Dissipative Quantum Phase Transition in an Interacting Many-Particle System: from Two-Level to Multilevel Spins

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LIÈGE université Sciences

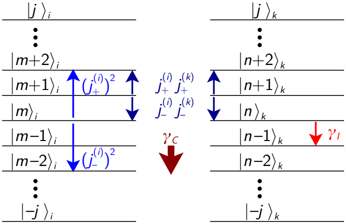
EOS THE EXCELLENCE OF SCIENCE

Spin-j Dissipative Lipkin-Meshkov-Glick Model

N spin-j particles, $J_\alpha = \sum_{i=1}^{N} j_\alpha^{(i)}$

Hamiltonian

$$H = \frac{V}{2N}\left(J_+^2 + J_-^2\right)$$



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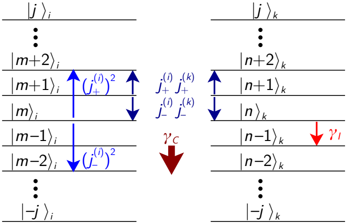

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$$L_i = \sum_{m=-j}^{j-1} \ell_m \left|m\right\rangle_i \left\langle m+1\right|_i \overset{j=\frac{1}{2}}{\sim} \sigma_-, \qquad L_C = \sum_{i=1}^{N} L_i \overset{j=\frac{1}{2}}{\sim} J_-$$

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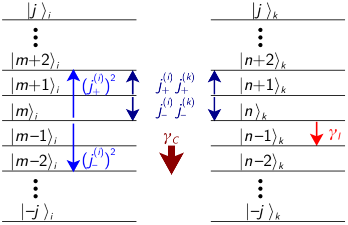

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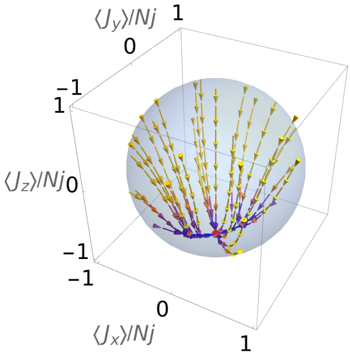
Lindblad master equation

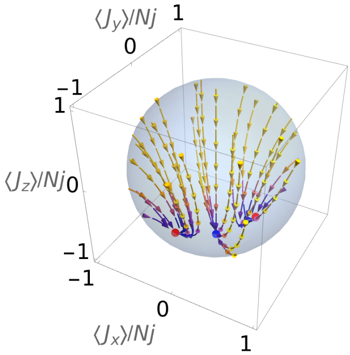
$$\dot{\rho} = -\mathrm{i}[H,\rho] + \gamma_I \sum_{i=1}^{N} \left(L_i \rho L_i^\dagger - \frac{1}{2}\left\{ L_i^\dagger L_i, \rho \right\} \right) + \frac{\gamma_C}{N}\left(L_C \rho L_C^\dagger - \frac{1}{2}\left\{ L_C^\dagger L_C, \rho \right\} \right)$$

Qubit Model in Mean-Field Limit $N \to \infty$

Dissipative (steady-state) phase transition at $\gamma_I + \gamma_C = 2|V|$:



$(\gamma_I, \gamma_C) = (2.3V, 0)$



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T. E. Lee et al., PRL **110**, 257204 (2013), T. E. Lee et al., PRA **90**, 052109 (2014)

Qubit Model in Mean-Field Limit $N \to \infty$

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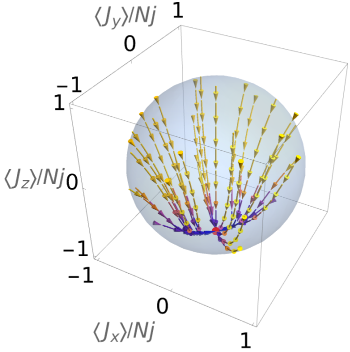
Different interaction-dominated phases for $\gamma_I \neq 0$ and $\gamma_I = 0$.



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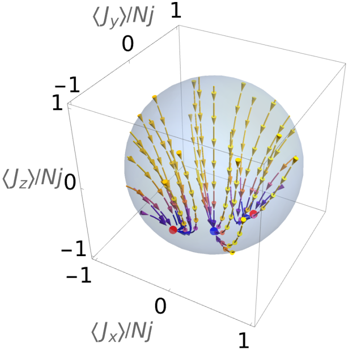

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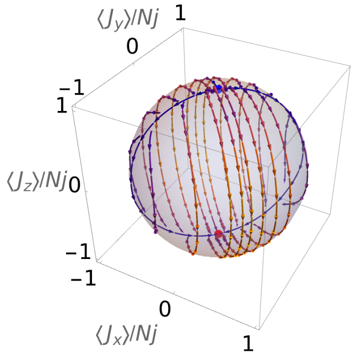
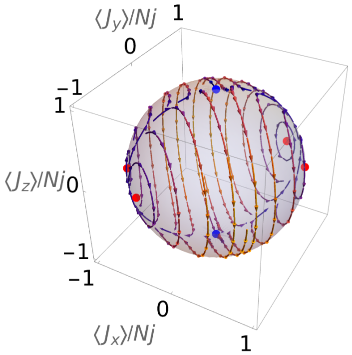


$(\gamma_I, \gamma_C) = (0, 2.3V)$



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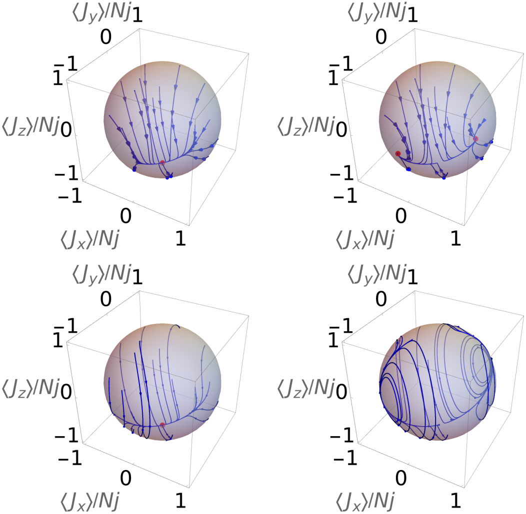
E.g. $j = 3/2$, $L_i = j_-^{(i)}$

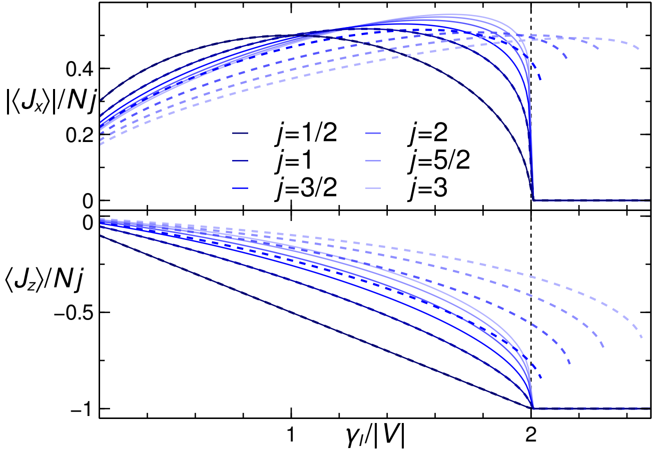
$(\gamma_I, \gamma_C) = (\gamma_0, 0)$:

$(\gamma_I, \gamma_C) = (0, \gamma_0)$:



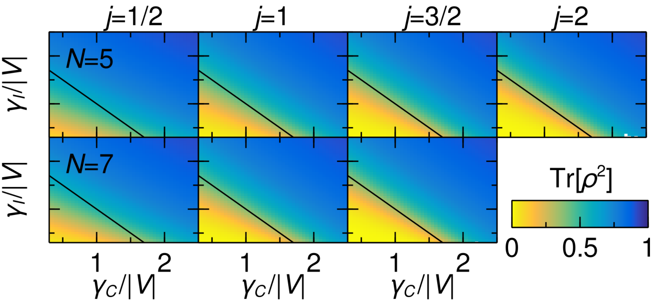
$\gamma_0 = 2.3V$ $\gamma_0 = 1.7V$

Steady States for Multilevel Systems ($N \to \infty$)

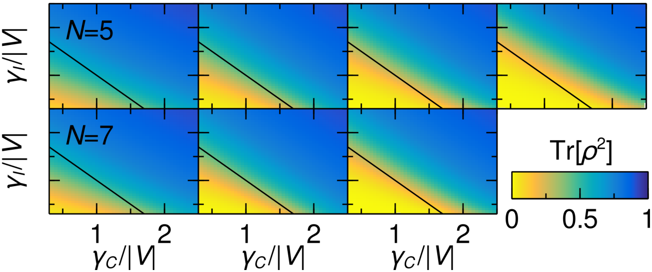


$$\gamma_C = 0, \quad L_i^{(1)} = j_-^{(i)}, \quad L_i^{(2)} = \sqrt{2j} \sum_{m=-j}^{j-1} |m\rangle \langle m+1|$$

Steady-State Purity $\mathrm{Tr}[\rho^2]$



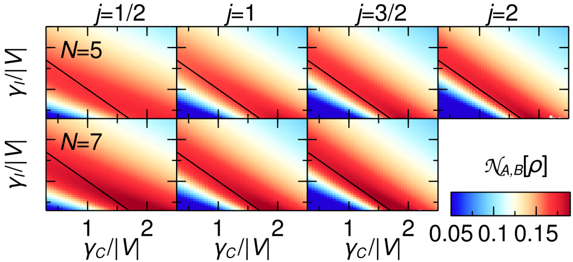
$L_i^{(1)}$



$L_i^{(2)}$

Steady-State Entanglement: Negativity $\mathcal{N}_{A|B}[\rho]$

Subsystems A, B, partial transpose ρ^{T_B}
with negative eigenvalues $\lambda_1, \cdots, \lambda_n$: $\quad \mathcal{N}_{A|B}[\rho] = \sum_{i=1}^{n} |\lambda_i|$



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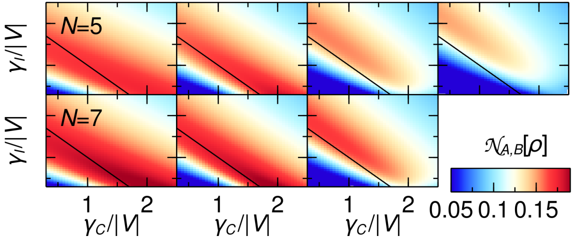
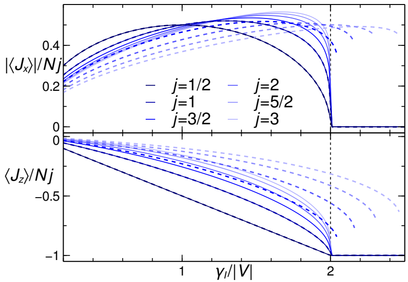


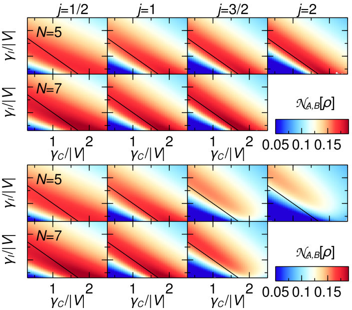
$L_i^{(2)}$

$A = \left\lceil \frac{N}{2} \right\rceil$ particles,
$B = \left\lfloor \frac{N}{2} \right\rfloor$ particles

Conclusions

1. Despite same critical point $\gamma_I + \gamma_C = 2|V|$ as for $j = \frac{1}{2}$:
   New critical region, change of $\langle J \rangle$ at transition stronger with j
2. Sharp change of steady-state purity & maximum of its entanglement negativity at phase transition





Supplemental Material: Details on Symmetries of $\mathcal{L}$

Symmetry: Superoperator Π such that $[\Pi, \mathcal{L}] = 0$.

1. For all $\pi \in \mathrm{S}_N$: $\Pi_\pi : \rho \mapsto \pi \rho \pi^\dagger$.
2. $\Pi_1 : \rho \mapsto e^{\mathrm{i}\pi J_z} \rho e^{-\mathrm{i}\pi J_z}$, with

$$\langle J_x \rangle_{\Pi_1[\rho]} = \mathrm{Tr}(J_x \Pi_1[\rho]) = - \langle J_x \rangle_\rho ,$$
$$\langle J_y \rangle_{\Pi_1[\rho]} = - \langle J_y \rangle_\rho ,$$
$$\langle J_z \rangle_{\Pi_1[\rho]} = \langle J_z \rangle_\rho .$$

3. If L_i and L_C are real operators, $\Pi_2 : \rho \mapsto e^{\mathrm{i}\pi J_z/2} \rho^* e^{-\mathrm{i}\pi J_z/2}$, with

$$\langle J_x \rangle_{\Pi_2[\rho]} = \langle J_y \rangle_\rho ,$$
$$\langle J_y \rangle_{\Pi_2[\rho]} = \langle J_x \rangle_\rho ,$$
$$\langle J_z \rangle_{\Pi_2[\rho]} = \langle J_z \rangle_\rho .$$

Spectrum of the Liouvillian and Symmetry Breaking

Master equation: $\dot{\rho} = \mathcal{L}[\rho]$, $\dim \mathcal{L} = \binom{N+4j(j+1)}{N} \times \binom{N+4j(j+1)}{N}$

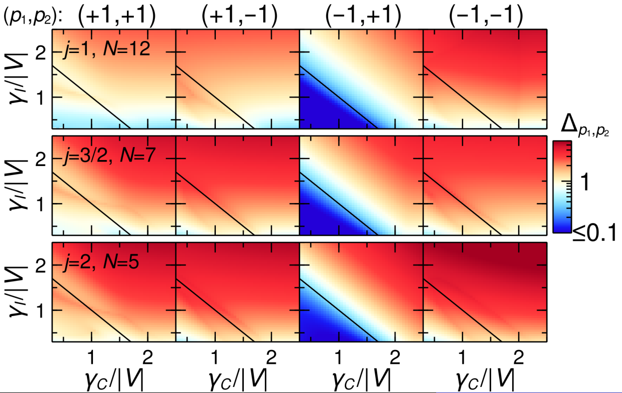
Steady state: $\mathcal{L}[\rho] = 0$

Symmetry breaking:

$\Delta = |\lambda_1| \overset{N\to\infty}{\to} 0$ (eigenvalue $\lambda_1 \neq 0$ closest to 0)

in corresponding symmetry sector

F. Minganti et al., PRA **98**, 042118 (2018).



$L_i^{(1)}$

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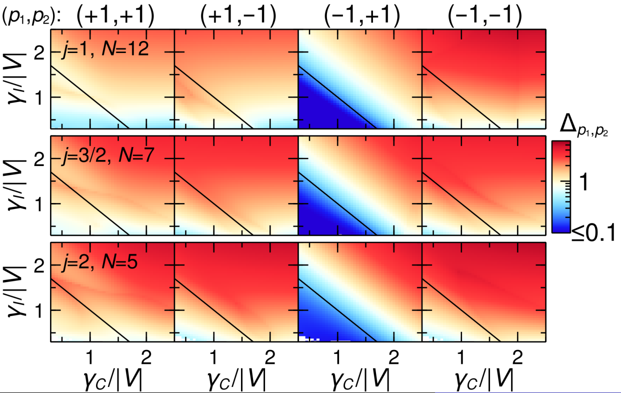
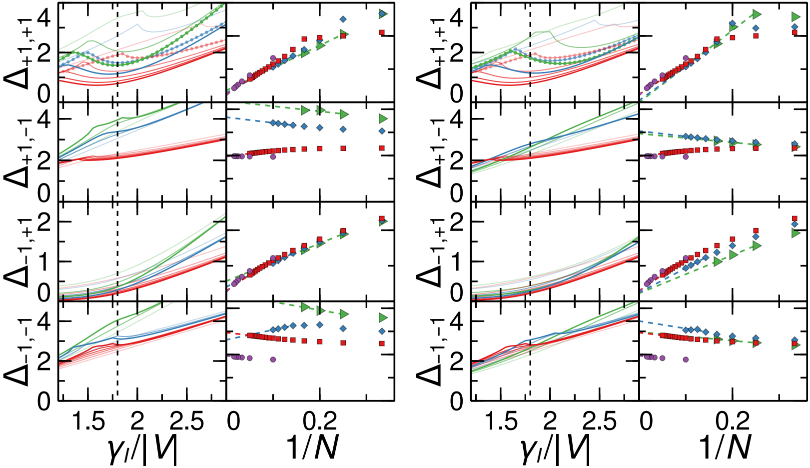


$L_i^{(2)}$

Supplemental Material: Details on Spectrum, $\gamma_C = 0.2$



$L_i = j_-^{(i)}$ $\qquad$ $L_i = \sqrt{2j}\sum_{m=-j}^{j-1} |m\rangle\langle m+1|$

$j = \frac{1}{2}$, $j = 1$, $j = \frac{3}{2}$, $j = 2$

Supplemental Material: Mean-Field for $j = 1/2$

Mean-field assumptions for $N \to \infty$:

$$\frac{1}{N^2} \langle \{J_\alpha, J_\beta\} \rangle \approx \frac{2}{N^2} \langle J_\alpha \rangle \langle J_\beta \rangle ,$$

justified by

$$\frac{\langle J_\alpha \rangle}{N} \overset{N\to\infty}{=} \mathcal{O}(1), \qquad \frac{\langle [J_\alpha, J_\beta] \rangle}{N^2} \overset{N\to\infty}{=} \mathcal{O}(N^{-1}).$$

With $X := 2 \langle J_x \rangle / N$, $Y := 2 \langle J_y \rangle / N$, $Z := 2 \langle J_z \rangle / N$ get

$$\begin{aligned} \dot{X} &= -2VYZ - \gamma_I X + \gamma_C XZ, \\ \dot{Y} &= -2VXZ - \gamma_I Y + \gamma_C YZ, \\ \dot{Z} &= 4VXY - 2\gamma_I (Z+1) - \gamma_C (X^2 + Y^2), \end{aligned}$$

Supplemental Material: Mean-Field for Arbitrary j

$$S^{(i)}_{x;m,n} = \frac{1}{2N}\sum_{i=1}^{N}\left(|m\rangle_i \langle n|_i + |n\rangle_i \langle m|_i\right),$$

$$S^{(i)}_{y;m,n} = -\frac{\mathrm{i}}{2N}\sum_{i=1}^{N}\left(|m\rangle_i \langle n|_i - |n\rangle_i \langle m|_i\right),$$

with

$$\langle\{S_{\alpha;m,n}, S_{\beta;o,p}\}\rangle \approx 2\langle S_{\alpha;m,n}\rangle\langle S_{\beta;o,p}\rangle \quad (N\to\infty).$$

Equations of motion:

$$\begin{aligned}\partial_t \langle S_{\alpha;\mu,\nu}\rangle =& \mathrm{i}\langle[H, S_{\alpha;\mu,\nu}]\rangle + \frac{\gamma_I}{2j}\sum_{i=1}^{N}\left\langle [L_i^\dagger, S_{\alpha;\mu,\nu}]L_i + L_i^\dagger[S_{\alpha;\mu,\nu}, L_i]\right\rangle \\ &+ \frac{\gamma_C}{2Nj}\left\langle [L_C^\dagger, S_{\alpha;\mu,\nu}]L_C + L_C^\dagger[S_{\alpha;\mu,\nu}, L_C]\right\rangle \\ =& V\ f_1(\langle S_{\beta;o,p}\rangle) + \gamma_I\ g(\langle S_{\beta;o,p}\rangle) \\ &+ \gamma_C\ f_2(\langle S_{\beta;o,p}\rangle)\end{aligned}$$

Supplemental Material: Mean-Field Unitary Part

$$
\begin{aligned}
\mathrm{i}[H, S_{x;m,n}] =& \frac{V}{2j}\left(\left\{\frac{J_x}{N}, \mathscr{A}^{y;m,n}_{+1,+1}\right\} + \left\{\frac{J_y}{N}, \mathscr{A}^{x;m,n}_{-1,+1}\right\}\right),\\
\mathrm{i}[H, S_{y;m,n}] =& \frac{V}{2j}\left(-\left\{\frac{J_x}{N}, \mathscr{A}^{x;m,n}_{+1,-1}\right\} + \left\{\frac{J_y}{N}, \mathscr{A}^{y;m,n}_{-1,-1}\right\}\right),
\end{aligned}
$$

with

$$
A_{j,m} := \sqrt{(j-m)(j+m+1)},
$$

$$
J_x/N = \sum_{m=-j}^{j-1} A_{j,m} S_{x;m+1,m}, \qquad J_y/N = \sum_{m=-j}^{j-1} A_{j,m} S_{y;m+1,m}
$$

and

$$
\begin{aligned}
\mathscr{A}^{\alpha;m,n}_{\sigma_1,\sigma_2} :=& A_{j,n-1} S_{\alpha;m,n-1} + \sigma_1 A_{j,n} S_{\alpha;m,n+1}\\
&+ \sigma_2 A_{j,m-1} S_{\alpha;n,m-1} + \sigma_1\sigma_2 A_{j,m} S_{\alpha;n,m+1}
\end{aligned}
$$

Supplemental Material: Mean-Field Individual Dissipation

$$
\begin{aligned}
\frac{1}{2}\sum_{i=1}^{N}[L_i^\dagger, S_{x;m,n}]L_i + \text{h.c.} = &\operatorname{Re}\left[\ell_m^*\ell_n\right] S_{x;m+1,n+1} - \operatorname{Im}\left[\ell_m^*\ell_n\right] S_{y;m+1,n+1} \\
&- \frac{|\ell_{n-1}|^2 + |\ell_{m-1}|^2}{2} S_{x;m,n}, \\
\frac{1}{2}\sum_{i=1}^{N}[L_i^\dagger, S_{y;m,n}]L_i + \text{h.c.} = &\operatorname{Re}\left[\ell_m^*\ell_n\right] S_{y;m+1,n+1} + \operatorname{Im}\left[\ell_m^*\ell_n\right] S_{x;m+1,n+1} \\
&- \frac{|\ell_{n-1}|^2 + |\ell_{m-1}|^2}{2} S_{y;m,n}
\end{aligned}
$$

$$\frac{1}{2N}[L_C^\dagger, S_{x;m,n}]L_C + \text{h.c.} = \frac{1}{4}\sum_{o=-j}^{j-1}\Big(-\Big\{\mathscr{R}^{y;m,n,o}_{+1,+1} + \mathscr{I}^{x;m,n,o}_{-1,+1}, S_{y;o+1,o}\Big\}$$
$$- \Big\{\mathscr{R}^{x;m,n,o}_{-1,+1} - \mathscr{I}^{y;m,n,o}_{+1,+1}, S_{x;o+1,o}\Big\}\Big)$$

$$\frac{1}{2N}[L_C^\dagger, S_{y;m,n}]L_C + \text{h.c.} = \frac{1}{4}\sum_{o=-j}^{j-1}\Big(-\Big\{\mathscr{R}^{y;m,n,o}_{-1,-1} + \mathscr{I}^{x;m,n,o}_{+1,-1}, S_{x;o+1,o}\Big\}$$
$$+ \Big\{\mathscr{R}^{x;m,n,o}_{+1,-1} - \mathscr{I}^{y;m,n,o}_{-1,-1}, S_{y;o+1,o}\Big\}\Big)$$

with

$$\mathscr{R}^{\alpha;m,n,o}_{\sigma_1,\sigma_2} := \text{Re}[\ell^*_{n-1}\ell_o]S_{\alpha;m,n-1} + \sigma_1\,\text{Re}[\ell^*_n\ell_o]S_{\alpha;m,n+1}$$
$$+ \sigma_2\,\text{Re}[\ell^*_{m-1}\ell_o]S_{\alpha;n,m-1} + \sigma_1\sigma_2\,\text{Re}[\ell^*_m\ell_o]S_{\alpha;n,m+1},$$
$$\mathscr{I}^{\alpha;m,n,o}_{\sigma_1,\sigma_2} := \text{Im}[\ell^*_{n-1}\ell_o]S_{\alpha;m,n-1} + \sigma_1\,\text{Im}[\ell^*_n\ell_o]S_{\alpha;m,n+1}$$
$$+ \sigma_2\,\text{Im}[\ell^*_{m-1}\ell_o]S_{\alpha;n,m-1} + \sigma_1\sigma_2\,\text{Im}[\ell^*_m\ell_o]S_{\alpha;n,m+1}.$$

Supplemental Material: Numerical Method

Permutation invariance: $\pi\mathcal{L}[\rho]\pi^\dagger = \mathcal{L}[\pi\rho\pi^\dagger]$ for all $\pi \in \mathrm{S}_N$

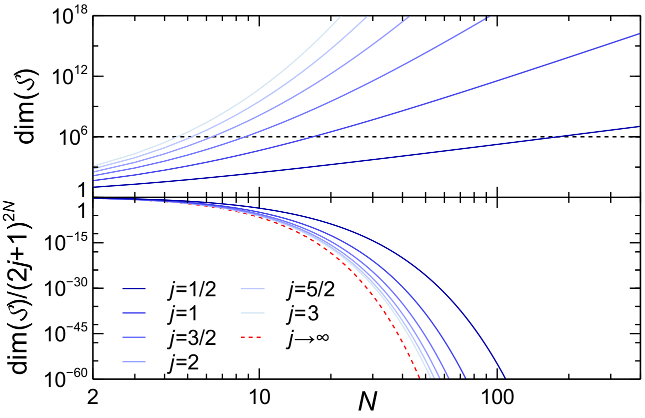
- Assume $\rho = \pi\rho\pi^\dagger$ for all $\pi \in \mathrm{S}_N$
- Write ρ as vector in basis

$$[n_{-j,-j}, n_{-j,-j+1}, \dots, n_{j,j}]$$
$$\sim \sum_{\pi\in\mathrm{S}_N} \pi |-j\rangle\langle -j|^{\otimes n_{-j,-j}} \otimes |-j\rangle\langle -j+1|^{\otimes n_{-j,-j+1}} \otimes \cdots \otimes |j\rangle\langle j|^{\otimes n_{j,j}} \pi^\dagger$$

- Dimension of ρ: $\binom{N+4j(j+1)}{N}$ instead of $(2j+1)^N \times (2j+1)^N$

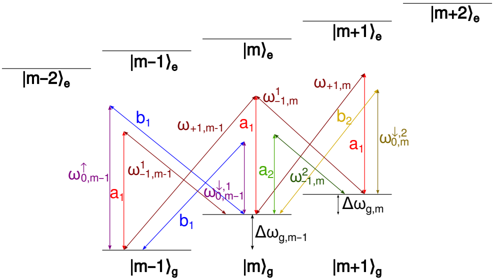
M. Gegg and M. Richter, NJP **18**, 043037 (2016)
D. Huybrechts, PhD Thesis (Universiteit Antwerpen, 2021)
V. Sukharnikov et al., PRA **107**, 053707 (2023)

Supplemental Material: Permutation-Invariant Space



$$\mathcal{S} = \left\{ \rho \in \mathbb{C}^{(2j+1)^N \times (2j+1)^N} \mid \forall \pi \in \mathrm{S}_N \ : \ \rho = \pi \rho \pi^\dagger \right\}$$

Supplemental Material: Proposal for Experiment



- N atoms with $d = 2j + 1$ ground states, $d + 2$ excited states
- 4 cavity modes $a_{1,2}, b_{1,2}$, $6(d-1)$ driving fields.
- Off-resonant driving: effective ground-state transitions
- Dissipative cavity dynamics: effective interactions & dissipation of the atomic ground states

Example: Transition from $5S_{1/2}, F = 2$ to $5P_{3/2}, F' = 3$ in ^{87}Rb,
E. Suarez et al., PRA **107**, 023714 (2023)