Substitutions and Cantor real numeration systems

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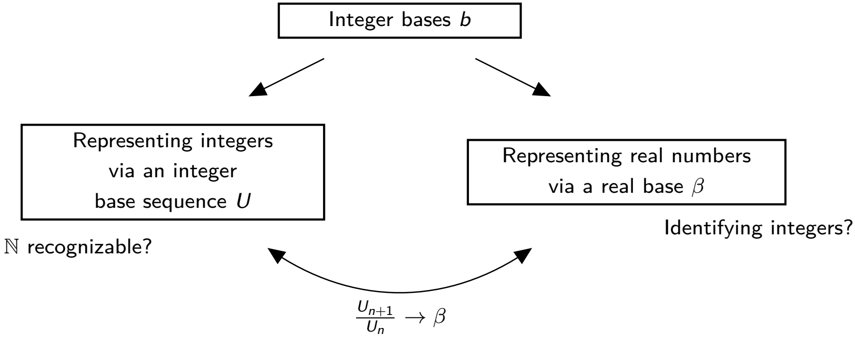
Motivation

In base 2, we write 78 as 1001110 and 7/3 as $10 \bullet 01010101 \cdots$.



When $\frac{U_{n+p}}{U_n} \to \beta$, there is a similar relationship with representations of real numbers via some alternate base $B = (\beta_{p-1}, \ldots, \beta_0)$.

Cantor real numeration systems

A Cantor real base is a biinfinite sequence $B = (\beta_n)_{n\in\mathbb{Z}}$ of bases such that

- $\beta_n \in \mathbb{R}_{>1}$ for all n
- $\prod_{n\geq 0} \beta_n = \prod_{n\geq 1} \beta_{-n} = +\infty$.

We consider biinfinite sequences $a = (a_n)_{n\in\mathbb{Z}}$ over $\mathbb{N}$ having a left tail of zeros, that is, there exists some $N \in \mathbb{Z}$ such that $a_n = 0$ for all $n \geq N$.

$$\begin{cases} a_{N-1}\cdots a_0 \bullet a_{-1}a_{-2}\cdots & \text{if } N \geq 1 \\ 0 \bullet 0^{-N} a_{N-1}a_{N-2}\cdots & \text{if } N \leq 0. \end{cases}$$

The associated value map is defined as

$$\mathrm{val}_B(a) = \cdots + a_3\beta_2\beta_1\beta_0 + a_2\beta_1\beta_0 + a_1\beta_0 + a_0 + \frac{a_{-1}}{\beta_{-1}} + \frac{a_{-2}}{\beta_{-1}\beta_{-2}} + \cdots$$

provided that the series is convergent.

If $x = \mathrm{val}_B(a)$, we say that a is a B-representation of x.

The greedy algorithm

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- For $x \in [0, 1)$:
  - We first set $r_{-1} = x$.
  - Then for $n < 0$, we iteratively compute $a_n = \lfloor \beta_n r_n \rfloor$ and $r_{n-1} = \beta_n r_n - a_n$.
  - Then $d_B(x) = 0 \bullet a_{-1} a_{-2} \cdots$

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  - Then $d_B(x) = 0 \bullet a_{-1} a_{-2} \cdots$
- For $x \geq 1$:
  - We let $N \geq 1$ be the minimal integer such that $x < \beta_{N-1} \cdots \beta_0$.
  - Let $S(B) = (\beta_{n+1})_{n \in \mathbb{Z}}$ and compute $d_{S^N(B)}\left(\frac{x}{\beta_{N-1} \cdots \beta_0}\right) = 0 \bullet a_{N-1} a_{N-2} \cdots$.
  - Then the B-expansion of x is defined as $d_B(x) = a_{N-1} \cdots a_0 \bullet a_{-1} a_{-2} \cdots$.

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In particular:

- The greedy digits a_n belong to the alphabet $\{0, \ldots, \lceil \beta_n \rceil - 1\}$ for all n.
- We have $d_B(1) = 1 \bullet 0^\omega$.

Let's look at a few examples

- $B = (1+2^n)_{n\in\mathbb{Z}}$ is not a Cantor real base since $\prod_{n\geq 1}(1+\frac{1}{2^n}) \sim 2.38423$.

  If we perform the greedy algorithm on $x = \frac{1}{2}$ then we obtain the digits $0 \bullet 0010^\omega$, although $\mathrm{val}_B(0 \bullet 0010^\omega) = \frac{64}{135} \neq \frac{1}{2}$.

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- $B = (2 + 2^n)_{n \in \mathbb{Z}}$ is a Cantor real base since $\prod_{n \geq 0}(2 + 2^n) = \infty$ and $\prod_{n \geq 1}(2 + \frac{1}{2^n}) = \infty$.

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An alternate base is a periodic Cantor real base. In this case, we simply write

$$B = (\beta_{p-1}, \dots, \beta_0)$$

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- For $B = (\sqrt{6}, 3, \frac{2+\sqrt{6}}{3})$, we have $d_B\left(1 - \frac{1}{\sqrt{6}}\right) = 0 \bullet 1(10)^\omega$.

$r_0 = 1 - \frac{1}{\sqrt{6}}$	$a_0 = \left\lfloor \sqrt{6} r_0 \right\rfloor = \left\lfloor -1 + \sqrt{6} \right\rfloor = 1$
$r_1 = \sqrt{6} r_0 - a_0 = -2 + \sqrt{6}$	$a_1 = \lfloor 3 r_1 \rfloor = \left\lfloor -6 - 3\sqrt{6} \right\rfloor = 1$
$r_2 = 3 r_1 - a_1 = -7 + 3\sqrt{6}$	$a_2 = \left\lfloor \frac{2+\sqrt{6}}{3} r_2 \right\rfloor = \left\lfloor \frac{4-\sqrt{6}}{3} \right\rfloor = 0$
$r_3 = \frac{2+\sqrt{6}}{3} r_2 - a_2 = \frac{4-\sqrt{6}}{3}$	$a_3 = \left\lfloor \sqrt{6} r_3 \right\rfloor = \left\lfloor \frac{-6+4\sqrt{6}}{3} \right\rfloor = 1$
$r_4 = \sqrt{6} r_3 - a_3 = \frac{-9+4\sqrt{6}}{3}$	$a_4 = \lfloor 3 r_4 \rfloor = \left\lfloor -9 + 4\sqrt{6} \right\rfloor = 0$
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$r_6 = \frac{2+\sqrt{6}}{3} r_5 - a_5 = \frac{3-\sqrt{6}}{3}$	$a_6 = \left\lfloor \sqrt{6} r_6 \right\rfloor = \left\lfloor -2 + \sqrt{6} \right\rfloor = 0$
$r_7 = \frac{2+\sqrt{6}}{3} r_6 - a_6 = -2 + \sqrt{6}$	$a_7 = \lfloor 3 r_7 \rfloor = \left\lfloor -6 - 3\sqrt{6} \right\rfloor = 1$

Parry's theorem for Cantor real bases

Theorem (C. & Cisternino 2021)

A sequence $0 \bullet a_{-1}a_{-2}\cdots$ *is the* B*-expansion of some number* $x \in [0,1)$ *if and only if* $a_{n-1}a_{n-2}\cdots <_{\mathrm{lex}} d^*_{S^n(B)}(1)$ *for all* n*.*

Here we used the quasi-greedy B-expansion of 1, which is given by

$$d^*_B(1) = d_1d_2d_3\cdots$$

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For $B = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$, we can compute

$$d^*_B(1) = 20(01)^\omega = 20010101\cdots \quad \text{and} \quad d^*_{S(B)}(1) = (10)^\omega = 101010\cdots.$$

The sequence

$$0 \bullet 20001020(001)^\omega$$

is the B-expansion of some $x \in [0,1)$, whereas it is not the case of the sequence

$$0 \bullet 2000120(001)^\omega.$$

The B-integers

A real number $x \geq 0$ is a B-integer if its B-expansion is of the form

$$d_B(x) = a_{n-1} \cdots a_0 \bullet 0^\omega \quad \text{with } n \in \mathbb{N}.$$

The set of all B-integers is denoted by $\mathbb{N}_B$.

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Proof of dicreteness: The B-expansion of a B-integer smaller than $\beta_{n-1} \cdots \beta_0$ is of the form $a_m a_{m-1} \cdots a_0 \bullet 0^\omega$ with $m \leq n$. Since $a_i < \beta_i$ for each i, there are only finitely many B-expansions having this property.

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Let $(x_k)_{k \in \mathbb{N}}$ be the increasing sequence of B-integers:

$$\mathbb{N}_B = \{x_k : k \in \mathbb{N}\}.$$

For every $n \in \mathbb{N}$, we define $M_{B,n} = \max\{x \in \mathbb{N}_B : x < \beta_{n-1} \cdots \beta_0\}$.

As a consequence of the characterization of admissible sequences, we obtain:

Proposition

For all $n \in \mathbb{N}$, if we write $d^*_{S^n(B)}(1) = d_{n,1} d_{n,2} d_{n,3} \cdots$, then $d_B(M_{B,n}) = d_{n,1} \cdots d_{n,n} \bullet 0^\omega$.

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$B = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$

Since $d^*_B(1) = 20(01)^\omega = 20010101\cdots$ and $d^*_{S(B)}(1) = (10)^\omega = 101010\cdots$, we can compute the numbers $M_{B,n}$ as follows:

n	$d_B(M_{B,n})$	$M_{B,n}$
0	ε	0
1	1	1
2	20	$\frac{5+\sqrt{13}}{3}$
3	101	$\frac{5+\sqrt{13}}{2}$
4	2001	$\frac{17+4\sqrt{13}}{3}$
5	10101	$8+2\sqrt{13}$
6	200101	$\frac{109+29\sqrt{13}}{6}$
7	1010101	$26+7\sqrt{13}$

Let us now compute the first B-integers x_k:

k	x_k	$d_B(x_k)$	k	x_k	$d_B(x_k)$	k	x_k	$d_B(x_k)$
0	0	ε	12	8.03	1100	24	16.64	100001
1	1	1	13	9.03	1101	25	17.07	100010
2	1.43	10	14	9.47	2000	26	18.07	100011
3	2.43	11	15	10.47	2001	27	18.51	100020
4	2.86	20	16	10.90	10000	28	18.94	100100
5	3.30	100	17	11.90	10001	29	19.94	100101
6	4.30	101	18	12.34	10010	30	20.38	101000
7	4.73	1000	19	13.34	10011	31	21.38	101001
8	5.73	1001	20	13.77	10020	32	21.81	101010
9	6.17	1010	21	14.21	10100	33	22.81	101011
10	7.17	1011	22	15.21	10101	34	23.25	101020
11	7.60	1020	23	15.64	100000	35	23.68	101100

Distances between B-integers

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The distances between consecutive B-integers take only values of the form

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accordingly to the first position $n \geq 0$ *where their B-expansions differ (from left to right).*

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Note that:

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We consider the infinite sequence

$$w_B = (w_k)_{k \in \mathbb{N}}$$

where

$$w_k = n$$

if $d_B(x_k)$ and $d_B(x_{k+1})$ differ at index n and not at greater indices.

$B = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$

We can compute a prefix of w_B by looking at the first position where consecutive B-integers differ:

k	x_k	$d_B(x_k)$	w_B	k	x_k	$d_B(x_k)$	w_B	k	x_k	$d_B(x_k)$	w_B
0	0	ε	0	12	8.03	1100	0	24	16.64	100001	1
1	1	1	1	13	9.03	1101	3	25	17.07	100010	0
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11	7.60	1020	2	23	15.64	100000	0	35	23.68	101100	0

$$w_B = 010120301012030401012050101203010120\cdots$$

The sequence w_B is S-adic

Proposition

We have $\psi_B(w_{S(B)}) = w_B$ *where* ψ_B *is the substitution over* $\mathbb{N}$ *defined by*

$$\psi_B \colon \mathbb{N} \to \mathbb{N}^*, \; n \mapsto 0^{a_{n+1}}(n+1)$$

where a_n *is the least significant digit of* $d_B(M_{B,n})$.

By the term substitution, we mean that $\psi_B(w_0 w_1 w_2 \cdots) = \psi_B(w_0)\psi_B(w_1)\psi_B(w_2)\cdots$.

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Corollary

- *For an alternate base* $B = (\beta_{p-1}, \dots, \beta_0)$*, the sequence* w_B *is fixed by the composition* $\psi_B \circ \cdots \circ \psi_{S^{p-1}(B)}$.
- *In general, the sequence* w_B *is the* S*-adic sequence given by the sequence of substitutions* $(\psi_{S^n(B)})_{n\in\mathbb{N}}$ *applied on the letter* 0*:*

$$w_B = \lim_{n\to+\infty} \psi_B \circ \psi_{S(B)} \circ \cdots \circ \psi_{S^n(B)}(0).$$

Computing $\psi_B \colon \mathbb{N} \to \mathbb{N}^*,\ n \mapsto 0^{a_{n+1}}(n+1)$ for $B = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$

We get that

$$d_B(M_{B,2n}) \text{ and } d_{S(B)}(M_{S(B),2n+1}) \text{ are prefixes of } d_B^*(1) = 20010101\cdots$$

and

$$d_B(M_{B,2n+1}) \text{ and } d_{S(B)}(M_{S(B),2n}) \text{ are prefixes of } d_{S(B)}^*(1)) = 101010\cdots .$$

We then obtain the two substitutions

$$\psi_B\colon \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 2 \\ n \mapsto 0(n+1) & \text{for } n \geq 2 \end{cases} \quad \text{and} \quad \psi_{S(B)}\colon \begin{cases} 0 \mapsto 001 \\ n \mapsto n+1 & \text{for } n \geq 1. \end{cases}$$

and their composition

$$\Phi_B = \varphi_B \circ \varphi_{S(B)}\colon \begin{cases} 0 \mapsto 01012 \\ n \mapsto 0(n+2) & \text{for } n \geq 1 \end{cases}$$

fixes w_B:

$$w_B = \Phi_B^\omega(0) = (01012)(03)(01012)(03)(04)(01012)(05)(01012)(03)(01012)(03)(04)\cdots$$

More can be said for alternate bases

Theorem (C., Cisternino, Masáková & Pelantová 2024+)

Let $B = (\beta_{p-1}, \ldots, \beta_0)$ *be an alternate base. There are finitely many possible distances between consecutive* B*-integers if and only if the base* B *is Parry, meaning that* $d^*_{S^i(B)}(1)$ *is eventually periodic for each* i*.*

For such a base B, we can encode the distances between consecutive B-integers by a sequence taking only finitely many values.

For $B = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$, we consider the writings

$$d^*_B(1) = 20(01)^\omega, \quad d^*_{S(B)}(1) = (10)^\omega = 10(10)^\omega.$$

in order to obtain common preperiods and periods multiple that are multiple of $p = 2$, and the projection

$$\pi\colon \mathbb{N} \to \{0,1,2,3\}, \; n \mapsto \begin{cases} n, & \text{if } n \in \{0,1\}; \\ 2, & \text{if } n \geq 2, \textit{even}; \\ 3, & \text{if } n \geq 2, \textit{odd}. \end{cases}$$

The projected sequence $v_B = \pi(w_B)$ also codes the distances between consecutive B-integers:
$v_k = v_{k'} \implies x_{k+1} - x_k = x_{k'+1} - x_{k'}$.

k	x_k	$d_B(x_k)$	w_B	v_B	k	x_k	$d_B(x_k)$	w_B	v_B	k	x_k	$d_B(x_k)$	w_B	v_B
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3	2.43	11	1	1	15	10.47	2001	4	2	27	18.51	100020	2	2
4	2.86	20	2	2	16	10.90	10000	0	0	28	18.94	100100	0	0
5	3.30	100	0	0	17	11.90	10001	1	1	29	19.94	100101	3	3
6	4.30	101	3	3	18	12.34	10010	0	0	30	20.38	101000	0	0
7	4.73	1000	0	0	19	13.34	10011	1	1	31	21.38	101001	1	1
8	5.73	1001	1	1	20	13.77	10020	2	2	32	21.81	101010	0	0
9	6.17	1010	0	0	21	14.21	10100	0	0	33	22.81	101011	1	1
10	7.17	1011	1	1	22	15.21	10101	5	3	34	23.25	101020	2	2
11	7.60	1020	2	2	23	15.64	100000	0	0	35	23.68	101100	0	0

$$w_B = 010120301012030401012050101203010120\cdots$$

$$v_B = 010120301012030201012030101203010120\cdots$$

The two projected substitutions over the finite alphabet $\{0, 1, 2, 3\}$ are

$$\varphi_B : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 2 \\ 2 \mapsto 03 \\ 3 \mapsto 02 \end{cases} \quad \text{and} \quad \varphi_{S(B)} : \begin{cases} 0 \mapsto 001 \\ 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 2. \end{cases}$$

and their composition

$$\Phi_B = \varphi_B \circ \varphi_{S(B)} : \begin{cases} 0 \mapsto 01012 \\ 1 \mapsto 03 \\ 2 \mapsto 02 \\ 3 \mapsto 03 \end{cases}$$

is a primitive substitution that fixes v_B:

$$v_B = \Phi_B^{\omega}(0) = (01012)(03)(01012)(03)(02)(01012)(03)(01012)(03)(01012)(03)(02)\cdots$$

Suppose that all $d_{S^i(B)}(1)$ have the same preperiod ℓ and period m, which are multiple of p.

We define a projection

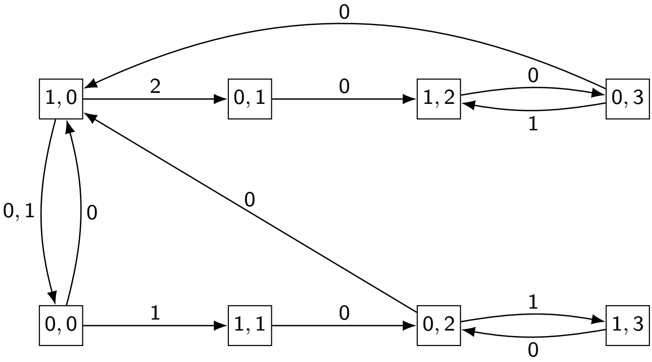
$$\pi\colon \mathbb{N} \to \{0,\ldots,\ell+m-1\},\ n \mapsto \begin{cases} n, & \text{if } 0 \le n \le \ell+m-1; \\ \ell + ((n-\ell) \bmod m), & \text{if } n \ge \ell+m. \end{cases}$$

Then we consider the projected sequence $v_B = \pi(w_B)$ and the substitution φ_B defined by $\varphi_B(n) = \pi(\psi_B(n))$ for $n \in \{0,\ldots,\ell+m-1\}$.

Theorem (C., Cisternino, Masáková & Pelantová 2024+)

The composition $\varphi_B \circ \varphi_{S(B)} \circ \cdots \circ \varphi_{S^{p-1}(B)}$ is a primitive substitution which fixes v_B.

A graph associated with $B = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$ is built from the quasi-greedy expansions $d_B^*(1) = 20(01)^\omega$ and $d_{S(B)}^*(1) = 10(10)^\omega$.



- We can see the subtitutions φ_B and $\varphi_{S(B)}$ in this graph.
- The primitiveness of the composition $\varphi_B \circ \varphi_{S(B)}$ corresponds to the strong connectiveness of the graph.

Combinatorial properties of v_B

A sequence $a_1 a_2 a_3 \cdots$ is sturmian if it has exactly $n+1$ length-n factors $a_i \cdots a_{i+n-1}$ for all n.

Proposition (C., Cisternino, Masáková & Pelantová 2024+)

Let $B = (\beta_{p-1}, \ldots, \beta_0)$ be a Parry alternate base. The sequence v_B is sturmian if and only if one of the following cases is satisfied.

Case 1. $p = 1$ *and* $d_B^*(1) = (d0)^\omega$ *with* $d \geq 1$.

Case 2. $p = 1$ *and* $d_B^*(1) = (d+1)d^\omega$ *with* $d \geq 1$.

Case 3. $p = 2$, $d_B^*(1) = (d0)^\omega$ *and* $d_{S(B)}^*(1) = (e0)^\omega$ *with* $d, e \geq 1$.

In all cases, one can derive frequencies ρ_0, ρ_1 of letters 0 and 1 in the sturmian sequence v_B from the primitive substitution.

We write $x = [a_0, a_1, a_2, \ldots]$ if

$$\gamma = \lim_{n \to +\infty} a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \; a_{n-1} + \cfrac{1}{a_n}}}}$$

and $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}_{\geq 1}$ for every $n > 0$.

If the sequence $a_0, a_1, a_2, \ldots$ is eventually periodic, then we use the notation

$$[a_0, a_1, \ldots, a_i, \overline{a_{i+1}, a_{i+2}, \ldots, a_{i+k}}].$$

Proposition (Continued)

Case 1. *We have* $(\rho_0, \rho_1) = \left(\frac{\beta_0}{\beta_0+1}, \frac{1}{\beta_0+1}\right)$ *and* $\rho_0 = [0, 1, \overline{d}]$.

Case 2. *We have* $(\rho_0, \rho_1) = \left(\frac{\beta_0-1}{\beta_0}, \frac{1}{\beta_0}\right)$ *and* $\rho_0 = [0, \overline{1, d}]$.

Case 3. *We have* $(\rho_0, \rho_1) = \left(\frac{\beta_1}{\beta_1+1}, \frac{1}{\beta_1+1}\right)$ *and* $\rho_0 = [0, 1, \overline{e, d}]$.

Surprisingly, one can obtain a sturmian sequence v_B with frequency $\rho_0 = [0, \overline{1, a}]$ in different numeration systems.

- For $p = 1$, this is only possible for $a = 1$ and the real bases τ and τ^2 where $\tau = \frac{1+\sqrt{5}}{2}$.
  - τ belongs to Case 1 with $d = 1$.
  - τ^2 belongs to Case 2 with $d = 1$.

- If we allow $p \in \{1, 2\}$ then there are infinitely many pairs of numeration systems giving the same frequency $\rho_0 = [0, \overline{1, a}]$.
  - $p = 1$ with $d_B^*(1) = (a + 1)a^\omega$.

    For $a = 2$, we obtain the real base $(2 + \sqrt{3})$.
    The sequence v_B is fixed by the substitution $0 \mapsto 0001$ and $1 \mapsto 001$.
  - $p = 2$ with $d_B^*(1) = (10)^\omega$ and $d_{S(B)}^*(1) = (a0)^\omega$.

    For $a = 2$, we get the alternate base $B = (\beta_1, \beta_0) = (\frac{1+\sqrt{3}}{2}, 1 + \sqrt{3})$.
    The sequence v_B is fixed by another substitution, namely, $0 \mapsto 0010$ and $1 \mapsto 001$.

Minimal alphabet

In our specific example $B = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$, since $\Delta_{B,1} = \Delta_{B,2} = \Delta_{B,3}$, the image

$$\sigma(v_B) = 0101101010110101\cdots$$

under the projection

$$\sigma\colon \{0,1,2,3\}^* \to \{0,1\}^*, \quad \begin{cases} 0 \mapsto 0 \\ 1,2,3 \mapsto 1 \end{cases}$$

contains enough information to encode the distances between consecutive B-integers.

This new infinite sequence $\sigma(v_B)$ is the fixed point of the projected substitution

$$\begin{cases} 0 \mapsto 01011 \\ 1 \mapsto 01. \end{cases}$$

and hence is sturmian.

Minimal alphabet

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Thank you!