Absolute separability of symmetric multiqubit systems under unitary transformations

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SciPost Phys. **15**, 120 (2023), PRA **109**, 022430 (2024)

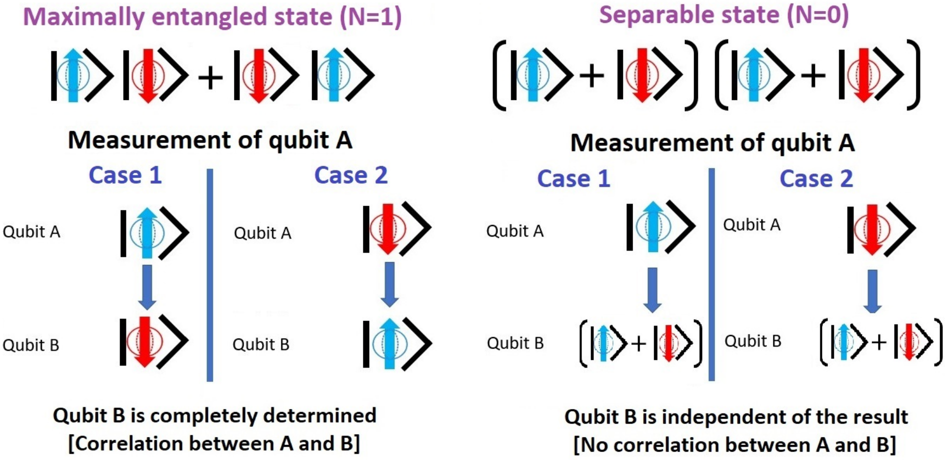
March 14, 2024

Outline of the talk

1. Statement of the problem
   a. Entangled and separable states
   b. Absolutely separable states
   c. Symmetric case: Symmetric absolutely separable (SAS) states
2. Results
   a. Symmetric 2-qubit system
   b. Symmetric 3-qubit system (Numerical results)
   c. SAS witnesses for symmetric N-qubit systems
      a. One linear SAS witness
      b. Two non-linear SAS witnesses
3. Conclusions

Entanglement

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$



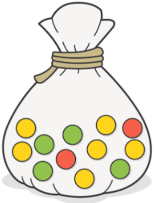
Entanglement of mixed states

Separable mixed states [Werner (1989)]

ρ is separable if

$$\rho = \int_{\mathcal{H}_2^{\otimes 2}} P(\boldsymbol{n}_1\,,\boldsymbol{n}_2)|\boldsymbol{n}_1\rangle|\boldsymbol{n}_2\rangle\langle\boldsymbol{n}_1|\langle\boldsymbol{n}_2|\,\mathrm{d}\boldsymbol{n}_1\,\mathrm{d}\boldsymbol{n}_2\,.$$

with $P(\boldsymbol{n}_1,\boldsymbol{n}_2)\geq 0$. Otherwise is entangled.



Negativity: Measure of entanglement for qubit-qubit and qubit-qutrit, [Peres (1996)], [Horodecki et al (1996)]

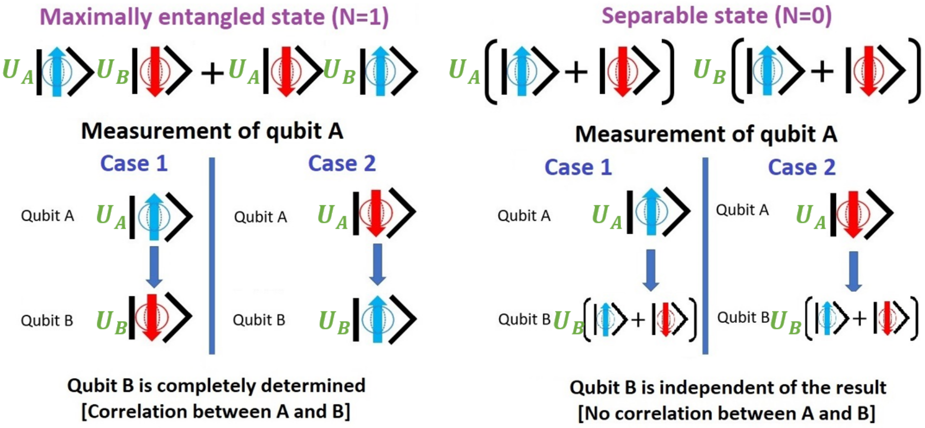
$\mathcal{N}$ is defined in terms of the negative eigenvalues Λ_k of ρ^{T_A}

$$\mathcal{N}(\rho) = -2\sum_{\Lambda_k<0}\Lambda_k\,,$$

- $\mathcal{N}(\rho_{sep}) = 0$.
- Invariant under local unitary transformations.

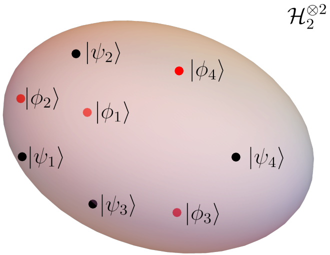
Entanglement

Invariant under local unitary transformations $U_A \otimes U_B \in SU(2) \otimes SU(2)$



Entanglement (Pure state case)

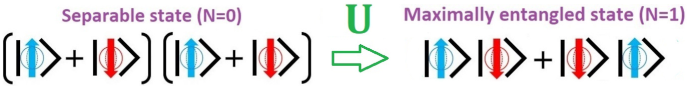
Not-invariant under **global** unitary transformations $SU(4)$



Global unitary transformation

$$\rho = \sum_{k=0}^{3} \lambda_k |\phi_k\rangle\langle\phi_k| \,,$$

$$U\rho U^\dagger = \sum_{k=0}^{3} \lambda_k |\psi_k\rangle\langle\psi_k| \,,$$

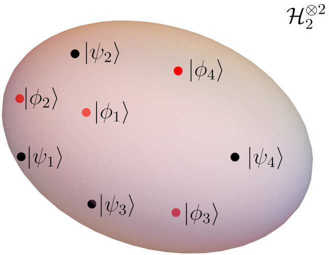


Pure state ρ_{pure}

$$\lambda_0 = 1 \,, \lambda_1 = \lambda_2 = \lambda_3 = 0 \,, \qquad \max_{U \in SU(4)} \mathcal{N}(U \rho_{pure} U^\dagger) = 1 \,,$$

Entanglement (Maximally mixed state case)

Not-invariant under **global** unitary transformations $SU(4)$



Global unitary transformation

$$\rho = \sum_{k=0}^{3} \lambda_k |\phi_k\rangle\langle\phi_k| \,,$$

$$U\rho U^\dagger = \sum_{k=0}^{3} \lambda_k |\psi_k\rangle\langle\psi_k| \,,$$

$$\rho_* = U\rho_* U^\dagger = \frac{1}{4}\mathbb{1} = \frac{1}{4}\int_{S^2\otimes S^2} |\boldsymbol{n}_1\rangle|\boldsymbol{n}_2\rangle\langle\boldsymbol{n}_1|\langle\boldsymbol{n}_2|\, \mathrm{d}^2\boldsymbol{n}_1\, \mathrm{d}^2\boldsymbol{n}_2 \,.$$

Maximally mixed state ρ_*

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1/4 \,, \qquad \max_{U\in SU(4)} \mathcal{N}(U\rho_* U^\dagger) = 0 \,,$$

Maximum entanglement in the unitary orbit of ρ

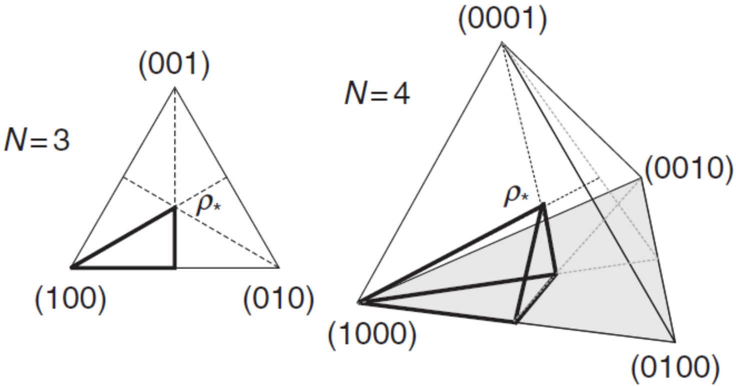


Figure taken from [Bengtsson and Życzkowski (2017)]

Questions

- What is the maximum entanglement of ρ attained in its $SU(4)$-orbit?
- Is ρ_* the unique state that is absolutely separable (AS) over all its unitary orbit?

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$

$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$

$$\max_{U \in SU(4)} \mathcal{N}\left(U\rho U^{\dagger}\right) = \max\left(0, \sqrt{(\lambda_0 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2} - \lambda_1 - \lambda_3\right),$$

ρ is AS iff $\lambda_0 \leqslant \lambda_2 + 2\sqrt{\lambda_1 \lambda_3}$.

[Verstraete, Audenart & De Moor (2001)].

Qubit-qutrit system $\mathcal{H}_2 \otimes \mathcal{H}_3$

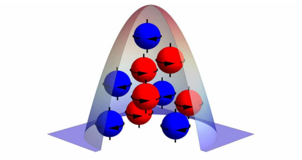
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$

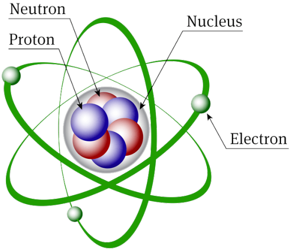
$$\max_{U \in SU(6)} \mathcal{N}\left(U\rho U^{\dagger}\right)$$

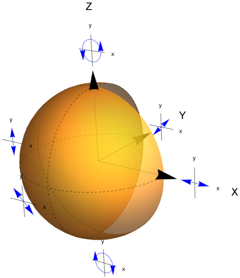
Open question. Partial results [Mendonça, Marchiolli, Herdemann (2017)]

Statement of the problem

Bosons: BEC, spin-j system, multiphotons systems, etc.







New question

For a symmetric qubit-qubit state ρ_S,

- What is the maximum entanglement achievable under a global unitary transformation U_S restricted in the symmetric subspace ?
- What is the spectrum of the symmetric states that remains separable after any global unitary transformation U_S?

Symmetric bipartite systems

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$	Symmetric 2-qubit system $\mathcal{H}_2^{\vee 2}$
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$	ρ_S-spectrum: $(\lambda_0, \lambda_1, \lambda_2, 0)$
$\max_{U \in SU(4)} \mathcal{N}\left(U\rho U^\dagger\right)$	$\max_{U_S \in SU(3)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right)$
Qubit-qutrit system $\mathcal{H}_2 \otimes \mathcal{H}_3$	**Symmetric 3-qubit system $\mathcal{H}_2^{\vee 3}$**
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$	ρ_S-spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, 0, 0)$
$\max_{U \in SU(6)} \mathcal{N}\left(U\rho U^\dagger\right)$	$\max_{U_S \in SU(4)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right)$

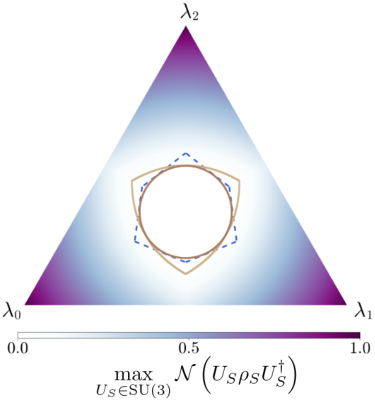
Symmetric 2-qubit system

Symmetric 2-qubit system

Theorem [ESE, Martin (2023)]

Let $\rho_S \in \mathcal{B}(\mathcal{H}_2^{\vee 2})$ with spectrum $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2$. It holds that

$$\max_{U_S \in SU(3)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right) = \max\left(0, \sqrt{\lambda_0^2 + (\lambda_1 - \lambda_2)^2} - \lambda_1 - \lambda_2\right).$$



Maximally entangled state

$$\rho_S = \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

SAS states

$\mathcal{A}$
Absolutely separable (AS) states [Życzkowski (1999)]

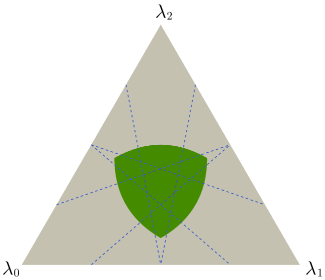
$$\max_{U \in SU(4)} \mathcal{N}(U \rho_S U^\dagger) = 0$$

$$\mathcal{A}(\mathcal{H}_2^{\vee 2}) = \{\rho_0\}$$

$\mathcal{A}_{\mathrm{sym}}$
Symmetric absolutely separable (SAS) states [Giraud et al (2008)]

$$\max_{U \in SU(3)} \mathcal{N}(U_S \rho_S U_S^\dagger) = 0$$

$$\mathrm{d}(\mathcal{A}_{\mathrm{sym}}(\mathcal{H}_2^{\vee 2})) = 2$$



Corollary [ESE, Martin (2023)]

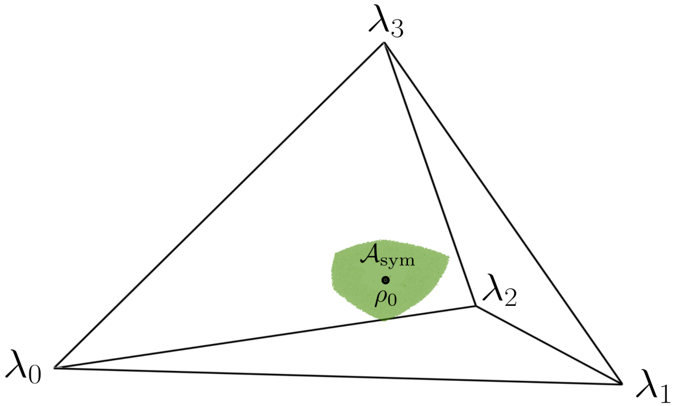
$\rho_S \in \mathcal{A}_{\mathrm{sym}}$ iff

$$\sqrt{\lambda_1} + \sqrt{\lambda_2} \geq 1\,.$$

Symmetric 3-qubit system

Symmetric 3-qubit system

$\mathcal{H}_2^{\vee 3} \subset \mathcal{H}_2 \otimes \mathcal{H}_3$, numerical results



Set of SAS states in the spectra polytope of symmetric 3-qubit states.

SAS witnesses for symmetric N-qubit states

SAS witnesses for symmetric N-qubit states

[ESE, Denis, Martin (2023)]

SAS states

Let $\rho \in \mathcal{B}(\mathcal{H}_2^{\vee N})$, $\rho \in \mathcal{A}_{\text{sym}} \Leftrightarrow$ there exists $P(U\rho U^\dagger; \boldsymbol{n})$ such that

$$U\rho U^\dagger = \int_{S^2} P(U\rho U^\dagger; \boldsymbol{n}) |\boldsymbol{n}\rangle^{\otimes n} \langle \boldsymbol{n}|^{\otimes n} \, \mathrm{d}^2\boldsymbol{n} \,,$$

and

$$\min_{\substack{U \in SU(N+1) \\ \boldsymbol{n} \in S^2}} P(U\rho U^\dagger; \boldsymbol{n}) \geqslant 0 \,,$$

SAS-witness $\mathcal{W}$ [Bohnet-Waldraff, Giraud, Braun (2017)]

$$\rho \in \mathcal{A}_{\text{sym}} \;\text{ if }\; \mathrm{Tr}(\rho^2) \leqslant \frac{1}{N+1}\left(1 + \frac{1}{2(2N+1)\binom{2N}{N} - (N+2)}\right) \,,$$

SAS witnesses for symmetric N-qubit states

Non-uniqueness of the P-function [Giraud, Braun, Braun (2008)]

$$P(\rho\,,\boldsymbol{n}) = \underbrace{\sum_{L=0}^{N}\sum_{M=-L}^{L} y_{LM}Y_{LM}(\boldsymbol{n})}_{P_0=\mathrm{Tr}(\rho\omega^{(1)}(\boldsymbol{n}))\ ,\ \text{unique for } \rho} + \underbrace{\sum_{L=N+1}^{\infty}\sum_{M=-L}^{L} y_{LM}Y_{LM}(\boldsymbol{n})}_{P'\ ,\ \text{arbitrary } y_{LM}}\,,$$

Proposal

To consider $P(\rho,\boldsymbol{n})$ such that

i) They are covariant

$$P(U\rho U^{\dagger},\boldsymbol{n}) = P(D(\mathrm{R})^{\dagger}U\rho U^{\dagger}D(\mathrm{R}),\boldsymbol{z}) = P(V\rho V^{\dagger},\boldsymbol{z})\,.$$

ii) We built $P(U\rho U^{\dagger},\boldsymbol{n})$ that their explicit expressions depend only on (or can be approximated) the unistochastic matrices $B\in\mathcal{U}_{N+1}\subset\mathcal{B}_{N+1}$

$$B_{ij} = |V_{ij}|^2\,,\quad B_{ij}\geqslant 0\,,\quad \sum_i B_{ij} = \sum_j B_{ij} = 1\,.$$

SAS witness $\mathcal{W}_1$: A polytope of SAS states

$P = P_0$ [Denis, Davis, Mann, Martin (2023)]

Observation 1

$$\min_{\substack{U \in SU(N+1) \\ \boldsymbol{n} \in S^2}} P_0(U\rho U^\dagger; \boldsymbol{n}) = \min_{V \in SU(N+1)} \mathrm{Tr}\left[\rho V \omega^{(1)}(\boldsymbol{z}) V^\dagger\right]$$

$$\left(\rho_{jk} = \tau_j \delta_{jk}\,, \omega^{(1)}(\boldsymbol{z})_{jk} = \Delta_j \delta_{jk}\right) = \min_{B \in \mathcal{U}_{N+1}} \boldsymbol{\lambda} B \boldsymbol{\Delta}^T\,,$$

B a unistochastic matrix, $B \in \mathcal{U}_{N+1} \subset \mathcal{B}_{N+1}$.

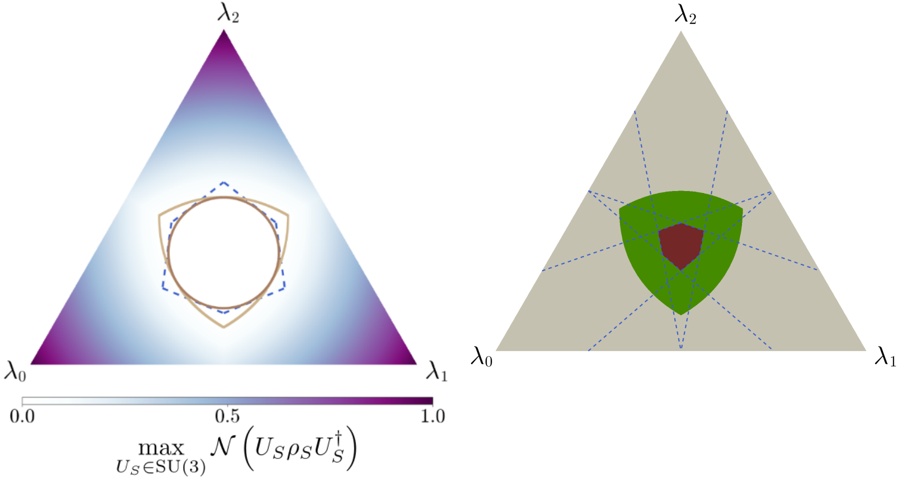
Observation 2 (Birkhoff's Theorem)

Permutations matrices achieve extremal values of a convex function $f(B)$

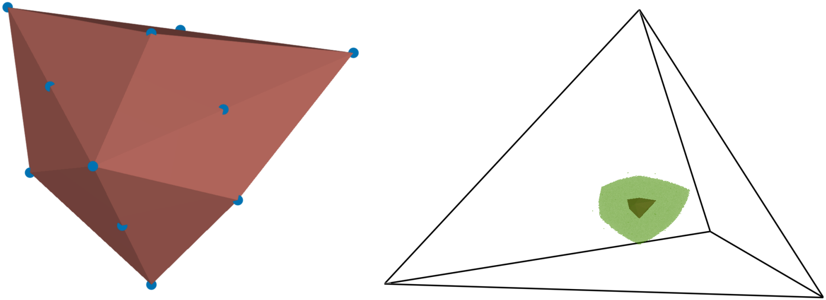
$$\min_{B \in \mathcal{U}_{N+1}} \boldsymbol{\lambda} B \boldsymbol{\Delta}^T = \min_{\Pi \in S_{N+1}} \boldsymbol{\lambda} \Pi \boldsymbol{\Delta}^T\,,$$

SAS witness $\mathcal{W}_1$

$$\rho \in \mathcal{A}_{\text{sym}} \qquad \text{if} \qquad \boldsymbol{\lambda}^{\downarrow} \boldsymbol{\Delta}^{\uparrow T} \geqslant 0\,, \qquad \Delta_k = (-1)^{N-k} \binom{N+1}{k}\,,$$



Polytope of SAS states detected by $\mathcal{W}_1$ for $N = 2$.



Polytope of SAS states detected by $\mathcal{W}_1$ for $N = 3$.

SAS witness $\mathcal{W}_2(\{y_L\})$

Non-uniqueness of the P-function [Giraud, Braun, Braun (2008)]

$$P(\rho\,,\boldsymbol{n}) = \underbrace{\sum_{L=0}^{N}\sum_{M=-L}^{L} y_{LM}\,Y_{LM}(\boldsymbol{n})}_{P_0=\mathrm{Tr}(\rho\omega^{(1)}(\boldsymbol{n}))\text{ , unique for }\rho} + \underbrace{\sum_{L=N+1}^{\infty}\sum_{M=-L}^{L} y_{LM}\,Y_{LM}(\boldsymbol{n})}_{P'\text{ , arbitrary }y_{LM}}\,,$$

We add some quadratic SU(2)-covariant terms of ρ (with $j = N/2$)

$$Q_L(\rho,\boldsymbol{n}) = \sum_{M=-L}^{L} \mathrm{Tr}\left(\rho T_{LM}^{(j)\dagger}\right) Y_{LM}(\boldsymbol{n})\,,$$

$$P_L(\rho,\boldsymbol{n}) \equiv Q_L^2 - \sum_{\sigma=0}^{N}\sum_{\nu=-\sigma}^{\sigma}\left(\int Q_L^2\, Y_{\sigma\nu}^{*}(\boldsymbol{n}')\,\mathrm{d}\boldsymbol{n}'\right) Y_{\sigma\nu}(\boldsymbol{n})\,,$$

$$P'(\rho,\boldsymbol{n}) = \sum_{L>N/2} y_L P_L\,,$$

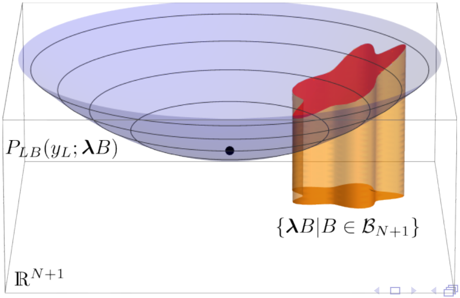
SAS witness $\mathcal{W}_2(\{y_L\})$

$P = P_0 + P'(\rho, \boldsymbol{n})$

$$\min_{\substack{U\in SU(N+1)\\ \boldsymbol{n}\in S^2}} P(U\rho U^\dagger;\boldsymbol{n}) \geqslant \min_{\substack{\boldsymbol{\lambda}B\\ B\in\mathcal{U}_{N+1}}} P_{LB}(y_L;\boldsymbol{\lambda}B) \geqslant \min_{\substack{\boldsymbol{\lambda}B\\ B\in\mathcal{B}_{N+1}}} P_{LB}(y_L;\boldsymbol{\lambda}B)$$

P_{LB} is a quadratic function on the entries of B and linear on the $\{y_L\}$'s added by P'. The second inequality is tight because

$$\{\boldsymbol{\lambda}B \in \mathbb{R}^{N+1} | B \in \mathcal{U}_{N+1}\} = \{\boldsymbol{\lambda}B \in \mathbb{R}^{N+1} | B \in \mathcal{B}_{N+1}\}\,.$$



SAS witnesses $\mathcal{W}_2(\{y_L\})$

SAS witness $\mathcal{W}_2(\{y_L\})$: A symmetric $2j = N$-qubit state ρ is SAS if for some values of $\{y_L\}$

$$\min_{\substack{\boldsymbol{\lambda} B \\ B\in\mathcal{B}_{N+1}}} P_{LB}(y_L;\boldsymbol{\lambda}B) = \min_{\substack{\boldsymbol{\lambda} B \\ B\in\mathcal{B}_{N+1}}} f + \sum_{L=1}^{2j}\left[g_L\boldsymbol{\lambda}B\,\mathbf{t}_L^T + h_L\left(\boldsymbol{\lambda}B\,\mathbf{t}_L^T\right)^2\right] \geqslant 0\,,$$

$$f = \frac{1}{N+1} + \left(\frac{y_N F(N,1)}{2}\right)\left(\mathrm{Tr}(\rho^2) - \frac{1}{N+1}\right)^2,$$

$$g_L = \sqrt{\frac{2L+1}{N+1}}\left(C^{jj}_{jjL0}\right)^{-1}, \quad h_L = y_L F(L,0)\Theta(L-j) - \frac{y_{2j}F(2j,1)}{2}\,,$$

$$\boldsymbol{t}_L = (C^{L0}_{jj,j-j}, -C^{L0}_{jj-1,j1-j}, \ldots, (-1)^{2j}C^{L0}_{j-j,jj})\,,$$

$$F(L,\mu) \equiv \begin{cases} 1 - \displaystyle\sum_{\substack{\sigma=0 \\ \sigma\ \text{even}}}^{2j}\left(C^{\sigma 0}_{L0L0}\right)^2 & \text{if } \mu = 0 \\ 2(-1)^{\mu+1}\displaystyle\sum_{\substack{\sigma=0 \\ \sigma\ \text{even}}}^{2j} C^{\sigma 0}_{L0L0}C^{\sigma 0}_{L\mu L-\mu} & \text{if } \mu \neq 0 \end{cases}$$

The variables h_L must be positive, restricting the domain of the free parameters $\{y_L\}$.

Example: $\mathcal{W}_2(\{y_2\})$ for $N = 2$

A symmetric 2-qubit state ρ with spectrum $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2)$ is SAS if

$$\min_{\substack{\boldsymbol{\lambda} B \\ B \in \mathcal{B}_3}} P_{LB}(y_L; \boldsymbol{\lambda} B) = \min_{\substack{\boldsymbol{\lambda} B \\ B \in \mathcal{B}_3}} f + \sum_{L=1}^{2} \left[g_L \boldsymbol{\lambda} B\, \mathbf{t}_L^T + h_L \left(\boldsymbol{\lambda} B\, \mathbf{t}_L^T \right)^2 \right] \geqslant 0$$

for some $y_2 \in \mathbb{R}^+$ and

$$f = \frac{1}{3} - \frac{12}{35} y_2 \left(\mathrm{Tr}(\rho^2) - \frac{1}{3} \right),$$

$$(g_1, g_2) = \left(\sqrt{2}, 5\sqrt{\frac{2}{3}} \right), \quad (h_1, h_2) = \frac{6}{35}(2y_2, 5y_2),$$

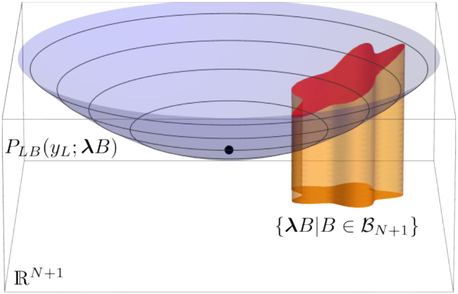
$$\boldsymbol{t}_L = (C^{L0}_{11,1-1}, -C^{L0}_{10,10}, C^{L0}_{1-1,11}),$$

Dear Eduardo: Don't forget the video. Best, your colleagues.

A ball of SAS states detected by the $\mathcal{W}_2(\{y_L\})$ witnesses

$$\min_{\substack{\boldsymbol{\lambda} B \\ B\in\mathcal{B}_{N+1}}} P_{LB}(y_L;\boldsymbol{\lambda} B) \geqslant \min_{\boldsymbol{v}\in\mathbb{R}^{N+1}} P_{LB}(y_L;\boldsymbol{v}) \geqslant 0\,.$$

A function that depends only in the purity of the state $\mathrm{Tr}(\rho^2)$. Moreover, we can maximize the purity attained over the $\{y_L\}$ variables.

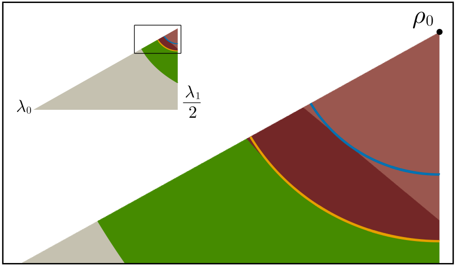


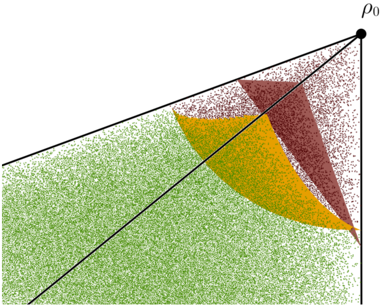
SAS witness $\mathcal{W}_3$

$\mathcal{W}_3$: A symmetric N-qubit state ρ is SAS if

$$r^2 \leqslant \frac{1}{(2j+1)^2} \left(\sum_{L=1}^{2j} \frac{g_L^2}{1 - 2\Theta(L-j)\frac{F(L,0)}{F(L,1)}} \right)^{-1},$$

where $r^2 \equiv \|\rho - \rho_0\|_{\text{HS}}^2 = \text{Tr}(\rho^2) - (N+1)^{-1}$.





Dark Brown $= \mathcal{S}_2(\{y_L\})$
Light Brown $= \mathcal{S}_1$

Orange surface $=$ Bound of $\mathcal{S}_3$
Blue surface $=$ Bound of $\mathcal{S}$
[Giraud, Braun, Braun (2017)]

Green $=$ Unwitnessed SAS states by $\mathcal{W}_k$

SAS witnesses for symmetric N-qubit states

[ESE, Denis, Martin (2024)]

Number of qubits $N = 2j$	$\begin{cases} \text{Witness } \mathcal{W}_1 \\ \text{Witness } \mathcal{W}_3 \end{cases}$
2	$\begin{cases} \boldsymbol{\lambda}\,(-3,\,1,\,3)^T \geqslant 0 \\ r^2 \leqslant \frac{1}{78} \approx 0.01282 \end{cases}$
3	$\begin{cases} \boldsymbol{\lambda}\,(-6,\,-1,\,4,\,4)^T \geqslant 0 \\ r^2 \leqslant \frac{1}{354} \approx 0.002825 \end{cases}$
4	$\begin{cases} \boldsymbol{\lambda}\,(-10,\,-5,\,1,\,5,\,10)^T \geqslant 0 \\ r^2 \leqslant \frac{11}{25390} \approx 0.0004332 \end{cases}$
5	$\begin{cases} \boldsymbol{\lambda}\,(-15,\,-15,\,-1,\,6,\,6,\,20)^T \geqslant 0 \\ r^2 \leqslant \frac{1595}{16058598} \approx 0.00009932 \end{cases}$

Table: SAS witnesses $\mathcal{W}_1$ and $\mathcal{W}_3$ for a state with eigenspectrum $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_N)$ sorted in descending order $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_N$.

Conclusions

Maximum entanglement (negativity) over the unitary orbit for $N = 2, 3$

- Characterization of the SAS states for symmetric 2-qubit system
- Numerical study of the SAS states for symmetric 3-qubit system

ESE and John Martin, SciPost Phys. **15**, 120 (2023)

SAS witnesses in terms of the spectrum or the purity

of the symmetric N-qubit states

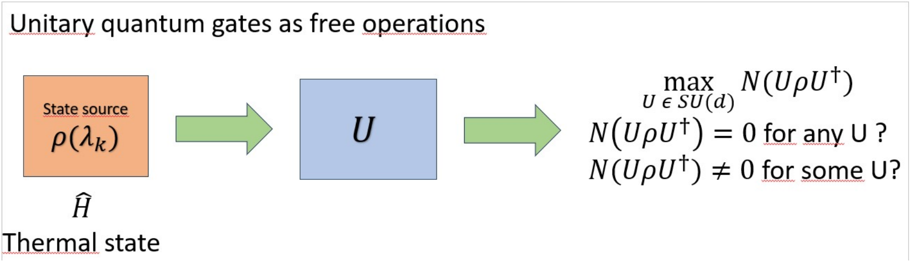
ESE, Jérôme Denis and John Martin, PRA **109**, 022430 (2024)

Future work

SAS witnesses with extra terms in the P-representation

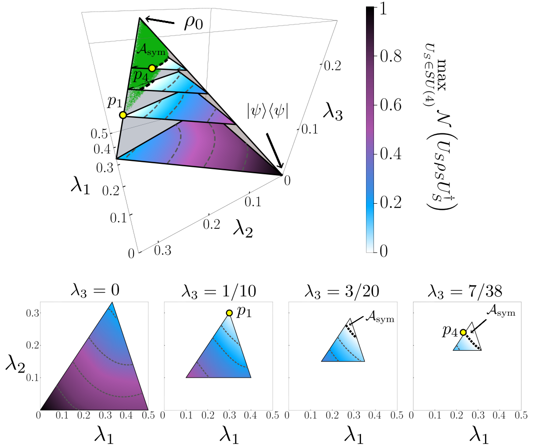
Thank you very much for your attention!

Physical implications

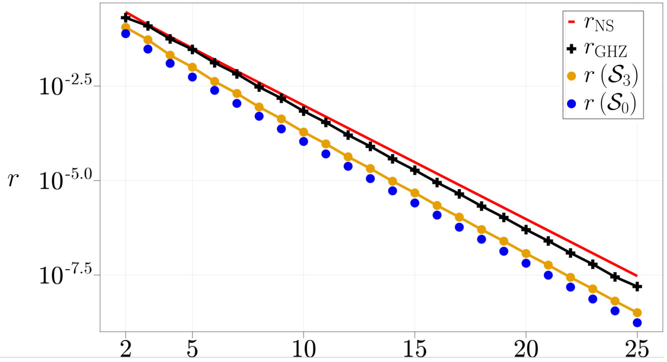


Symmetric 3-qubit system

$\mathcal{H}_2^{\vee 3} \subset \mathcal{H}_2 \otimes \mathcal{H}_3$, numerical results

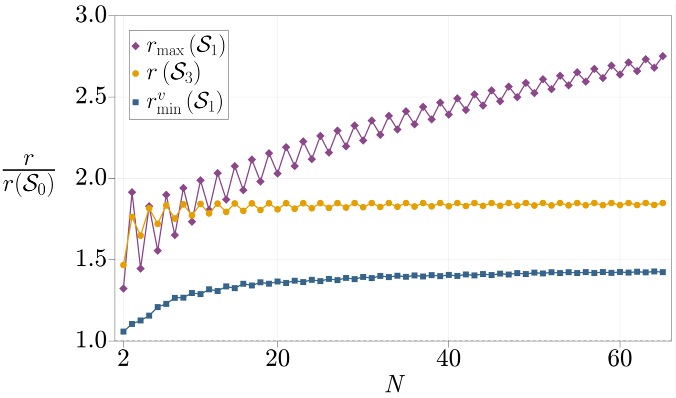


Maximum negativity in the $SU(4)$ orbit of ρ.



Comparison between the maximal distances of several sets of SAS states. The black crosses are defined by the furthest away SAS state in the ray ρ_0 and the GHZ pure state, with distance $r_{\rm GHZ}$. The red line shows the radius $r_{\rm NS} = (2^N(2^N - 1))^{-1/2}$ of the largest ball containing only AS states in the full Hilbert space [Gurvits and Barnum (2002)].

Comparison between the SAS witnesses



Distances $r_{\max}(\mathcal{S}_1)$ (purple), $r^v_{\min}(\mathcal{S}_1)$ (blue) and $r(\mathcal{S}_3)$ (orange), rescaled by the distance of the witness $r(\mathcal{S}_0)$.