INTEGERS IN REAL NUMERATION SYSTEMS

ÉMILIE CHARLIER[1], CÉLIA CISTERNINO[1], ZUZANA MASÁKOVÁ[2] AND EDITA PELANTOVÁ[2]

A *real numeration system* is defined by a strictly increasing biinfinite sequence $\mathcal{U} = (\mathcal{U}_n)_{n\in\mathbb{Z}}$ of positive real numbers such that

$$\mathcal{U}_0 = 1, \quad \lim_{n\to+\infty} \mathcal{U}_n = +\infty, \quad \lim_{n\to-\infty} \mathcal{U}_n = 0. \tag{1}$$

A *representation* of a real number $x \geq 0$ in the system $\mathcal{U}$ is a biinfinite sequence $(x_n)_{n\in\mathbb{Z}}$ of non-negative integers such that $x = \sum_{n\in\mathbb{Z}} x_n \mathcal{U}_n$. By (1), necessarily there exists $N \in \mathbb{Z}$ such that $x_n = 0$ for each $n \geq N$. The lexicographically largest of all representations of $x \geq 0$ in the system $\mathcal{U}$ is obtained by the greedy algorithm and it is called $\mathcal{U}$*-expansion* of x. The $\mathcal{U}$-expansion of x is denoted $(x)_\mathcal{U}$ and we write

$$(x)_\mathcal{U} = \begin{cases} x_{N-1}x_{N-2}\cdots x_0 \bullet x_{-1}x_{-2}\cdots & \text{if } N > 0,\\ 0 \bullet 0^N x_{N-1}x_{N-2}\cdots & \text{if } N \leq 0.\end{cases}$$

The above concept includes the classical b-ary expansions, with $b \in \mathbb{N}_{\geq 2}$, if $\mathcal{U}_n = b^n$, or more general Rényi β-expansions, with $\beta \in \mathbb{R}_{>1}$, if $\mathcal{U}_n = \beta^n$ for $n \in \mathbb{Z}$.

A non-negative real number x is called a $\mathcal{U}$*-integer* if the $\mathcal{U}$-expansion of x is of the form

$$(x)_\mathcal{U} = x_N x_{N-1} \cdots x_0 \bullet 0^\omega .$$

The set of all $\mathcal{U}$-integers is denoted by $\mathbb{N}_\mathcal{U}$. We thus generalize the notion of β-integers, introduced for the Rényi numeration system by Burík et al. [1].

Our first aim is to study the generalization of the classical properties of the positional numeration systems [3] to the real numeration system framework. Next, we study certain properties of the set $\mathbb{N}_\mathcal{U}$. We describe the distances between consecutive elements of $\mathbb{N}_\mathcal{U}$. We show that the infinite word $w_\mathcal{U}$ (over an infinite alphabet) coding the ordering of the distances in $\mathbb{N}_\mathcal{U}$ is an S-adic word. As the main tool we use the results on the recently defined Cantor real base and alternate base systems [2]. In fact, we link the $\mathcal{U}$-expansion of a non-negative real number x and the $\boldsymbol{\beta}$-expansion of $x/\mathcal{U}_N$ where $\boldsymbol{\beta}$ is the Cantor real base $(\mathcal{U}_n/\mathcal{U}_{n-1})_{n\leq N}$. We give a necessary and sufficient condition so that the distances between consecutive $\mathcal{U}$-integers take only finitely many values. In that case we show that the word $w_\mathcal{U}$ can be projected to an infinite word over a finite alphabet which is a fixed point of a substitution. The incidence matrix of the substitution is irreducible and, as a consequence, we may derive using the Perron-Frobenius theorem a result on uniqueness in alternate base systems, i.e., in the systems where $(\mathcal{U}_n/\mathcal{U}_{n-1})_{n\leq 0}$ is purely periodic.

References

[1] Č. Burdík, C. Frougny, J. P. Gazeau, and R. Krejcar. Beta-integers as natural counting systems for quasicrystals. *J. Phys. A*, 31(30):6449–6472, 1998.

[2] É. Charlier and C. Cisternino. Expansions in Cantor real bases. *Monatsh. Math.*, 195:585–610, 2021.

[3] M. Lothaire. *Algebraic combinatorics on words*, volume 90 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.

[1]Department of Mathematics, University of Liège, Allée de la Découverte 12, 4000 Liège, Belgium,
[2]Department of Mathematics, Czech Technical University in Prague, Trojanova 13, 120 00 Praha 2, Czech Republic

E-mail address: echarlier@uliege.be, ccisternino@uliege.be zuzana.masakova@fjfi.cvut.cz and edita.pelantova@fjfi.cvut.cz
.