb{N}).$$

Observe that this alphabet always contains the letter 0, but it is not necessarily made of consecutive integers as the following example shows.

Example 18. Consider the alternate base $B = (\beta_1, \beta_0)$ with $d_B^*(1) = 1101^\omega$ and $d_{S(B)}^*(1) = 21101^\omega$. This happens choosing β_1 to be the real root of $9x^3 - 15x^2 + 3x - 1$ and $\beta_0 = 3$. Denote by Δ_n distances between consecutive B-integers and by $\widetilde{\Delta}_n$ distances in $S(B)$-integers. According to Definition 8

$$\Delta_0 = \Delta_1, \quad \Delta_2 = \Delta_3, \quad \text{and} \quad \Delta_n = \Delta_4 \quad \text{for every } n \geq 4$$

Therefore, the alphabet of $\mathbf{v}_B$ is $\mathcal{A}_B = \{0, 2, 4\}$. On the other hand, since

$$\widetilde{\Delta}_0 > \widetilde{\Delta}_1 = \widetilde{\Delta}_2 \quad \text{and} \quad \widetilde{\Delta}_n = \widetilde{\Delta}_3 \quad \text{for every } n \geq 3,$$

the alphabet of $\mathbf{v}_{S(B)}$ is $\mathcal{A}_{S(B)} = \{0, 1, 3\}$.

Lemma 19. *Let ψ_B be the substitution introduced in Definition 11. Then*

$$\pi_B \, \psi_B \, \pi_{S(B)} = \pi_B \, \psi_B.$$

Proof. Let $(\Delta_n)_{n\in\mathbb{N}}$ and $(\tilde{\Delta}_n)_{n\in\mathbb{N}}$ be the sequences of distances assigned to $\mathbb{N}_B$ and $\mathbb{N}_{S(B)}$, respectively. Let $n \in \mathbb{N}$ and let $n' = \pi_{S(B)}(n)$. Then $n' \leq n$ and $\tilde{\Delta}_{n'} = \tilde{\Delta}_n$. We have to show that $\pi_B \, \psi_B(n') = \pi_B \, \psi_B(n)$. By Lemma 9 applied to the shifted base $S(B)$, we know that $d_{n'+1,n'+1} = d_{n+1,n+1}$. By Corollary 13, we get that $\Delta_{n'+1} = \Delta_{n+1}$, which in turns gives us that $\pi_B(n'+1) = \pi_B(n+1)$. Therefore

$$\pi_B \, \psi_B(n') = 0^{d_{n'+1,n'+1}} \pi_B(n'+1) = 0^{d_{n+1,n+1}} \pi_B(n+1) = \pi_B \, \psi_B(n)$$

as desired. □

Let

$$\varphi_B \colon \mathcal{A}_{S(B)}^* \to \mathcal{A}_B^*$$

be the morphism defined as the restriction of $\pi_B\psi_B$ to the monoid $\mathcal{A}_{S(B)}^*$.

Proposition 20. *We have $\varphi_B(\mathbf{v}_{S(B)}) = \mathbf{v}_B$.*

Proof. By the definitions of $\mathbf{v}_B$ and $\mathbf{v}_{S(B)}$, Proposition 12 and Lemma 19, we get

$$\mathbf{v}_B = \pi_B(\mathbf{w}_B) = \pi_B\psi_B(\mathbf{w}_{S(B)}) = \pi_B\psi_B\pi_{S(B)}(\mathbf{w}_{S(B)}) = \pi_B\psi_B(\mathbf{v}_{S(B)}) = \varphi_B(\mathbf{v}_{S(B)})$$

as announced. □

For $m \in \mathbb{Z}$, we let

$$\mathcal{A}_m := \mathcal{A}_{S^m(B)}, \quad \pi_m := \pi_{S^m(B)} \colon \mathbb{N}^* \to \mathcal{A}_m^* \quad \text{and} \quad \varphi_m := \varphi_{S^m(B)} \colon \mathcal{A}_{m+1}^* \to \mathcal{A}_m^*. \tag{5.1}$$

Corollary 21. *The infinite word $\mathbf{v}_B$ has the S-adic expansion $((\varphi_m, 0))_{m\in\mathbb{N}}$, i.e., we have*

$$\mathbf{v}_B = \lim_{m\to\infty} \varphi_0\varphi_1\cdots\varphi_{m-1}(0).$$

In particular, if the sequence of morphisms $(\varphi_m)_{m\in\mathbb{N}}$ is purely periodic, say with period p, then $\mathbf{v}_B$ is a fixed point of the substitution $\varphi_0\cdots\varphi_{p-1}$.

Proof. We derive from Lemma 19 that for each $m \geq 1$, the composition $\varphi_0\varphi_1\cdots\varphi_{m-1}$ coincides with the restriction of $\pi_0\psi_0\psi_1\cdots\psi_{m-1}$ to the monoid $\mathcal{A}_m^*$. The result then follows from Corollary 14. □

Example 22. Consider the Cantor real base from Example 16. The distances Δ_n, $n \in \mathbb{N}$, from Definition 8 take values $\Delta_0 = 1$ and $\Delta_n = \mathrm{val}_B(0 \bullet 01^\omega)$ for $n \geq 1$. The faithful coding $\mathbf{v}_B = \pi_B(\mathbf{w}_B)$ is therefore an infinite word over the alphabet $\{0, 1\}$. Moreover, since $\psi_m = \psi_0$ for $m \in \mathbb{N}$, we get that $\mathbf{v}_B$ is a fixed point of the substitution $\varphi = \pi_0\psi_0$, namely $\varphi \colon 0 \mapsto 001,\ 1 \mapsto 1$. This substitution is not primitive and its fixed point

$$\mathbf{v}_B = 0010011001001110010011001001111\cdots$$

is not uniformly recurrent since it contains arbitrarily long blocks of consecutive 1's.

In the previous example, we see that there can be only finitely many possible distances between consecutive B-integers. We show that if this happens for the base B then it also happens for all its shifts $S^m(B)$.

Proposition 23. *Suppose that the distances between consecutive B-integers take finitely many values. Then so do the distances between consecutive $S^m(B)$-integers for all $m \in \mathbb{Z}$.*

Proof. It is sufficient to show this for $m \in \{-1, 1\}$ (we can then do inductions to the left and to the right). In view of Lemma 9, it is enough to show that the finiteness of the set of strings $D_0 = \{d_{n,n+1}d_{n,n+2}\cdots : n \in \mathbb{N}\}$ implies the finiteness of the sets $D_{-1} = \{d_{n-1,n+1}d_{n-1,n+2}\cdots : n \in \mathbb{N}\}$ and $D_1 = \{d_{n+1,n+1}d_{n+1,n+2}\cdots : n \in \mathbb{N}\}$. The set D_{-1} is clearly finite if D_0 is finite since for $n \geq 1$, $d_{n-1,n+1}d_{n-1,n+2}\ldots$ is the first shift of $d_{n-1,n}d_{n-1,n+2}\ldots \in D_0$. In order to show the finiteness of D_1, we need to bound the first letter of its strings. Using (4.4), we have that $d_{n+1,n+1} \leq d_{1,1}$ for each $n \geq 0$, hence there are only finitely many strings in D_1 if D_0 is finite. □

6. B-integers in alternate bases

A two-way Cantor real base $B = (\beta_n)_{n\in\mathbb{Z}}$ is called an *alternate base* if it is periodic, that is, if there exists a positive integer p such that $\beta_{n+p} = \beta_n$ for all $n \in \mathbb{Z}$. In this case, we simply denote $B = (\beta_{p-1}, \dots, \beta_0)$ and the integer p is called the *period* of B.

Definition 24. An alternate base B is called *Parry* if the sequence $d^*_{S^i(B)}(1)$ is eventually periodic for every $i \in \{0, \dots, p-1\}$.

The above notion allows us to characterize alternate bases B for which the set of B-integers has a finite number of distances between consecutive elements.

Theorem 25. *Let $B = (\beta_{p-1}, \dots, \beta_0)$ be an alternate base. Then the set of distances between consecutive B-integers is finite if and only if the alternate base B is Parry. In such a case, the infinite word $\mathbf{v}_B$ faithfully coding the B-integers is a fixed point of the primitive substitution $\varphi_0 \cdots \varphi_{p-1}$.*

Proof. By Proposition 7, we have to show that the set

$$D = \{\mathrm{val}_B(0 \bullet d_{n,n+1}d_{n,n+2}\cdots) : n \in \mathbb{N}\}$$

is finite if and only if B is Parry. For $i \in \{0, \dots, p-1\}$ we let

$$D_i = \{\mathrm{val}_B(0 \bullet d_{i,n+1}d_{i,n+2}\cdots) : n \in \mathbb{N},\ n \equiv i \bmod p\}.$$

Since $D = \cup_{i=0}^{p-1} D_i$, the set D is finite if and only if D_i is finite for every i. We will show that for every i, the sequence $d^*_{S^i(B)}(1)$ is eventually periodic if and only if D_i is finite. Let us fix $i \in \{0, \dots, p-1\}$.

First, suppose that $d^*_{S^i(B)}(1)$ is eventually periodic. Without loss of generality, we may assume that there exist $\ell \geq 0$ and $m \geq 1$ such that $d^*_{S^i(B)}(1)$ has preperiod ℓp and period mp. Then for all $n \geq \ell p$ such that $n \equiv i \bmod p$, we have $d_{i,n+1}d_{i,n+2}\cdots = d_{i,n+mp+1}d_{i,n+mp+2}\cdots$. Hence, the set D_i has at most $\ell + m$ elements.

Conversely, suppose that the set D_i is finite. Then there exist $n, n' \in \mathbb{N}$ such that $n < n'$, $n \equiv i \bmod p$, $n' \equiv i \bmod p$ and $\Delta_n = \mathrm{val}_B(0{\bullet}d_{i,n+1}d_{i,n+2}\cdots) = \mathrm{val}_B(0{\bullet}d_{i,n'+1}d_{i,n'+2}\cdots) = \Delta_{n'}$. By Lemma 9, we have equality of the strings $d_{i,n+1}d_{i,n+2}\cdots = d_{i,n'+1}d_{i,n'+2}\cdots$, which means that the sequence $d_{i,n+1}d_{i,n+2}\cdots$ is purely periodic and thus $d^*_{S^i(B)}(1) = d_{i,1}d_{i,1}\cdots$ is eventually periodic.

Since B is an alternate base with period p, the sequence of morphisms $(\varphi_m)_{m\in\mathbb{Z}}$ defined in (5.1) is also periodic with period p. Corollary 21 implies that $\mathbf{v}_B$ is a fixed point of the substitution $\varphi = \varphi_0 \cdots \varphi_{p-1}$ over the finite alphabet $\mathcal{A}_0$. Note that the size of $\mathcal{A}_0$ is precisely the number of possible distances between consecutive B-integers. It remains to be shown that the substitution φ is primitive. Since each φ_i is just a restriction of ψ_i to a finite alphabet, Proposition 15 implies that for a sufficiently large k the image of $\varphi^k(0)$ contains all letters occurring in the fixed point of φ. As B is an alternate base, the assumption of the second part of Proposition 15 is satisfied as well. It means that for sufficiently large k the image $\varphi^k(n)$ of every letter n occurring in the fixed point of φ contains 0. In other words, the first column and the first row of the incidence matrix of the substitution φ^k are positive. Consequently each entry of the incidence matrix of the substitution φ^{2k} is positive, i.e., the substitution φ is primitive. □

We conclude this section by proving an important consequence of Theorem 25 about the uniqueness of the alternate base. We first prove a lemma which is of interest by itself.

Lemma 26. *Let $B = (\beta_{p-1}, \dots, \beta_0)$ be a Parry alternate base. The Perron–Frobenius eigenvalue of the incidence matrix of the substitution $\varphi_0 \cdots \varphi_{p-1}$ is given by the product $\beta_0 \cdots \beta_{p-1}$ of the bases.*

Proof. For each $i \in \{0, \dots, p-1\}$, we denote by $\Delta^{(i)} = (\Delta_{i,a} : a \in \mathcal{A}_i)$ the row vector (indexed by the letters in $\mathcal{A}_i$) formed by the distances between $S^i(B)$-integers. Note that all components of the vectors are positive and distinct. The incidence matrix of the morphism φ_i, which maps the faithful coding of $S^{i+1}(B)$-integers to the faithful coding of $S^i(B)$-integers, is denoted by M_i.

Let us show that the following matrix equalities hold:

$$\beta_i \Delta^{(i+1)} = \Delta^{(i)} M_i, \quad i \in \{0, 1, \dots, p-1\}. \tag{6.1}$$

Equivalently, we have to show that for all letter $b \in \mathcal{A}_{i+1}$, we have

$$\beta_i \Delta_{i+1,b} = \sum_{a \in \mathcal{A}_i} \Delta_{i,a} (M_i)_{a,b}, \quad i \in \{0, 1, \dots, p-1\}.$$

By Corollary 13, we know that

$$\beta_i \Delta_{i+1,n} = \Delta_{i,n+1} + d_{i+n+1,n+1} \tag{6.2}$$

for all $i \in \mathbb{Z}$ and all $n \in \mathbb{N}$. Let us consider some fixed $i \in \{0, \dots, p-1\}$. By definition of φ_i, one has $\varphi_i(b) = 0^{d_{i+b+1,b+1}} \pi_i(b+1)$ for all $b \in \mathcal{A}_i$. We thus consider two cases.

First, suppose that $\pi_i(b+1) = 0$. This means that $\Delta_{i,b+1} = \Delta_{i,0} = 1$. In this case, we get that $(M_i)_{0,b} = d_{i+b+1,b+1} + 1$ and that $(M_i)_{a,b} = 0$ for $a \neq 0$. By using (6.2), we get

$$\sum_{a \in \mathcal{A}_i} \Delta_{i,a} (M_i)_{a,b} = \Delta_{i,0} \cdot (d_{i+b+1,b+1} + 1) = d_{i+b+1,b+1} + \Delta_{i,b+1} = \beta_i \Delta_{i+1,b}.$$

As our second case, suppose that $\pi_i(b+1) = a_0 \neq 0$. Then $\Delta_{i,b+1} = \Delta_{i,a_0}$, $(M_i)_{0,b} = d_{i+b+1,b+1}$ and $(M_i)_{a,b}$ equals 1 if $a = a_0$ and equals 0 otherwise. By using (6.2) again, we get

$$\sum_{a \in \mathcal{A}_i} \Delta_{i,a} (M_i)_{a,b} = \Delta_{i,0} \cdot d_{i+b+1,b+1} + \Delta_{i,a_0} \cdot 1 = d_{i+b+1,b+1} + \Delta_{i,b+1} = \beta_i \Delta_{i+1,b}.$$

Hence (6.1) is satisfied in both cases, as announced.

From this, we derive

$$\beta_0 \cdots \beta_{p-1} \Delta^{(0)} = \beta_0 \cdots \beta_{p-1} \Delta^{(p)} = \Delta^{(0)} M_0 \cdots M_{p-1}.$$

In other words, the row vector $\Delta^{(0)}$ is a positive (left) eigenvector of the square matrix $M_0 \cdots M_{p-1}$ corresponding to the eigenvalue $\beta_0 \cdots \beta_{p-1}$. By Theorem 25, the composition $\varphi_0 \varphi_1 \cdots \varphi_{p-1}$ is primitive and hence the matrix $M_0 \cdots M_{p-1}$ is primitive as well. The Perron-Frobenius theorem implies that this eigenvalue is precisely the Perron-Frobenius eigenvalue. □

Theorem 27. *Let $B = (\beta_{p-1}, \dots, \beta_0)$ be a Parry alternate base. Then no other alternate base of length p has the same list of quasi-greedy expansions of* 1.

Proof. Assume that the list of quasi-greedy expansions $d^*_{S^i(B)}(1)$ for $i \in \{0, \dots, p-1\}$ also gives the list of quasi-greedy expansions of 1 associated with an alternate base $C = (\gamma_{p-1}, \gamma_{p-2}, \dots, \gamma_0)$. The list of substitutions associated with an alternate base depends on the quasi-greedy expansions, but not on the values β_i themselves. Therefore, the compositions of the substitutions associated with B and those associated with C coincide. By

Lemma 26, we obtain that the products $\beta_0 \cdots \beta_{p-1}$ and $\gamma_0 \cdots \gamma_{p-1}$ are the same. Then by [13, Proposition 20], we derive that $C = B$. □

Let us mention that the question of the uniqueness of an alternate base given a list of quasi-greedy expansions of 1 remains open in case that the base is not assumed to be a Parry alternate base.

7. From the point of view of automata theory

In this section, we build an automaton Aut_B associated with a Parry alternate base B which encodes all the information studied in Sections 5 and 6. Let B be a fixed Parry alternate base. The set of states is given by

$$Q = \{(i, a) : i \in \{0, \ldots, p-1\},\ a \in \mathcal{A}_i\}.$$

Thus, the number of states is given by $|\mathcal{A}_0| + \cdots + |\mathcal{A}_{p-1}|$, which is the sum of the number of possible distances between consecutive $S^i(B)$-integers. There are p initial states, given by $(i, 0)$ for $i \in \{0, \ldots, p-1\}$, and all states are final. The transitions are defined as follows: for all $(i, a) \in Q$, we have one transition of the form

$$(i, a) \xrightarrow{d_{i+a,a+1}} (i-1, \pi_{i-1}(a+1)) \tag{7.1}$$

and $d_{i+a,a+1}$ transitions of the form

$$(i, a) \xrightarrow{c} (i-1, 0), \qquad \text{for } c \in \{0, \ldots, d_{i+a,a+1} - 1\}. \tag{7.2}$$

Our aim is to show that this automaton accepts the language of the B-shift, and that the p substitutions φ_i, for $i \in \{0, \ldots, p-1\}$, defined in Section 6 can be seen in this automaton. Recall that in general for a Cantor real base, the B-shift is defined as the closure of the set of strings $\cup_{n\in\mathbb{Z}}\{d_{S^n(B)}(x) : x \in [0, 1)\}$, with respect to the product topology, and its language is the set all factors occurring in some $d_{S^n(B)}(x)$ with $n \in \mathbb{Z}$ and $x \in [0, 1)$. For an alternate base, the language of the B-shift is simply the set of factors of the set $\{d_B(x) : x \in [0, 1)\}$.

We will show that the automaton described above can be obtained as a projection of the automaton accepting the language of the B-shift given in [11]. In order to prove that Aut_B keeps the same accepted language, we will then prove that this projection is in fact a conformal morphism of automata; see [23].

In order to do so, we have to give a description of the automaton from [11], but adapted to the notation of the present framework (in particular, the bases were shifted in the opposite direction in [11]). This automaton, which we call here Aut'_B, is defined as follows. The base B being Parry, for each $i \in \{0, \ldots, p-1\}$, there exist (minimal) integers m_i, n_i such that $d^*_{S^i(B)}(1) = d_{i,1} \ldots d_{i,m_i}(d_{i,m_i+1} \ldots d_{i,m_i+n_i p})^\omega$. Note that $m_i \geq 0$ and $n_i \geq 1$. The set of states of Aut'_B is then given by

$$Q' = \{(i, k) : i \in \{0, \ldots, p-1\},\ k \in \{0, \ldots, m_i + n_i p - 1\}\}.$$

The initial states are given by $(i, 0)$ for $i \in \{0, \ldots, p-1\}$, and all states are final. For all $(i, k) \in Q'$, we have the following transitions:

$$(i, k) \xrightarrow{d_{i,k+1}} (i, k+1) \qquad \text{if } k < m_i + n_i p - 1 \tag{7.3}$$

$$(i, k) \xrightarrow{d_{i,k+1}} (i, m_i) \qquad \text{if } k = m_i + n_i p - 1 \tag{7.4}$$

$$(i, k) \xrightarrow{c} (i - k - 1, 0) \qquad \text{for each } c \in \{0, \ldots, d_{i,k+1} - 1\}. \tag{7.5}$$

In general, the set of states Q' of the automaton Aut'_B is larger that the set of states Q of the automaton Aut_B. In the following proposition, we show that the former can be projected on the latter, while keeping the same accepted language.

Proposition 28. *The automaton* Aut_B *accepts the language of the* B*-shift.*

Proof. We consider the following mapping of the states of Aut'_B to the states of Aut_B:

$$h\colon Q' \to Q, \quad (i,k) \mapsto (i-k, \pi_{i-k}(k))$$

(where as usual, $i-k$ is considered modulo p). We claim that this mapping defines a morphism of automata (see for example [23]), i.e., that

- Initial states are mapped to initial states;
- Final states are mapped to final states;
- For all transitions (q', c, q) of Aut'_B, the transition $(h(q'), c, h(q))$ is present in Aut_B.

The first two items are evident in our case. Let us check that the third item is also satisfied.

Consider a transition in Aut'_B starting from a state $(i,k) \in Q'$. Let $j = i-k$ and $a = \pi_j(k)$, so that $h(i,k) = (j,a)$. This means that $\Delta_{j,k} = \Delta_{j,a}$ (where $(\Delta_{j,n})_{n\in\mathbb{N}}$ denotes the sequence of distances between $S^j(B)$-integers). By Lemma 9, we get the string equality

$$d_{i,k+1}d_{i,k+2}d_{i,k+2}\cdots = d_{j+a,a+1}d_{j+a,a+2}d_{j+a,a+3}\cdots .$$

This implies that $d_{i,k+1} = d_{j+a,a+1}$ and $\Delta_{j-1,k+1} = \Delta_{j-1,a+1}$, hence also $\pi_{j-1}(k+1) = \pi_{j-1}(a+1)$.

Consider now the three types of transitions of Aut'_B separately. For transitions of the form (7.3), we are good since $h(i,k+1) = (j-1, \pi_{j-1}(a+1))$. For transitions of the form (7.5), we are fine too since $h(i-k-1,0) = (j-1,0)$. Now, assume that we deal with a transition of the form (7.4). Thus, we have $k = m_i + n_i p - 1$ and $j - 1 \equiv i - m_i \bmod p$. By definition of m_i and n_i, we obtain that $\Delta_{i-m_i,m_i} = \Delta_{i-m_i,m_i+n_ip}$, hence $\Delta_{j-1,m_i} = \Delta_{j-1,k+1} = \Delta_{j-1,a+1}$. This yields $h(i,m_i) = (j-1, \pi_{j-1}(a+1))$.

In order to positively assert that the image automaton $\mathrm{Aut}_B = h(\mathrm{Aut}'_B)$ accepts the same language as Aut'_B, we still have to argue that any transition in Aut_B is the image of a transition in Aut'_B; such a morphism of automata is said to be *conformal*, see [23, Section 3.1.2]. For this, we note that $\Delta_{i,a} = \Delta_{i,a-n_ip}$ for all $a \geq m_{i+a} + n_{i+a}p$. Therefore, for all $a \in \mathcal{A}_i$, we have $a < m_{i+a} + n_{i+a}p$. This shows that if $(i,a) \in Q$, then $(i+a,a) \in Q'$, hence $h(i+a,a) = (i,a)$. Consider a transition of the form (7.1) in Aut_B. If $a < m_{i+a} + n_{i+a}p - 1$ then this transition comes from the transition

$$(i+a,a) \xrightarrow{d_{i+a,a+1}} (i+a,a+1)$$

in Aut'_B. In the case where a happens to coincide with $m_{i+a} + n_{i+a}p - 1$, then the transition comes from the transition

$$(i+a,a) \xrightarrow{d_{i+a,a+1}} (i+a,m_{i+a})$$

in Aut'_B since $\pi_{i+a-m_{i+a}}(m_{i+a}) = \pi_{i-1}(a+1)$. Finally, any transition of the form (7.2) in Aut_B comes from the transition $(i+a,a) \xrightarrow{c} (i-1,0)$ in Aut'_B.

The conclusion follows since we know from [12, 11] that the automaton Aut'_B accepts the language of the B-shift. □

Remark 29. To be a little more precise, let us emphasize that, from the proofs of [12, Theorem 48] adapted to the automaton Aut'_B and of Proposition 28, we actually get that for every $i \in \{0,\ldots,p-1\}$, the language accepted from the state $(i,0)$ in Aut_B is the set of all prefixes of the language $\{d_{S^i(B)}(x) : x \in [0,1)\}$.

Let us now show how to find the p substitutions $\varphi_i\colon \mathcal{A}_{i+1}^* \to \mathcal{A}_i^*$ from the automaton Aut_B. For any $i \in \{0,\dots,p-1\}$ and $a \in \mathcal{A}_{i+1}$, the image $\varphi_i(a)$ can be obtained by inspecting the automaton Aut_B as follows. Among all transitions going out of the state $(i+1,a)$, pick that with maximal label. If this label is d and the reached state is $(i-1,b)$, then $\varphi_i(a) = 0^d b$.

To end this section, let us also point out some nice properties of the automaton Aut_B. This automaton is invariant under shifts of the base, i.e., one has $\mathrm{Aut}_B = \mathrm{Aut}_{S(B)}$. It is also strongly connected and p-partite with respect to the state partition $\{(i,a) : a \in \mathcal{A}_i\}$ for $i \in \{0,\dots,p-1\}$.

8. Examples

In this section, we illustrate our results on three examples of alternate bases, going from all the details for the first example to only a few enlightening remarks for the next two.

First, consider the alternate base $B = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. It is Parry since $d_B^*(1) = 20(01)^\omega$ and $d_{S(B)}^*(1) = (10)^\omega$. There are two possible distances between consecutive B-integers since

$$\Delta_0 = \mathrm{val}_B(0 \bullet 20(01)^\omega) = 1$$

$$\Delta_n = \Delta_1 = \mathrm{val}_B(0 \bullet (01)^\omega) = \frac{-1+\sqrt{13}}{6} \sim 0.434259 \quad \text{for } n \neq 0.$$

Thus, the projection π_0 is given by

$$\pi_0\colon \begin{cases} 0 \mapsto 0 \\ n \mapsto 1 \quad \text{for } n \neq 0 \end{cases}$$

and the alphabet $\mathcal{A}_0 = \pi_0(\mathbb{N})$ is $\{0,1\}$. In Table 1 are given the B-integers whose B-expansions have lengths at most 5 up to two decimal digits, (the left part of) their B-expansions, the corresponding prefixes of the sequence $\mathbf{w}_B$ and its projection $\mathbf{v}_B = \pi_0(\mathbf{w}_B)$ onto the alphabet $\mathcal{A}_0$. The first B-integers, the B-expansions of which are given in Table 1,

k	x_k	$\langle x_k\rangle_B$	$\mathbf{w}_B$	$\mathbf{v}_B$	k	x_k	$\langle x_k\rangle_B$	$\mathbf{w}_B$	$\mathbf{v}_B$	k	x_k	$\langle x_k\rangle_B$	$\mathbf{w}_B$	$\mathbf{v}_B$
0	0	0	0	0	12	8.03	1100	0	0	24	16.64	100001	1	1
1	1	1	1	1	13	9.03	1101	3	1	25	17.07	100010	0	0
2	1.43	10	0	0	14	9.47	2000	0	0	26	18.07	100011	1	1
3	2.43	11	1	1	15	10.47	2001	4	1	27	18.51	100020	2	1
4	2.86	20	2	1	16	10.90	10000	0	0	28	18.94	100100	0	0
5	3.30	100	0	0	17	11.90	10001	1	1	29	19.94	100101	3	1
6	4.30	101	3	1	18	12.34	10010	0	0	30	20.38	101000	0	0
7	4.73	1000	0	0	19	13.34	10011	1	1	31	21.38	101001	1	1
8	5.73	1001	1	1	20	13.77	10020	2	1	32	21.81	101010	0	0
9	6.17	1010	0	0	21	14.21	10100	0	0	33	22.81	101011	1	1
10	7.17	1011	1	1	22	15.21	10101	5	1	34	23.25	101020	2	1
11	7.60	1020	2	1	23	15.64	100000	0	0	35	23.68	101100	0	0

Table 1. B-expansions of the first B-integers and the corresponding prefixes of $\mathbf{w}_B$ and $\mathbf{v}_B$, with respect to the alternate base $B = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

are represented in Figure 1.

FIGURE 1. The first B-integers, with respect to the alternate base $B = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

Now, we consider the shifted base $S(B) = (\frac{5+\sqrt{13}}{6}, \frac{1+\sqrt{13}}{2})$. Let us denote the k-th $S(B)$-integer by $\tilde{x}_k$ and the distance between consecutive $S(B)$-integers differing in (maximal) position n by $\tilde{\Delta}_n$. We get also two possible distances between consecutive $S(B)$-integers:

$$\tilde{\Delta}_n = \tilde{\Delta}_0 = \mathrm{val}_{\tilde{B}}(0 \bullet (10)^\omega) = 1 \quad \text{for } n \neq 1$$

$$\tilde{\Delta}_1 = \mathrm{val}_{\tilde{B}}(0 \bullet 0(01)^\omega) = \frac{-3+\sqrt{13}}{2} \sim 0.302776.$$

Thus, the projection π_1 is given by

$$\pi_1 \colon \begin{cases} 1 \mapsto 1 \\ n \mapsto 0 \quad \text{for } n \neq 1 \end{cases}$$

and the alphabet $\mathcal{A}_1 = \pi_1(\mathbb{N})$ is $\{0, 1\}$ again. Table 2 is the analogue of Table 1 for the shifted base $S(B)$. The first $S(B)$-integers, the $S(B)$-expansions of which are given in

k	$\tilde{x}_k$	$\langle \tilde{x}_k \rangle_{\tilde{B}}$	$\mathbf{w}_{\tilde{B}}$	$\mathbf{v}_{\tilde{B}}$	k	$\tilde{x}_k$	$\langle \tilde{x}_k \rangle_{\tilde{B}}$	$\mathbf{w}_{\tilde{B}}$	$\mathbf{v}_{\tilde{B}}$	k	$\tilde{x}_k$	$\langle \tilde{x}_k \rangle_{\tilde{B}}$	$\mathbf{w}_{\tilde{B}}$	$\mathbf{v}_{\tilde{B}}$
0	0	ε	0	0	12	9.90	1010	4	0	24	20.51	11002	1	1
1	1	1	0	0	13	10.90	10000	0	0	25	20.81	11010	4	0
2	2	2	1	1	14	11.90	10001	0	0	26	21.81	20000	0	0
3	2.30	10	2	0	15	12.90	10002	1	1	27	22.81	20001	0	0
4	3.30	100	0	0	16	13.21	10010	2	0	28	23.81	20002	1	1
5	4.30	101	0	0	17	14.21	10100	0	0	29	24.11	20010	5	0
6	5.30	102	1	1	18	12.21	10101	0	0	30	25.11	100000	0	0
7	5.60	110	2	0	19	16.21	10102	1	1	31	26.11	100001	0	0
8	6.60	200	3	0	20	16.51	10110	2	0	32	27.11	100002	1	1
9	7.60	1000	0	0	21	17.51	10200	3	0	33	27.42	100010	2	0
10	8.60	1001	0	0	22	18.51	11000	0	0	34	28.42	100100	0	0
11	9.60	1002	1	1	23	19.51	11001	0	0	35	29.42	100101	0	0

TABLE 2. $S(B)$-expansions of the first $S(B)$-integers and the corresponding prefixes of $\mathbf{w}_{S(B)}$ and $\mathbf{v}_{S(B)}$, with respect to the alternate base $S(B) = \tilde{B} = (\frac{5+\sqrt{13}}{6}, \frac{1+\sqrt{13}}{2})$.

Table 2, are represented in Figure 2.

FIGURE 2. The first $S(B)$-integers, with respect to the alternate base $S(B) = (\frac{5+\sqrt{13}}{6}, \frac{1+\sqrt{13}}{2})$.

We first illustrate Proposition 12, which says that $\psi_B(\mathbf{w}_{S(B)}) = \mathbf{w}_B$. The substitution ψ_B is given by $0 \mapsto 01,\ 1 \mapsto 2$ and $n \mapsto 0(n+1)$ for all $n \geq 2$. Computing a prefix

$$\begin{aligned}\psi_B(\mathbf{w}_{S(B)}) &= \psi_B(00120012300140012001230014001 5\cdots)\\ &= 01012030101203040101205010120301012030401012050101206\cdots\end{aligned}$$

of $\psi_B(\mathbf{w}_{S(B)})$, we see that it indeed coincides with a prefix of $\mathbf{w}_B$.

Using the projections π_0 and π_1, we get the substitutions φ_0 and φ_1, both over the alphabet $\{0,1\}$, which are given by

$$\varphi_0\colon \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 1 \end{cases} \qquad \text{and} \qquad \varphi_1\colon \begin{cases} 0 \mapsto 001 \\ 1 \mapsto 0. \end{cases}$$

Their incidence matrices are given by

$$M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrix

$$M_0 M_1 = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$$

is the incidence matrix of the substitution

$$\varphi = \varphi_0\varphi_1\colon \begin{cases} 0 \mapsto 01011 \\ 1 \mapsto 01. \end{cases} \tag{8.1}$$

This matrix is clearly primitive as it has only positive entries. Computing a prefix

$$01011 \cdot 01 \cdot 01011 \cdot 01 \cdot 01 \cdot 01011 \cdot 01 \cdot 01011 \cdot 01 \cdot 01011 \cdot 01 \cdot 01\cdots$$

of the fixed point of the substitution $\varphi_0\varphi_1$, we see that it indeed corresponds to the prefix of the infinite word v_B coding the distances between B-integers given in Table 1.

Moreover, the characteristic polynomial of M_0M_1 is given by $X^2 - 3X - 1$. The eigenvalues are thus given by $\frac{3+\sqrt{13}}{2}$ and $\frac{3-\sqrt{13}}{2}$. The dominant eigenvalue is $\frac{3+\sqrt{13}}{2}$, which is precisely the product $\beta_1\beta_0$ as stated in Lemma 26.

In this example, as we have 2 possible distances between consecutive B-integers, and also 2 possible distances between consecutive $S(B)$-integers, we get 4 states in the automaton Aut_B. This automaton is depicted in Figure 3.

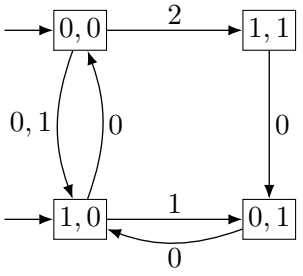

FIGURE 3. The automaton Aut_B associated with the base $B = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

Finally, with this example, we also illustrate the conformal morphism of automata from Aut'_B to Aut_B described in the proof of Proposition 28. The automaton Aut'_B is drawn in

Figure 4. The mapping $h\colon Q' \to Q$ where $Q' = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1)\}$ and $Q = \{(0,0),(0,1),(1,0),(1,1)\}$ is given by

$$h\colon \begin{cases} (0,0) \mapsto (0,0) \\ (0,1) \mapsto (1,1) \\ (0,2),(1,1) \mapsto (0,1) \\ (0,3),(1,0) \mapsto (1,0) \end{cases}$$

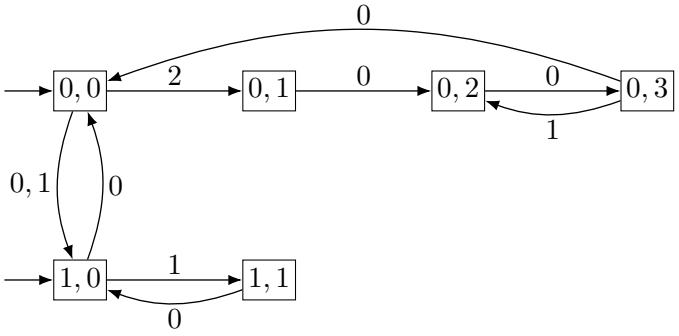

FIGURE 4. The automaton Aut'_B associated with the base $B = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

Our second example is the alternate base $B = (\frac{5+2\sqrt{5}}{5}, \sqrt{5})$ for which $d^*_B(1) = 1^\omega$ and $d^*_{S(B)}(1) = 2001^\omega$. There are two possible distances between consecutive B-integers since

$$\Delta_n = \Delta_0 = \mathrm{val}_B(0 \bullet 1^\omega) = 1 \quad \text{for } n \neq 1$$
$$\Delta_1 = \mathrm{val}_B(0 \bullet 001^\omega) = -2 + \sqrt{5} \simeq 0.236068.$$

For the shifted base $S(B)$, we get three possible distances between consecutive $S(B)$-integers, as

$$\tilde{\Delta}_0 = \mathrm{val}_{S(B)}(0 \bullet 2001^\omega) = 1$$
$$\tilde{\Delta}_n = \tilde{\Delta}_1 = \mathrm{val}_{S(B)}(0 \bullet 1^\omega) = \frac{2\sqrt{5}}{5} \simeq 0.894427 \quad \text{for } n \neq 0, 2$$
$$\tilde{\Delta}_2 = \mathrm{val}_{S(B)}(0 \bullet 01^\omega) = \frac{\sqrt{5}}{5} \simeq 0.447214.$$

The corresponding projections π_0 and π_1 are given by

$$\pi_0\colon \begin{cases} 1 \mapsto 1 \\ n \mapsto 0 \quad \text{for } n \neq 1 \end{cases} \quad \text{and} \quad \pi_1\colon \begin{cases} 0 \mapsto 0 \\ 2 \mapsto 2 \\ n \mapsto 1 \quad \text{for } n \neq 0, 2 \end{cases}$$

and we get $\mathcal{A}_0 = \pi_0(\mathbb{N}) = \{0, 1\}$ and $A_1 = \pi_1(\mathbb{N}) = \{0, 1, 2\}$. The automaton Aut_B associated with this alternate base B is depicted in Figure 5.

As a last example, we consider the same alternate base as in Example 18 and construct the automaton Aut_B. It has 6 states as there are 3 possible distances between consecutive B-integers and also 3 possible distances between consecutive $S(B)$-integers. The automaton Aut_B is depicted in Figure 6.

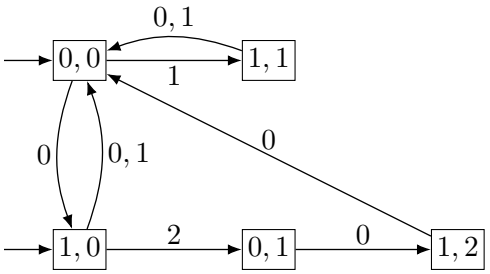

FIGURE 5. The automaton Aut_B associated with the base $B = (\frac{5+2\sqrt{5}}{5}, \sqrt{5})$.

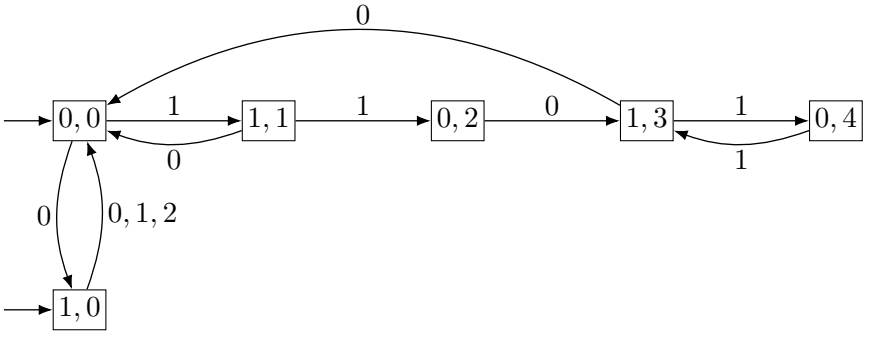

FIGURE 6. The automaton Aut_B associated with the base B of Example 18.

9. B-integers from the dynamical point of view

In this section, we present an alternative definition of the set of B-integers based on the dynamical point of view of B-expansions. The results of the previous sections can then be reinterpreted in terms of this alternative point of view. For $\beta > 1$, the *β-transformation* is the map $T_\beta\colon [0,1) \to [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor$. In our context of two-way Cantor real bases $B = (\beta_n)_{n\in\mathbb{Z}}$, the greedy digits can be obtained by applying the maps T_{β_n} as follows. For a real number $x \in [0,1)$ with $\langle x\rangle_B = 0 \bullet a_{-1}a_{-2}a_{-1}\cdots$, the digits a_n with $n < 0$ are given by $a_n = \left\lfloor \beta_n T_{\beta_{n+1}} T_{\beta_{n+2}} \cdots T_{\beta_{-1}}(x) \right\rfloor$. For a real number $x \geq 1$ with $\langle x\rangle_B = a_{N-1}\cdots a_0 \bullet a_{-1}a_{-2}a_{-1}\cdots$, the digits a_n with $n < N$ are given by $a_n = \left\lfloor \beta_n T_{\beta_{n+1}} T_{\beta_{n+2}} \cdots T_{\beta_{N-1}}(\frac{x}{\beta_0\cdots\beta_{N-1}}) \right\rfloor$. Using these notation, the set $\mathbb{N}_B$ of B-integers can be defined as

$$\mathbb{N}_B = \{x \in [0,+\infty) : \exists N \in \mathbb{N},\ x < \beta_0\cdots\beta_{N-1} \text{ and } T_{\beta_0}T_{\beta_1}\cdots T_{\beta_{N-1}}\left(\frac{x}{\beta_0\cdots\beta_{N-1}}\right) = 0\}.$$

We now define sequences $(\Delta_{m,n})_{n\in\mathbb{N}}$, for all $m \in \mathbb{Z}$, which will later be interpreted as encodings of the distances between consecutive $S^m(B)$-integers. These sequences are defined recursively as follows. We first set $\Delta_{m,0} = 1$ for all $m \in \mathbb{Z}$, and then, we let

$$\Delta_{m,n} = \beta_m \Delta_{m+1,n-1} - (\lceil \beta_m \Delta_{m+1,n-1} \rceil - 1)$$

for all $m, n \in \mathbb{Z}$ with $n \geq 1$. Clearly, we have $\Delta_{m,n} \in (0,1]$. Let us denote the integers in the previous definition as

$$c_{m,n} = \lceil \beta_m \Delta_{m+1,n-1} \rceil - 1.$$

It is easily seen by using an induction on n that

$$\Delta_{m,n} = \beta_m \cdots \beta_{m+n-1} - \mathrm{val}_{S^m(B)}(c_{m+n-1,1}c_{m+n-2,2}\cdots c_{m,n} \bullet 0^\omega) \tag{9.1}$$

for all $m, n \in \mathbb{Z}$ with $n \geq 0$. This allows us to prove that the integers $c_{m,n}$ coincide with the quasi-greedy digits defined in (4.3).

Lemma 30. *For all $m, n \in \mathbb{Z}$ with $n \geq 1$, we have $c_{m,n} = d_{m+n,n}$.*

Proof. We do an induction on $n \geq 1$. For $n = 1$ and all $m \in \mathbb{Z}$, we have $c_{m,1} = \lceil \beta_m \Delta_{m+1,0} \rceil - 1 = \lceil \beta_m \rceil - 1 = d_{m+1,1}$. Now, let $n > 1$ and suppose that $c_{m,n'} = d_{m+n',n'}$ for all $m, n' \in \mathbb{Z}$ with $1 \leq n' < n$. From (9.1), we know that

$$1 = \frac{c_{m+n-1,1}}{\beta_{m+n-1}} + \frac{c_{m+n-2,2}}{\beta_{m+n-2}\beta_{m+n-1}} + \cdots + \frac{c_{m,n}}{\beta_m \cdots \beta_{m+n-1}} + \frac{\Delta_{m,n}}{\beta_m \cdots \beta_{m+n-1}}.$$

By definition of the quasi-greedy expansion $d^*_{S^{m+n}(B)}(1)$, it is not difficult to see that there exists $r \in (0, 1]$ such that

$$1 = \frac{d_{m+n,1}}{\beta_{m+n-1}} + \frac{d_{m+n,2}}{\beta_{m+n-2}\beta_{m+n-1}} + \cdots + \frac{d_{m+n,n}}{\beta_m \cdots \beta_{m+n-1}} + \frac{r}{\beta_m \cdots \beta_{m+n-1}}.$$

Using the induction hypothesis, we can replace the digits $d_{m+n,k}$ by $c_{m+n-k,k}$ for each $k \in \{1, 2, \ldots, n-1\}$. Hence we obtain $c_{m,n} + \Delta_{m,n} = d_{m+n,n} + r$. Since $c_{m,n}$ and $d_{m+n,n}$ are integers, and since both r and $\Delta_{m,n}$ belong to $(0, 1]$, we get that $c_{m,n} = d_{m+n,n}$ as desired. □

Therefore and by using Corollary 13, the sequence $(\Delta_{m,n})_{n\in\mathbb{N}}$ coincide with the sequence of distances defined in Section 4 with respect to the shifted base $S^m(B)$.

For all $m \in \mathbb{Z}$, the substitution $\psi_m := \psi_{S^m(B)}$ can be rewritten with the notation of this section as

$$\psi_m \colon n \mapsto 0^{c_{m,n+1}}(n+1).$$

Our aim is now to reobtain the results of Section 4 by the lens of the β_m-transformations. This alternative approach, although less intuitive since we understand the link with distances of consecutive B-integers only afterwards, has the advantage of being shorter in terms of presentation. Note that here we work with all shifts $S^m(B)$ of the base B at once whereas in Section 4, we only worked with the base B and its first shift $S(B)$.

For all $m \in \mathbb{Z}$, the integers $c_{m,1}$ are greater than 1, hence the images $\psi_m(0)$ start with 0. Then the limits

$$\mathbf{w}_m := \lim_{n\to+\infty} \psi_m \psi_{m+1} \cdots \psi_{m+n-1}(0)$$

exist. Moreover, we clearly have

$$\psi_m(\mathbf{w}_{m+1}) = \mathbf{w}_m. \tag{9.2}$$

The following proposition shows that the sequences $(\Delta_{m,n})_{n\in\mathbb{N}}$ encode the distances between consecutive $S^m(B)$-integers.

Proposition 31. *Let $m \in \mathbb{Z}$ and write $\mathbf{w}_m = w_{m,0}w_{m,1}w_{m,2}\cdots$ where the $w_{m,k}$'s are letters. Then the set of $S^m(B)$-integers can be computed as*

$$\mathbb{N}_{S^m(B)} = \left\{ \Delta_{m,w_{m,0}} + \cdots + \Delta_{m,w_{m,k-1}} : k \in \mathbb{N} \right\}.$$

Proof. It suffices to show that

$$\Delta_{m,w_{m,0}} + \cdots + \Delta_{m,w_{m,k}} \leq \beta_m \cdots \beta_{m+n-1}$$

and

$$T_{\beta_m} \cdots T_{\beta_{m+n-1}} \left(\frac{\Delta_{m,w_{m,0}} + \cdots + \Delta_{m,w_{m,k-1}} + y}{\beta_m \cdots \beta_{m+n-1}} \right) = y$$

for all $m \in \mathbb{Z}$, all $n \in \mathbb{N}$, all $k \in \mathbb{N}$ such that such that $w_{m,0} \cdots w_{m,k}$ is a prefix of $\psi_m \cdots \psi_{m+n-1}(0)$ and all $y \in [0, \Delta_{m,w_{m,k}})$. We proceed by induction on n. For $n = 0$, this is trivially true since we must have $k = 0$ and $\Delta_{m,0} = 1$. Now, pick some $n > 0$ and suppose that the claim is true up to $n-1$. Let m, k, y satisfying the given constraints. In view of (9.2), we can write

$$w_{m,0} \cdots w_{m,k-1} = \psi_m(w_{m+1,0} \cdots w_{m+1,\ell-1})0^i$$

with $\ell \in \mathbb{N}$ and $i \leq c_{m,w_{m+1,\ell}+1}$.

By using the definition of the substitution ψ_m and the values $\Delta_{m,n}$, we get that

$$\sum_{j=0}^{k-1} \Delta_{m,w_{m,j}} = \sum_{j=0}^{\ell-1} (c_{m,w_{m+1,j}+1} + \Delta_{m,w_{m+1,j}+1}) + i = \beta_m \sum_{j=0}^{\ell-1} \Delta_{m+1,w_{m+1,j}} + i. \tag{9.3}$$

Clearly, the word $w_{m+1,0} \cdots w_{m+1,\ell-1}$ is a prefix of $\psi_{m+1} \cdots \psi_{m+n-1}(0)$. Thus, the first part of the induction hypothesis gives us that

$$\Delta_{m+1,w_{m+1,0}} + \cdots + \Delta_{m+1,w_{m+1,\ell}} \leq \beta_{m+1} \cdots \beta_{m+n-1}. \tag{9.4}$$

Let us argue that $i + \Delta_{m,w_{m,k}} \leq \beta_m \Delta_{m+1,w_{m+1,\ell}}$. Note that $w_{m,k}$ is equal to either 0 or to $w_{m+1,\ell} + 1$. If $w_{m,k} = 0$, it should be that $i \leq c_{m,1+w_{m+1,\ell}} - 1$ and $\Delta_{m,0} = 1$ whereas if $w_{m,k} = w_{m+1,\ell} + 1$, it should be that $i = c_{m,w_{m+1,\ell}+1}$. In both cases, we obtain

$$i + \Delta_{m,w_{m,k}} \leq c_{m,1+w_{m+1,\ell}} + \Delta_{m,w_{m+1,\ell}+1} = \beta_m \Delta_{m+1,w_{m+1,\ell}}. \tag{9.5}$$

Now, by combining (9.3), (9.4) and (9.5), we obtain that

$$\sum_{j=0}^{k} \Delta_{m,w_{m,j}} = \beta_m \sum_{j=0}^{\ell} \Delta_{m+1,w_{m+1,j}} - \beta_m \Delta_{m+1,w_{m+1,\ell}} + i + \Delta_{m,w_{m,k}} \leq \beta_m \cdots \beta_{m+n-1}$$

which is the first part of our induction claim.

Since $y \in [0, \Delta_{m,w_{m,k}})$, we get from (9.5) that $\frac{i+y}{\beta_m} < \Delta_{m+1,w_{m+1,\ell}}$. Hence the second part of the induction hypothesis tells us that

$$T_{\beta_{m+1}} \cdots T_{\beta_{m+n-1}} \left(\frac{\Delta_{m+1,w_{m+1,0}} + \cdots + \Delta_{m+1,w_{m+1,\ell-1}} + \frac{i+y}{\beta_m}}{\beta_{m+1} \cdots \beta_{m+n-1}} \right) = \frac{i+y}{\beta_m}.$$

By applying the transformation T_{β_m} on both sides of the latter equality, and by using (9.3) again, we obtain that

$$T_{\beta_m} \cdots T_{\beta_{m+n-1}} \left(\frac{\Delta_{m,w_{m,0}} + \cdots + \Delta_{m,w_{m,k-1}} + y}{\beta_m \cdots \beta_{m+n-1}} \right) = T_{\beta_m} \left(\frac{i+y}{\beta_m} \right) = y$$

which is the second part of our induction claim. □

10. B-integers and sturmian sequences

Balanced sequences over a binary alphabet that are not eventually periodic are called *sturmian*. Recall that a sequence $\mathbf{v} = (v_n)_{n \in \mathbb{N}}$ over $\{a, b\}$ is *balanced* if for every length $n \in \mathbb{N}$ and every $i, j \in \mathbb{N}$ the number of occurrences of the letter a in the words $v_i v_{i+1} \cdots v_{i+n-1}$ and $v_j v_{j+1} \cdots v_{j+n-1}$ differs at most by 1. Sturmian sequences can be defined by many equivalent ways, see [22].

We use the result which characterizes sturmian sequences by their S-adic representation using morphisms from the so-called special sturmian monoid, i.e., the monoid of binary morphisms generated by the following morphisms:

$$G\colon \begin{cases} 0\mapsto 0 \\ 1\mapsto 01 \end{cases} \qquad \tilde{G}\colon \begin{cases} 0\mapsto 0 \\ 1\mapsto 10 \end{cases} \qquad D\colon \begin{cases} 0\mapsto 10 \\ 1\mapsto 1 \end{cases} \qquad \tilde{D}\colon \begin{cases} 0\mapsto 01 \\ 1\mapsto 1 \end{cases} \tag{10.1}$$

See [22] for details. Every sturmian sequence has an S-adic expansion in the form (2.1), where $\sigma_n \in \{D, \tilde{D}, G, \tilde{G}\}$ and the sequence $(a_n)_{n\in\mathbb{N}}$ with values in $\{0,1\}$ may be taken constant. The monoid generated by the four morphisms is not free and thus the S-adic expansion of a sturmian need not to be unique. On the other hand, if the sequence $(\sigma_n)_{n\in\mathbb{N}}$ contains infinite number of the morphisms D or $\tilde{D}$ and also infinite number of G or $\tilde{G}$, the sequence obtained by the S-adic expansion of $(\sigma_n)_{n\in\mathbb{N}}$ is sturmian.

Applying the exchange morphism $E : 0 \mapsto 1, 1 \mapsto 0$ to a sturmian sequence with S-adic expansion $(\sigma_n)_{n\in\mathbb{N}}$ yields a sturmian sequence whose S-adic expansion is obtained from $(\sigma_n)_{n\in\mathbb{N}}$ by interchanging $G \leftrightarrow D$ and $\tilde{G} \leftrightarrow \tilde{D}$. The morphisms $E, G, D, \tilde{G}, \tilde{D}$ satisfy the relations $GE = ED$ and $\tilde{G}E = E\tilde{D}$.

If the S-adic expansion of a sturmian sequence contains only morphisms G and D, we say that the sturmian sequence is standard. In that case there exists a sequence of positive integers $(c_n)_{n\in\mathbb{N}}$ such that $(\sigma_n)_{n\in\mathbb{N}}$ can be written as $G^{c_0}D^{c_1}G^{c_2}D^{c_3}\cdots$ or $D^{c_0}G^{c_1}D^{c_2}G^{c_3}\cdots$.

Note that the faithful coding $\mathbf{v}_B$ of B-integers starts with the letter 0, and thus its S-adic expansion cannot be of the latter form.

Proposition 32. *For every standard sturmian sequence* $\mathbf{v}$ *over* $\{0,1\}$ *with a prefix* 0*, there exists a two-way Cantor base* B *such that* $\mathbf{v}$ *is the faithful coding of the* B*-integers.*

Proof. Let $G^{c_0}D^{c_1}G^{c_2}D^{c_3}\cdots$ be an S-adic expansion of the standard sturmian sequence $\mathbf{v}$. Choose arbitrarily the positive integers $c_{-1}, c_{-2}, c_{-3}, \ldots$. The two-way sequence $(c_n)_{n\in\mathbb{Z}}$ will be used to define a suitable Cantor real base. Consider the real numbers β_n with regular continued fraction expansion

$$\beta_n = [c_n; c_{n-1}, c_{n-2}, \ldots] \quad \text{for every } n \in \mathbb{Z}.$$

This choice implies that β_n satisfy the relation $\beta_n = c_n + \frac{1}{\beta_{n-1}}$. Consequently, the quasi-greedy expansion of 1 is of the form

$$d^*_{S^n(B)}(1) = c_{n-1}0c_{n-3}0c_{n-5}0\cdots .$$

This ensures that the distances between consecutive elements of $\mathbb{N}_B$ take two values, namely $\Delta_{2n} = \Delta_0 = \mathrm{val}_B(0 \bullet c_{-1}0c_{-3}0\cdots) = 1$ and $\Delta_{2n+1} = \Delta_1 = \mathrm{val}_B(0 \bullet 0c_{-2}0c_{-4}0\cdots) = \frac{1}{\beta_{-1}}$ for all $n \in \mathbb{N}$. By symmetry, we obtain that the distances between consecutive elements of $\mathbb{N}_{S(B)}$ take two values as well. According to Definition 11 and Proposition 20, the morphism which maps the faithful coding of $\mathbb{N}_{S^{n+1}(B)}$ onto the faithful coding of $\mathbb{N}_{S^n(B)}$ is $\varphi_n\colon 0 \mapsto 0^{c_n}1$ and $1 \mapsto 0$. It is easily verified that $\varphi_n = G^{c_n}E$. Since $\varphi_0\varphi_1\cdots\varphi_{2n+1} = G^{c_0}EG^{c_1}E\cdots G^{c_{2n+1}}E = G^{c_0}D^{c_1}\cdots G^{c_{2n}}D^{c_{2n+1}}$ for all $n \in \mathbb{N}$, the standard sturmian word $\mathbf{v}$ has the S-adic expansion $(\varphi_n)_{n\in\mathbb{N}}$, hence is equal to $\mathbf{v}_B$ by Corollary 21. □

Let us recall here another definition of sturmian sequences using exchange of two intervals. Let $\alpha \in (0,1) \setminus \mathbb{Q}$, $\varrho \in [0,1)$. The mapping $T : [0,1) \to [0,1)$ given by $T(x) = x + \alpha \bmod 1$ is the exchange of intervals $I_0 = [0, 1-\alpha)$, $I_1 = [1-\alpha, 1)$. The binary sequence $\mathbf{v} = v_0v_1v_2\cdots$ defined by $v_n = a \in \{0,1\}$ if $T^n(\rho) \in I_a$ is the sturmian sequence with slope α and intercept ϱ. It is known that a sturmian sequence with the slope α is standard if and only if the intercept ϱ is equal to α.

Sturmian sequences with the same slope α have the same language. The value of the slope can be determined from the S-adic expansion $G^{c_0}D^{c_1}G^{c_2}D^{c_3}\cdots$ of the standard sturmian sequence using the continued fraction expansion. We have $\frac{1-\alpha}{\alpha} = [c_0; c_1, c_2, \ldots]$.

In the following proposition we show that among the faithful codings of B-integers one can find also non-standard sturmian sequences, i.e., with intercept different from α.

Proposition 33. *Let* $\mathbf{v}$ *be a sturmian sequence with intercept* $\varrho = 0$*. Then there exists a two-way Cantor base* B *such that* $\mathbf{v}$ *is the faithful coding of the* B*-integers.*

Proof. As follows from Section 2.2.2 in [22], any sturmian sequence $\mathbf{v}$ with the intercept $\varrho = 0$ has an S-adic expansion in the form $G^{b_0}\tilde{D}^{b_1}G^{b_2}\tilde{D}^{b_3}\cdots$, where $b_i \in \mathbb{N}$ for every $i \in \mathbb{N}$, with $b_i > 0$ for $i > 0$.

Since $\tilde{D} = \tilde{D}G^0 = G^0\tilde{D}$, there exists a sequence of positive integers $(c_n)_{n\in\mathbb{N}}$ such that

$$G^{b_0}\tilde{D}^{b_1}G^{b_2}\tilde{D}^{b_3}\cdots = (G^{c_0-1}\tilde{D})(G^{c_1-1}\tilde{D})(G^{c_2-1}\tilde{D})\cdots \tag{10.2}$$

Choose arbitrarily the positive integers $c_{-1}, c_{-2}, c_{-3}, \ldots$. The two-way sequence $(c_n)_{n\in\mathbb{Z}}$ is used to define a suitable Cantor real base. We will find $B = (\beta_n)_{n\in\mathbb{Z}}$ so that the sequence satisfies

$$\beta_n = c_n + 1 - \frac{1}{\beta_{n-1}} \quad \text{for every } n \in \mathbb{Z}. \tag{10.3}$$

This relation will be valid for real numbers β_n given by the backward continued fraction, where

$$\beta_n = c_n + 1 - \cfrac{1}{c_{n-1} + 1 - \cfrac{1}{c_{n-2}+1-\frac{1}{\cdots}}}.$$

Equality (10.3) then implies that the quasi-greedy expansions of 1 is of the form $d^*_{S^n(B)}(1) = c_{n-1}(c_{n-2}-1)(c_{n-3}-1)(c_{n-4}-1)\cdots$ for $n \in \mathbb{Z}$. From there, we derive that the distances in $\mathbb{N}_{S^n(B)}$ take two values, namely 1 and $\beta_n - c_n$. According to Definition 11 and Proposition 20, the morphism which maps the faithful coding of $\mathbb{N}_{S^{n+1}(B)}$ onto the faithful coding of $\mathbb{N}_{S^n(B)}$ is $\varphi_n\colon 0 \mapsto 0^{c_n}1$ and $1 \mapsto 0^{c_n-1}1$. We can check that $\varphi_n = G^{c_n-1}\tilde{D}$. Therefore $\varphi_0\varphi_1\cdots\varphi_n$ equals $(G^{c_0-1}\tilde{D})(G^{c_1-1}\tilde{D})\cdots(G^{c_n-1}\tilde{D})$, hence by (10.2) and Corollary 21, the sturmian sequence $\mathbf{v}$ is equal to $\mathbf{v}_B$. □

Remark 34. The morphism φ given in (8.1) fixes the faithful coding $\mathbf{v}_B$ in the case $B = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. It is straightforward to check that $\varphi = \tilde{D}ED^2$ and $\varphi^2 = \tilde{D}G^2\tilde{G}D^2$. Hence $\mathbf{v}_B$ is a sturmian sequence with the S-adic representation $(\tilde{D}G^2\tilde{G}D^2)^\omega$. As all four morphisms from the list (10.1) occur in the S-adic representation of $\mathbf{v}_B$, the sturmian sequence $\mathbf{v}_B$ does not belong either to the class of standard sturmian sequences or the class of sturmian sequences with the intercept 0.

11. Comments and open problems

In Theorem 27, we have proved that at most one Parry alternate base can be associated with a given candidate list of quasi-greedy expansions of 1. It would be interesting to know whether this result extends to all alternate bases, i.e., that are not necessarily Parry. For a given list of p sequences that satisfy the inequalities (4.4) we cannot decide whether they correspond to any alternate base B.

The infinite symbolic sequences associated with β-integers in the context of Rényi numeration systems may have affine factor complexity. The characterization of such cases is given in [5] using a detailed description of special factors. This allows one to determine

which Parry sequences are sturmian, or more generally, Arnoux-Rauzy. It turns out that Arnoux-Rauzy sequences originated from β-integers are standard. Nevertheless, the set of standard Arnoux-Rauzy sequences which are coded by β-integers is very modest. Let us illustrate it on the example of binary Arnoux-Rauzy sequences, i.e. sturmian sequences. The slope α of a standard sturmian sequence, coding β-integers is necessarily an algebraic unit. At the same time, a sturmian sequence is a fixed point of a substitution for every quadratic slope α with conjugate $\alpha' \notin (0,1)$, see Yasutomi [26]. It follows from Proposition 32 that every standard sturmian sequence (after possible interchange $0 \leftrightarrow 1$) is a coding of B-integers for some Cantor real base B.

As Remark 34 demonstrates, Propositions 32 and 33 do not cover all sturmian sequences that serve as a faithful coding of B-integers. Identification of all sturmian sequences with the described property remains an open question.

Infinite symbolic sequences $\mathbf{v}_B$ coding B-integers as defined in this paper represent a very rich family of infinite sequences. Similarly as we have identified sturmian sequences among the infinite words $\mathbf{v}_B$, it would be interesting to find which of those infinite words $\mathbf{v}_B$ belong to a recently introduced class of dendric sequences [6]. Dendric sequences appear to be S-adic [8] and generalize two very extensively studied classes of infinite sequences, namely sequences coding interval exchange transformations and episturmian sequences.

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