Absolute separability of symmetric multiqubit systems under unitary transformations

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Outline of the talk

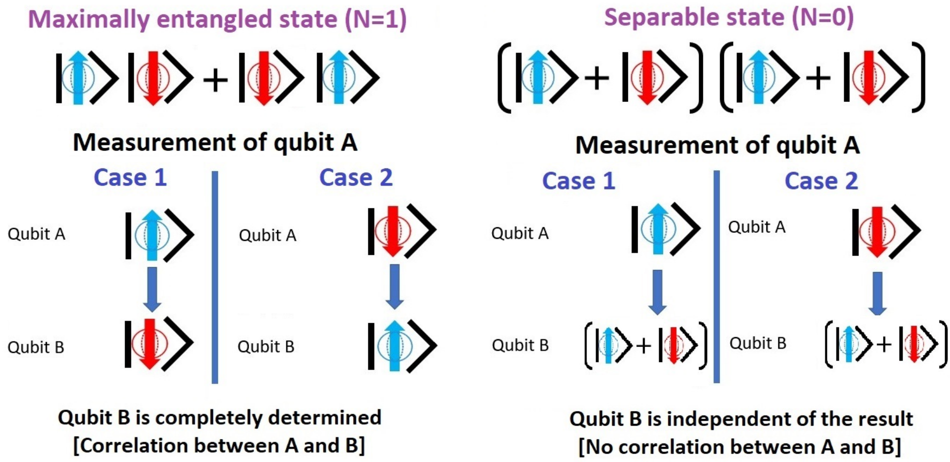
1. Statement of the problem
    a. Entangled and separable states
    b. Absolutely separable states
    c. Symmetric case: Symmetric absolutely separable (SAS) states

2. Results
    a. Symmetric 2-qubit system
    b. Symmetric 3-qubit system (Numerical results)
    c. SAS witnesses for symmetric N-qubit systems
        a. One linear SAS witness
        b. Two non-linear SAS witnesses

3. Conclusions

Entanglement

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$



Maximally entangled state (N=1)

Measurement of qubit A

Qubit B is completely determined
[Correlation between A and B]

Separable state (N=0)

Measurement of qubit A

Qubit B is independent of the result
[No correlation between A and B]

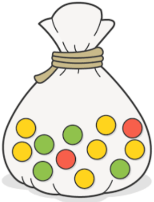
Entanglement of mixed states

Separable mixed states [Werner (1989)]

ρ is separable if

$$\rho = \int_{\mathcal{H}_2^{\otimes 2}} P(\boldsymbol{n}_1\,, \boldsymbol{n}_2) |\boldsymbol{n}_1\rangle |\boldsymbol{n}_2\rangle \langle \boldsymbol{n}_1 | \langle \boldsymbol{n}_2 | \,\mathrm{d}\boldsymbol{n}_1 \,\mathrm{d}\boldsymbol{n}_2 \,.$$

with $P(\boldsymbol{n}_1, \boldsymbol{n}_2) \geq 0$. Otherwise is entangled.



Measure of entanglement

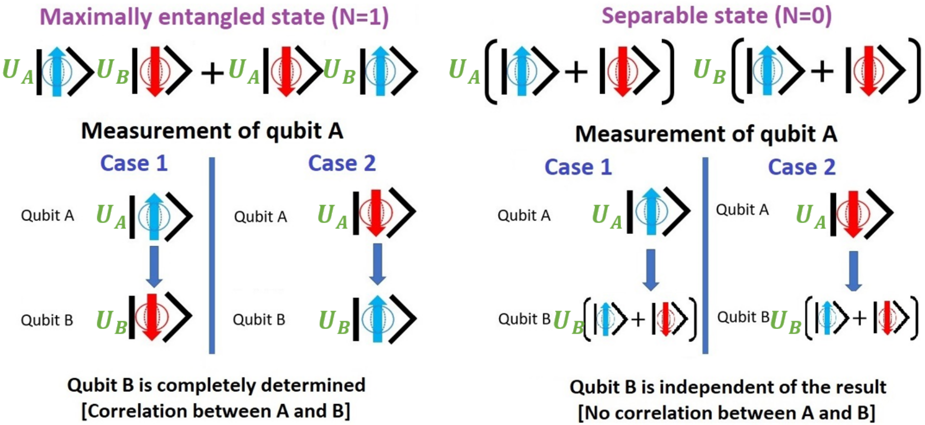
- $E(\rho_{sep}) = 0$.
- Invariant under local unitary transformations.
- Other properties...

Example for qubit-qubit and qubit-qutrit: **Negativity** [Peres (1996)], [Horodecki et al (1996)]
The sum of the negative eigenvalues Λ_k of ρ^{T_A}

$$\mathcal{N}(\rho) = -2 \sum_{\Lambda_k < 0} \Lambda_k \,,$$

Entanglement

Invariant under local unitary transformations $U_A \otimes U_B \in SU(2) \otimes SU(2)$

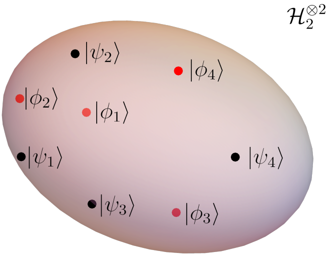


Qubit B is completely determined
[Correlation between A and B]

Qubit B is independent of the result
[No correlation between A and B]

Entanglement (Pure state case)

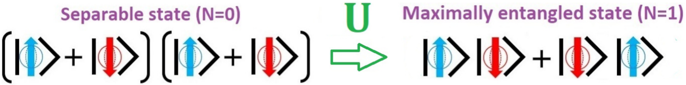
Not-invariant under **global** unitary transformations $SU(4)$



Global unitary transformation

$$\rho = \sum_{k=0}^{3} \lambda_k |\phi_k\rangle\langle\phi_k| \,,$$

$$U\rho U^\dagger = \sum_{k=0}^{3} \lambda_k |\psi_k\rangle\langle\psi_k| \,,$$



Separable state (N=0)

U

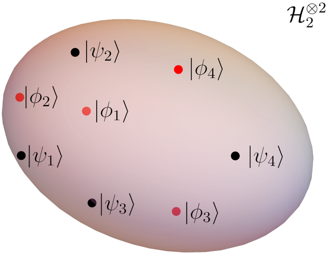
Maximally entangled state (N=1)

Pure state ρ_{pure}

$$\lambda_0 = 1 \,, \lambda_1 = \lambda_2 = \lambda_3 = 0 \,, \qquad \max_{U \in SU(4)} \mathcal{N}(U\rho_{pure}U^\dagger) = 1 \,,$$

Entanglement (Maximally mixed state case)

Not-invariant under **global** unitary transformations $SU(4)$



Global unitary transformation

$$\rho = \sum_{k=0}^{3} \lambda_k |\phi_k\rangle\langle\phi_k| \,,$$

$$U\rho U^\dagger = \sum_{k=0}^{3} \lambda_k |\psi_k\rangle\langle\psi_k| \,,$$

$$\rho_* = U\rho_* U^\dagger = \frac{1}{4}\mathbb{1} = \frac{1}{4}\int_{S^2\otimes S^2} |\boldsymbol{n}_1\rangle|\boldsymbol{n}_2\rangle\langle\boldsymbol{n}_1|\langle\boldsymbol{n}_2|\, \mathrm{d}^2\boldsymbol{n}_1\, \mathrm{d}^2\boldsymbol{n}_2 \,.$$

Maximally mixed state ρ_*

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1/4 \,, \qquad \max_{U\in SU(4)} \mathcal{N}(U\rho_* U^\dagger) = 0 \,,$$

Maximum entanglement in the unitary orbit of ρ

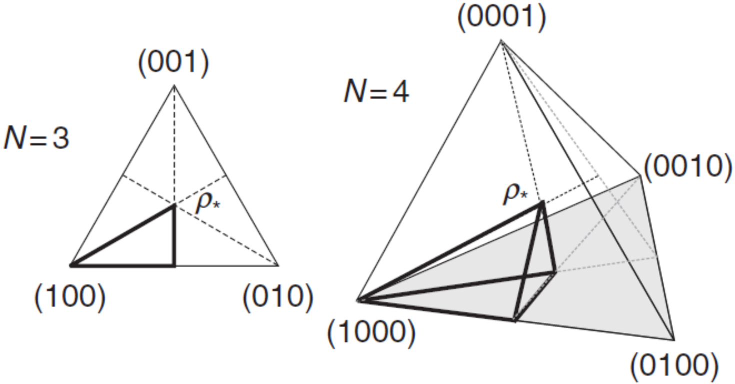


Figure taken from [Bengtsson and Życzkowski (2017)]

Question

- Is ρ_* the unique state that is absolutely separable (AS) over all its unitary orbit?

Maximum entanglement in the unitary orbit of ρ

Results for qubit-qubit and qubit-qutrit systems

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$

$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$

ρ is AS iff $\lambda_0 \leqslant \lambda_2 + 2\sqrt{\lambda_1 \lambda_3}$.

[Verstraete, Audenart & De Moor (2001)].

Qubit-qutrit system $\mathcal{H}_2 \otimes \mathcal{H}_3$

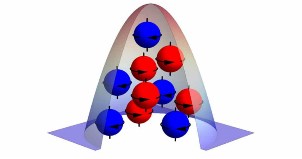
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$

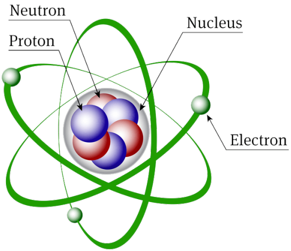
$$\max_{U \in SU(6)} \mathcal{N}\left(U \rho U^{\dagger}\right)$$

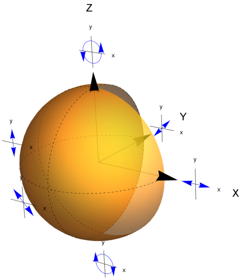
Open question. Partial results [Mendonça, Marchiolli, Herdemann (2017)]

Statement of the problem

Bosons: BEC, spin-j system, multiphotons systems, etc.







New question

For a symmetric qubit-qubit state ρ_S,

- What is the spectrum of the symmetric states that remains separable after any global unitary transformation U_S?

Symmetric bipartite systems

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$	Symmetric 2-qubit system $\mathcal{H}_2^{\vee 2}$
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$	ρ_S-spectrum: $(\lambda_0, \lambda_1, \lambda_2, 0)$
$\max_{U \in SU(4)} \mathcal{N}\left(U \rho U^\dagger\right)$	$\max_{U_S \in SU(3)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right)$
Qubit-qutrit system $\mathcal{H}_2 \otimes \mathcal{H}_3$	Symmetric 3-qubit system $\mathcal{H}_2^{\vee 3}$
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$	ρ_S-spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, 0, 0)$
$\max_{U \in SU(6)} \mathcal{N}\left(U \rho U^\dagger\right)$	$\max_{U_S \in SU(4)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right)$

Symmetric 2-qubit system

SAS states

$\mathcal{A}$
Absolutely separable (AS) states [Życzkowski (1999)]

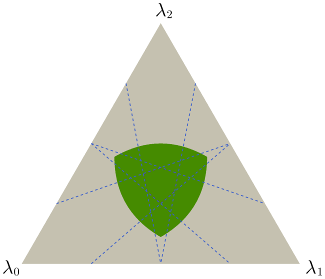
$$\max_{U\in SU(4)} \mathcal{N}(U\rho_S U^\dagger) = 0$$

$$\mathcal{A}(\mathcal{H}_2^{\vee 2}) = \{\rho_0\}$$

$\mathcal{A}_{\mathrm{sym}}$
Symmetric absolutely separable (SAS) states [Giraud et al (2008)]

$$\max_{U\in SU(3)} \mathcal{N}(U_S\rho_S U_S^\dagger) = 0$$

$$\mathrm{d}(\mathcal{A}_{\mathrm{sym}}(\mathcal{H}_2^{\vee 2})) = 2$$



Corollary [ESE, Martin (2023)]

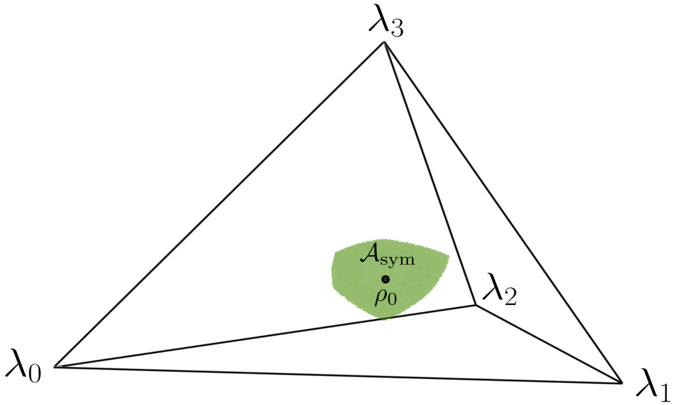
$\rho_S \in \mathcal{A}_{\mathrm{sym}}$ iff

$$\sqrt{\lambda_1} + \sqrt{\lambda_2} \geq 1\,.$$

Symmetric 3-qubit system

Symmetric 3-qubit system

$\mathcal{H}_2^{\vee 3} \subset \mathcal{H}_2 \otimes \mathcal{H}_3$, numerical results



Set of SAS states in the spectra polytope of symmetric 3-qubit states.

SAS witnesses for symmetric N-qubit states

SAS witnesses for symmetric N-qubit states

[ESE, Denis, Martin (2024)]

SAS states

Let $\rho \in \mathcal{B}(\mathcal{H}_2^{\vee N})$, $\rho \in \mathcal{A}_{\text{sym}} \Leftrightarrow$ there exists $P(U\rho U^\dagger; \boldsymbol{n})$ such that

$$U\rho U^\dagger = \int_{S^2} P(U\rho U^\dagger; \boldsymbol{n}) |\boldsymbol{n}\rangle^{\otimes n} \langle \boldsymbol{n}|^{\otimes n} \, \mathrm{d}^2\boldsymbol{n} \,,$$

and

$$\min_{\substack{U \in SU(N+1) \\ \boldsymbol{n} \in S^2}} P(U\rho U^\dagger; \boldsymbol{n}) \geqslant 0 \,,$$

SAS-witness $\mathcal{W}$ [Bohnet-Waldraff, Giraud, Braun (2017)]

$$\rho \in \mathcal{A}_{\text{sym}} \;\; \text{if} \;\; \mathrm{Tr}(\rho^2) \leqslant \frac{1}{N+1}\left(1 + \frac{1}{2(2N+1)\binom{2N}{N} - (N+2)}\right) \,,$$

SAS witnesses for symmetric N-qubit states

Non-uniqueness of the P-function [Giraud, Braun, Braun (2008)]

$$P(\rho\,,\boldsymbol{n}) = \underbrace{\sum_{L=0}^{N}\sum_{M=-L}^{L} y_{LM}Y_{LM}(\boldsymbol{n})}_{P_0=\mathrm{Tr}(\rho\omega^{(1)}(\boldsymbol{n}))\text{ , unique for }\rho} + \underbrace{\sum_{L=N+1}^{\infty}\sum_{M=-L}^{L} y_{LM}Y_{LM}(\boldsymbol{n})}_{P'\text{ , arbitrary }y_{LM}}\,,$$

Proposal: We build $P(\rho, \boldsymbol{n})$ such that

i) They are covariant functions on SU(2) transformations (to merge both sets of the minimization)
ii) We built $P(U\rho U^\dagger, \boldsymbol{n})$ that their explicit expressions depend only (or can be approximated) on the unistochastic matrices $B \in \mathcal{U}_{N+1} \subset \mathcal{B}_{N+1}$

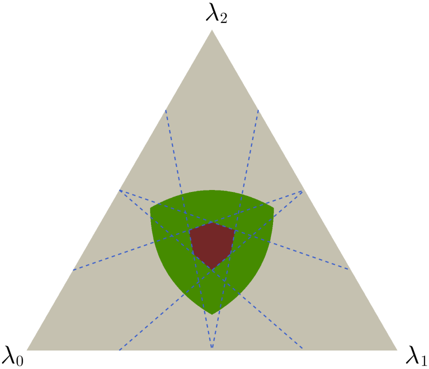
$$B_{ij} = |V_{ij}|^2\,, \quad B_{ij} \geqslant 0\,, \quad \sum_i B_{ij} = \sum_j B_{ij} = 1\,.$$

(to minimize over bistochastic matrices, and use Birkohff's theorem, majorization tools, etc...)

SAS Witness $\mathcal{W}_1$ for $N = 2$

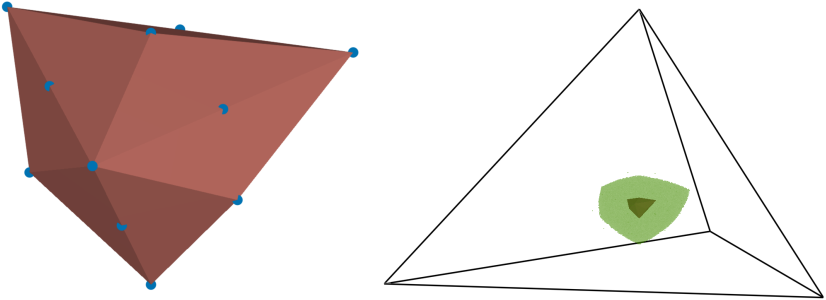
SAS witness $\mathcal{W}_1$: $P = P_0$

$$\rho \in \mathcal{A}_{\mathrm{sym}} \qquad \text{if} \qquad \boldsymbol{\lambda}^{\downarrow}\boldsymbol{\Delta}^{\uparrow T} \geqslant 0\,, \qquad \Delta_k = (-1)^{N-k}\binom{N+1}{k}\,,$$



Polytope of SAS states detected by $\mathcal{W}_1$ for $N = 2$.

SAS Witness $\mathcal{W}_1$ for $N = 3$



Polytope of SAS states detected by $\mathcal{W}_1$ for $N = 3$.

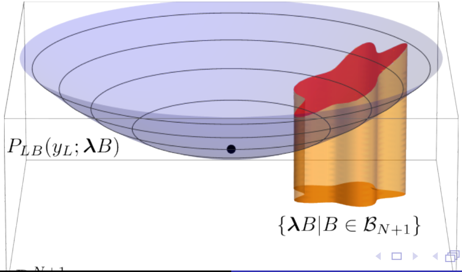
SAS witnesses $\mathcal{W}_2(\{y_L\})$

SAS witness $\mathcal{W}_2(\{y_L\})$: $P = P_0 + P(\rho^{\otimes 2}, \{y_L\}) \geqslant P_{LB}$

$$\min_{\substack{U\in SU(N+1)\\ \boldsymbol{n}\in S^2}} P(U\rho U^\dagger, \boldsymbol{n}; y_L) \geqslant \min_{\substack{\boldsymbol{\lambda} B\\ B\in\mathcal{B}_{N+1}}} P_{LB}(y_L; \boldsymbol{\lambda} B)$$

$$= \min_{\substack{\boldsymbol{\lambda} B\\ B\in\mathcal{B}_{N+1}}} f + \sum_{L=1}^{2j} \left[g_L \boldsymbol{\lambda} B\, \mathbf{t}_L^T + h_L \left(\boldsymbol{\lambda} B\, \mathbf{t}_L^T \right)^2 \right] \geqslant 0\,.$$

P_{LB} a quadratic function with nonegative Hessian that can be optimized in the set of the unistochastic matrices (bistochastic matrices, Birkohff's theorem, majorization, etc.).



SAS witnesses $\mathcal{W}_2(\{y_L\})$

SAS witness $\mathcal{W}_2(\{y_L\})$: A symmetric $2j = N$-qubit state ρ is SAS if for some values of $\{y_L\}$

$$\min_{\substack{\boldsymbol{\lambda} B\\ B\in\mathcal{B}_{N+1}}} P_{LB}(y_L;\boldsymbol{\lambda} B) = \min_{\substack{\boldsymbol{\lambda} B\\ B\in\mathcal{B}_{N+1}}} f + \sum_{L=1}^{2j}\left[g_L \boldsymbol{\lambda} B\,\mathbf{t}_L^T + h_L\left(\boldsymbol{\lambda} B\,\mathbf{t}_L^T\right)^2\right] \geqslant 0\,,$$

$$f = \frac{1}{N+1} + \left(\frac{y_N F(N,1)}{2}\right)\left(\mathrm{Tr}(\rho^2) - \frac{1}{N+1}\right)^2\,,$$

$$g_L = \sqrt{\frac{2L+1}{N+1}}\left(C^{jj}_{jjL0}\right)^{-1}\,, \qquad h_L = y_L F(L,0)\Theta(L-j) - \frac{y_{2j}F(2j,1)}{2}\,,$$

$$\boldsymbol{t}_L = \left(C^{L0}_{jj,j-j}, -C^{L0}_{jj-1,j1-j}, \dots, (-1)^{2j} C^{L0}_{j-j,jj}\right)\,,$$

$$F(L,\mu) \equiv \begin{cases} 1 - \displaystyle\sum_{\substack{\sigma=0\\ \sigma\ \text{even}}}^{2j} \left(C^{\sigma 0}_{L0L0}\right)^2 & \text{if } \mu = 0 \\[2em] 2(-1)^{\mu+1} \displaystyle\sum_{\substack{\sigma=0\\ \sigma\ \text{even}}}^{2j} C^{\sigma 0}_{L0L0} C^{\sigma 0}_{L\mu L-\mu} & \text{if } \mu \neq 0 \end{cases}$$

The variables h_L must be positive, restricting the domain of the free parameters $\{y_L\}$.

Example: $\mathcal{W}_2(\{y_2\})$ for $N = 2$

A symmetric 2-qubit state ρ with spectrum $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2)$ is SAS if

$$\min_{\substack{\boldsymbol{\lambda}B \\ B\in\mathcal{B}_3}} P_{LB}(y_L; \boldsymbol{\lambda}B) = \min_{\substack{\boldsymbol{\lambda}B \\ B\in\mathcal{B}_3}} f + \sum_{L=1}^{2} \left[g_L \boldsymbol{\lambda}B\, \mathbf{t}_L^T + h_L \left(\boldsymbol{\lambda}B\, \mathbf{t}_L^T \right)^2 \right] \geqslant 0$$

for some $y_2 \in \mathbb{R}^+$ and

$$f = \frac{1}{3} - \frac{12}{35} y_2 \left(\mathrm{Tr}(\rho^2) - \frac{1}{3} \right),$$

$$(g_1, g_2) = \left(\sqrt{2}, 5\sqrt{\frac{2}{3}} \right), \quad (h_1, h_2) = \frac{6}{35}(2y_2, 5y_2),$$

$$\boldsymbol{t}_L = (C^{L0}_{11,1-1}, -C^{L0}_{10,10}, C^{L0}_{1-1,11}),$$

Dear Eduardo: Don't forget the video. Best, your colleagues.

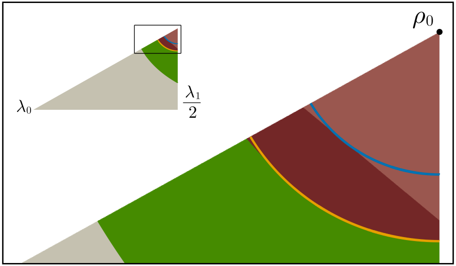
SAS witness $\mathcal{W}_3$

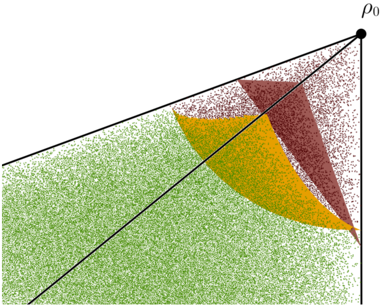
$\mathcal{W}_3$: A symmetric N-qubit state ρ is SAS if

$$r^2 \leqslant \frac{1}{(2j+1)^2}\left(\sum_{L=1}^{2j} \frac{g_L^2}{1-2\Theta(L-j)\frac{F(L,0)}{F(L,1)}}\right)^{-1},$$

where $r^2 \equiv \left\|\rho-\rho_0\right\|_{\mathrm{HS}}^2 = \mathrm{Tr}(\rho^2)-(N+1)^{-1}$.

Set of SAS states $\mathcal{S}_k$ witnessed by $\mathcal{W}_k$ in $N = 2, 3$





Dark Brown $= \mathcal{S}_2(\{y_L\})$
Light Brown $= \mathcal{S}_1$

Orange surface $=$ Bound of $\mathcal{S}_3$
Blue surface $=$ Bound of $\mathcal{S}$
[Giraud, Braun, Braun (2017)]

Green $=$ Unwitnessed SAS states by $\mathcal{W}_k$

SAS witnesses for symmetric N-qubit states

[ESE, Denis, Martin (2024)]

Number of qubits $N = 2j$	Witness $\mathcal{W}_1$ Witness $\mathcal{W}_3$
2	$\boldsymbol{\lambda}\,(-3,\,1,\,3)^T \geqslant 0$ $r^2 \leqslant \frac{1}{78} \approx 0.01282$
3	$\boldsymbol{\lambda}\,(-6,\,-1,\,4,\,4)^T \geqslant 0$ $r^2 \leqslant \frac{1}{354} \approx 0.002825$
4	$\boldsymbol{\lambda}\,(-10,\,-5,\,1,\,5,\,10)^T \geqslant 0$ $r^2 \leqslant \frac{11}{25390} \approx 0.0004332$
5	$\boldsymbol{\lambda}\,(-15,\,-15,\,-1,\,6,\,6,\,20)^T \geqslant 0$ $r^2 \leqslant \frac{1595}{16058598} \approx 0.00009932$

Table: SAS witnesses $\mathcal{W}_1$ and $\mathcal{W}_3$ for a state with eigenspectrum $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_N)$ sorted in descending order $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_N$.

Conclusions

Maximum entanglement (negativity) over the unitary orbit for $N = 2, 3$

- Characterization of the SAS states for symmetric 2-qubit system
- Numerical study of the SAS states for symmetric 3-qubit system

ESE and John Martin, SciPost Phys. **15**, 120 (2023)

SAS witnesses in terms of the spectrum or the purity

of the symmetric N-qubit states

ESE, Jérôme Denis and John Martin, PRA **109**, 022430 (2024)

Future work

SAS witnesses with extra terms in the P-representation

Thank you very much for your attention!