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PAPER

Speed excess and total acceleration: a kinematical approach to entanglement

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Abstract

The total variance of a spin state is defined as the average of the variances of spin projection measurements along three orthogonal axes. We show that this quantity also gives the squared rotational speed of the state in projective space, averaged over all rotation axes. We compute the addition law, under system composition, for this quantity and find that, in the case of separable states, it is of simple pythagorean form. In the presence of entanglement, we find that the composite state 'rotates faster than its parts', thus unveiling a kinematical origin for the correlation of total variance with entanglement. We analyze a similar definition for the acceleration of a state under rotations, for both pure and mixed states, and probe numerically its relation with a wide array of entanglement related measures.

1. Introduction

Quantum entanglement has been the focus of intense activity, both experimental and theoretical, for several decades now. This is due to its intrinsic appeal as the epitome of quantum counterintuitiveness as well as its relevance in quantum technology applications [1–3]. Central to its very definition, is the division of a physical system in subsystems—subsequent to that, quantifying entanglement, and related concepts, is a surprisingly multifaceted affair, that becomes increasingly convoluted as the number of subsystems considered grows.

Among the plethora of available measures of entanglement (see, *e.g.*, [4]), one that stands out is Klyachko's 'total variance' [5]. When applied to spin states (which is the case we focus on here) it averages the variance of spin projection measurements over three orthogonal axes (see (21) for the exact formula). In the author's words '…It measures the total level of quantum fluctuations of the system…'. There are many instances where this quantity appears naturally, for example, it is easily shown that a spin state is coherent (in many respects 'most classical') if and only if it minimizes total variance [6]. This is a desirable property for an entanglement measure, since coherent states, viewed as symmetrized states of spin-1/2 subsystems, are separable. At the other extreme, 1-anticoherent spin states [7], which have vanishing spin expectation value, maximize total variance, in accord with their (informal) status as the 'most quantum' states. Thus, Klyachko's proposal passes some basic consistency checks, but a lingering question is 'what has spin variance to do with entanglement?'—this is the starting point of the present work.

Other authors before us have focused their attention to total variance, finding, *e.g.*, an intriguing relation between its critical sets and SLOCC classes [8], a result that further motivated us to understand the definition at an intuitive level. What we found can be summarized as follows:

- The total variance of a state is its average squared speed (in projective space, using the Fubini-Study metric) under rotations.
- In a composite system, entangled states attain higher speeds, on average, than separable ones when rotated.
- 'Addition laws' for the average squared rotational speed, and similar quantities, involving higher order time derivatives, provide a kinematical point of view on entanglement that is worth exploring.

Guided by these initial findings, we came up with a 'speed excess' measure of entanglement: if $|v|^2$ denotes the average squared rotational speed of a bipartite state, and $|v_1|^2$, $|v_2|^2$ the analogous quantity for its subsystems, then the extent to which the separable state pythagorean addition law $|v|^2 = |v_1|^2 + |v_2|^2$ is violated can be taken as a measure of entanglement. Following further this line of reasoning, we looked for an addition law of average squared rotational acceleration. We found that, in the separable case, the subsystems contribute to the acceleration of the bipartite state not only through their own acceleration, but also through their speed. For entangled states, where the reduced states of the subsystems are mixed, the problem became more complicated, as the Fubini-Study metric got replaced by the Bures metric. We resorted to numerical methods in order to explore correlations of these kinematical quantities with several other physically relevant measures.

Entanglement has been related to 'speed of evolution' before, starting with the study of energy-time uncertainty relations [9], the subsequent realization that energy uncertainty relates to speed of evolution [10], followed by inquiries into the maximum attainable speed [11–14], and the role of entanglement in achieving it [15, 16]. Our present contribution complements the above by focusing on the average rotational speed, giving its precise quantitative relation with entanglement, in the form of an addition law, and generalizing these concepts to higher covariant derivatives of the curve traced in quantum state space.

The structure of the paper is as follows: in section 2 we provide some standard background material. In section 3 we interpret the total variance of a state as average squared rotational speed and generalize the concept for mixed states. In section 4 we introduce the total average squared acceleration of a pure state and give general expressions for it for any spin. Section 5 derives the acceleration addition law for pure separable states and then treats the mixed state case. In section 6 we present an extensive collection of plots exploring the correlation among the newly defined quantities and well known related measures like linear and von Neumann entropy, concurrence, negativity, *etc.* Some final remarks appear in section 7.

2. Mathematical preliminaries

2.1. The projective space $\mathbb{P}$

We use the notation and conventions in [17], which we briefly review here. Quantum states of a spin-s system are represented by a vector (ket) $|\psi\rangle \in \mathcal{H} \equiv \mathbb{C}^{n+1}$, with $n = 2s$. States that differ by (complex) rescaling are in a certain physical sense equivalent, there is therefore a natural projection Π to the equivalence class $[\psi] \in \mathbb{C}P^n$,

$$\Pi: \mathcal{H} \to \mathbb{C}P^n, \quad |\psi\rangle = (\psi^0, \psi^1, \ldots, \psi^n)^T \mapsto [\psi] = (z^1, \ldots, z^n), \tag{1}$$

with $z^i = \psi^i/\psi^0$, together with their complex conjugates $\bar{z^i} \equiv w^i$, serving as coordinates in the U_0 chart, where $\psi^0 \neq 0$—we will denote them collectively by z^A, with A ranging over $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$. $\mathbb{C}P^n$, in its turn, may be embedded into $\mathfrak{u}(n+1)$ as the $U(n+1)$-adjoint orbit of the density matrix $\rho_0 = \mathrm{diag}(1, 0, \ldots, 0)$ (see, *e.g.*, [18]), the latter living naturally in the unitary Lie algebra $\mathfrak{u}(n+1)$ (in its Hermitian version preferred by physicists),

$$\varphi: \mathbb{C}P^n \hookrightarrow \mathfrak{u}(n+1), \quad [\psi] \mapsto \rho_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} = \Delta^{-1}\begin{pmatrix} 1 & w^1 & \ldots & w^n \\ z^1 & z^1 w^1 & \ldots & z^1 w^n \\ \vdots & & & \vdots \\ z^n & z^n w^1 & \ldots & z^n w^n \end{pmatrix}, \tag{2}$$

with $\Delta \equiv 1 + \sum_{i=1}^n z^i w^i$. We abbreviate the image of $\mathbb{C}P^n$ under φ by $\mathbb{P} \subset \mathfrak{u}(n+1)$. Thus, $\mathbb{P}$ is the locus of $(n+1)$-dimensional complex matrices ρ satisfying

$$\rho^\dagger = \rho, \qquad \mathrm{Tr}\,\rho = 1, \qquad \rho^2 = \rho. \tag{3}$$

We use Greek indices, ranging from 0 to n, for the components of vectors and density matrices, and slightly abuse that notation writing $(z^\mu) = (1, z^1, \ldots, z^n)$, so that, *e.g.*,

$$\rho^{\mu\nu} = \Delta^{-1} z^\mu w^\nu. \tag{4}$$

The coordinate basis in the tangent space $T_\rho\mathbb{P}$ is given by $\rho_A \equiv \partial\rho/\partial z^A$, which are *not* Hermitian matrices. Real tangent vectors v are constrained to satisfy $v^{\bar{a}} = \overline{v^a}$, with $v^{\bar{a}}$ denoting the component of v along ∂_{w^a}, $v = v^a\partial_{z^a} + v^{\bar{a}}\partial_{w^a} \equiv v^a\partial_a + v^{\bar{a}}\partial_{\bar{a}}$. Tangent vectors to $\mathbb{P}$, like the above v, are $(n+1)$-dimensional complex matrices satisfying the infinitesimal versions of (3),

$$v^\dagger = v, \qquad \mathrm{Tr}\, v = 0, \qquad \rho v + v\rho = v. \tag{5}$$

The natural (Fubini-Study (FS)) metric on $\mathbb{P}$ is $\langle v, v' \rangle = \frac{1}{2}\mathrm{Tr}(vv')$. In the above coordinate basis, the FS metric and its inverse have components

$$g_{a\bar{b}} = \frac{1}{2}\Delta^{-2}(\Delta\delta^a{}_b - z^b w^a), \qquad g^{a\bar{b}} = 2\Delta(\delta^a{}_b + z^a w^b), \tag{6}$$

with $g_{\bar{b}a} = g_{a\bar{b}}$ (*i.e.*, g_{AB} is symmetric), and $g_{b\bar{a}} = \bar{g}_{a\bar{b}}$ (*i.e.*, $(g_{a\bar{b}})$ is Hermitian). Similar statements holding true for the inverse metric. The Christoffel symbols are found to be

$$\Gamma^c{}_{ab} = g^{c\bar{r}}\partial_a g_{b\bar{r}} = -\Delta^{-1}(\delta^c{}_b w^a + \delta^c{}_a w^b), \qquad \Gamma^{\bar{c}}{}_{\bar{a}\bar{b}} = -\Delta^{-1}(\delta^c{}_b z^a + \delta^c{}_a z^b), \tag{7}$$

with all mixed components vanishing, while the non-zero Riemann tensor components are

$$R_{a\bar{b}c\bar{d}} = \frac{1}{2}(g_{a\bar{b}}g_{c\bar{d}} + g_{a\bar{d}}g_{c\bar{b}}). \tag{8}$$

Given a curve ρ_t in $\mathbb{P}$, its velocity $v_t = \dot{\rho}_t$ is tangent to $\mathbb{P}$ at ρ_t, but its second time derivative is, in general not. The acceleration of ρ_t, which we define as the covariant time derivative of v_t, using the Levi-Civita connection of the FS metric, can be obtained from $\ddot{\rho}_t$ by projecting it onto $T_{\rho_t}\mathbb{P}$ (see [17] for details)

$$a_t = \nabla_t v_t = \ddot{\rho}_t^{\parallel} = \rho_t\ddot{\rho}_t\tilde{\rho}_t + \tilde{\rho}_t\ddot{\rho}_t\rho_t, \tag{9}$$

where $\tilde{\rho}_t \equiv I - \rho_t$ (in what follows we often omit writing the subindex t).

2.2. Hilbert-Schmidt space and the Bures metric

2.2.1. Horizontality in the Hilbert-Schmidt space

The Hilbert-Schmidt space $\mathcal{HS}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ is the (also Hilbert) space of linear operators acting on $\mathcal{H}$, equipped with the Hermitian inner product $\langle A, B\rangle = \frac{1}{2}\mathrm{Tr}(A^\dagger B)$. The latter gives rise to the $\mathcal{HS}$ metric

$$\bar{g}(A, B) = \frac{1}{2}\mathrm{Tr}(A^\dagger B + BA^\dagger). \tag{10}$$

There is a natural projection $\pi\colon \mathcal{HS}^* \to P$ (where $\mathcal{HS}^*$ denotes the subset of invertible operators, and P the positive cone, *i.e.*, the space of positive (and, hence, Hermitian) operators), given by

$$\pi(A) = AA^\dagger, \qquad \pi_*(\dot{A}) = \dot{A}A^\dagger + A\dot{A}^\dagger, \tag{11}$$

where π_* denotes the corresponding pushforward. The unitary group acts from the right on $\mathcal{HS}$, $A \triangleleft U = AU$ and the orbit of A under this action is the fiber of π, *i.e.*, the locus of operators that share the same projection. The vectors tangent to the fiber are declared to be vertical—a Hermitian matrix H gives rise to a vertical vector field $H^\sharp(A) \equiv -iAH$. Tangent vectors V that are perpendicular to all vertical vectors are declared horizontal,

$$\bar{g}(V, H^\sharp) = 0 \text{ for all Hermitian H} \Rightarrow V \text{ horizontal}. \tag{12}$$

Unpacking the previous expression we get

$$\bar{g}(V, H^\sharp) = 0 \Leftrightarrow \mathrm{Tr}(iVHA^\dagger - iV^\dagger AH) = 0 \Leftrightarrow \mathrm{Tr}[H(A^\dagger V - V^\dagger A)] = 0, \tag{13}$$

which, being true for all Hermitian H, implies that $A^\dagger V = V^\dagger A$, that is, $A^\dagger V = F$ is Hermitian. Solving for V gives

$$V = (A^{-1})^\dagger F = (A^{-1})^\dagger FA^{-1}A = GA, \tag{14}$$

where $G = (A^{-1})^\dagger FA^{-1}$ is also Hermitian. Thus, horizontality of $V \in T_A\mathcal{HS}^*$ implies V=GA, with G Hermitian—it is easily seen that the converse is also true. Given a curve $P = \pi(A)$ in the space of positive operators, with A horizontal (*i.e.*, with its tangent vector $\dot{A}$ horizontal), we get

$$\dot{P} = GP + PG. \tag{15}$$

More details can be consulted in [19] and references therein, see also [1].

2.2.2. The Bures metric

We restrict the previous discussion to operators A on the sphere S defined by $\mathrm{Tr}(AA^\dagger) = 1$. The image of S under π gives the space of density matrices ρ, where the Bures metric is defined [5]. The tangent space $T_A S$ consists of all vectors V orthogonal to A,

$$\mathrm{Tr}(VA^\dagger + V^\dagger A) = 0 \Leftrightarrow \bar{g}(V, A) = 0. \tag{16}$$

[5] There is a set of measure zero of singular operators, *i.e.*, operators without full rank in the Hilbert-Schmidt space. Over this set, the Bures metric can also be defined (see, for instance, [20]). Throughout this work, we only use either non-singular density operators or density operators defined from pure states (rank-1) where the Bures metric reduces to the Fubini-Study metric.

Given a vector $V \in T_A\mathcal{HS}^*$, its component orthogonal to $T_A S$ is

$$V^{\perp} = V - \frac{1}{2}\mathrm{Tr}(AV^{\dagger} + VA^{\dagger})A, \tag{17}$$

so that, given a curve $A \in S$, its intrinsic acceleration a can be computed by projecting $\ddot{A}$ onto $T_A(S)$,

$$\bar{a} = \ddot{A} - \frac{1}{2}\mathrm{Tr}(A\ddot{A}^{\dagger} + \ddot{A}A^{\dagger})A. \tag{18}$$

Finally, the Bures metric is defined by $g(\dot{\rho}_1, \dot{\rho}_2) = \bar{g}(\dot{A}_1, \dot{A}_2)$, where A_i are horizontal lifts of ρ_i. It follows that

$$\begin{aligned} g(\dot{\rho}_1, \dot{\rho}_2) &= \bar{g}(\dot{A}_1, \dot{A}_2) \\ &= \frac{1}{2}\mathrm{Tr}(\dot{A}_1^{\dagger}\dot{A}_2 + \dot{A}_2^{\dagger}\dot{A}_1) \\ &= \frac{1}{2}\mathrm{Tr}(A_1^{\dagger}G_1\dot{A}_2 + \dot{A}_2^{\dagger}G_1A_1) \\ &= \frac{1}{2}\mathrm{Tr}(G_1(\dot{A}_2A_1^{\dagger} + A_1\dot{A}_2^{\dagger})) \\ &= \frac{1}{2}\mathrm{Tr}(G_1\dot{\rho}_2), \end{aligned} \tag{19}$$

where $G_1A_1 = \dot{A}_1$, and $A_1(0) = A_2(0) = A$, $\rho_1(0) = \rho_2(0) = \rho$, *i.e.*, both curves A_i emanate from A, and similarly for ρ_i. Alternatively,

$$g(\dot{\rho}_1, \dot{\rho}_2) = \frac{1}{2}\mathrm{Tr}(G_1(G_2\rho + \rho G_2)) = \frac{1}{2}\mathrm{Tr}(\rho(G_1G_2 + G_2G_1)). \tag{20}$$

3. Total variance, entanglement, and speed excess

3.1. The many facets of total variance

3.1.1. Total variance as a measure of quantum fluctuations

Given a Lie group $G \subset U(n)$ which acts on $\mathbb{P}$, and a linear basis $\{e_A, A = 1,\ldots,k\}$ of the corresponding Lie algebra $\mathfrak{g} \subset \mathfrak{u}(n)$, orthonormal w.r.t. an $\mathrm{ad}_{\mathfrak{g}}$-invariant metric, the *total* $\mathfrak{g}$*-variance* of $\rho = |\psi\rangle\langle\psi| \in \mathbb{P}$ (with $|\psi\rangle$ in $\mathcal{H}$) is defined in [5] (see also [6]) as

$$\mathbb{D}_{\mathfrak{g}}(\rho) = \sum_{A=1}^{k}\langle\psi|T_A^2|\psi\rangle - \langle\psi|T_A|\psi\rangle^2, \tag{21}$$

where T_A is the matrix representing the action of e_A on $\mathcal{H}$. The obvious physical interpretation of this quantity is as a measure of '…the total level of quantum fluctuations of the system in state $|\psi\rangle$' [5]. In the case of $G = SU(2)$ and $\mathbb{P} = \mathbb{C}P^{2s}$, the first term in the r.h.s. of (21) gives the $SU(2)$ Casimir operator in the spin-s representation, so that

$$\mathbb{D}_{\mathfrak{su}(2)}(\rho) = s(s+1) - \sum_{A=1}^{3}\langle\psi|S_A|\psi\rangle^2, \tag{22}$$

implying that the total variance is minimized by coherent states, and maximized by anticoherent ones (the latter defined by the vanishing of the spin expectation value $\langle\psi|\mathbf{S}|\psi\rangle$ [7]). An additional, less than obvious, physical interpretation of $\mathbb{D}_{\mathfrak{su}(2)}(\rho)$ is also put forth in [5]: considering a spin-s state as a multipartite symmetric state of $2s$ spin-$1/2$ subsystems, the suggestion is made that its total $\mathfrak{su}(2)$-variance be considered as a measure of its entanglement. The idea has been further explored in [8], where the critical sets of $\mathbb{D}_{\mathfrak{su}(2)}$ are used to classify all SLOCC classes of multipartite pure states. For the case of spin-s pure states, viewed as totally symmetric states of $2s$ qubits, we derive in appendix A a monotonic relation between total variance and entropy of entanglement, showing that the former can be considered a measure of the latter (see more details in section 6).

3.1.2. Total variance as a measure of rotational speed

We propose here an alternative characterization of the total variance, that, in turn, suggests an explanation of its relation with entanglement. When a spin-s system is rotated in physical space, around an axis $\mathbf{n}$, the velocity of its quantum state in $\mathbb{P}$ is[6]

[6] The rotation is generated by the Hamiltonian $H = \mathbf{n}\cdot\mathbf{S}$, giving for the density matrix $\rho_t = e^{-itH}\rho_0 e^{itH}$, so that its derivative, at $t = 0$, is given by (23).

$$v_{\mathbf{n}} = -i[\mathbf{n}\cdot\mathbf{S},\, \rho], \tag{23}$$

with modulus squared

$$\begin{aligned}|v_{\mathbf{n}}|^2 &= \frac{1}{2}\operatorname{Tr} v_{\mathbf{n}}^2 \\ &= -\frac{1}{2}\operatorname{Tr}([\mathbf{n}\cdot\mathbf{S},\,\rho][\mathbf{n}\cdot\mathbf{S},\,\rho]) \\ &= \langle\psi|(\mathbf{n}\cdot\mathbf{S})^2|\psi\rangle - \langle\psi|\mathbf{n}\cdot\mathbf{S}|\psi\rangle^2.\end{aligned} \tag{24}$$

Averaging over the rotation axis (with $\int_{S^2} n_i n_j \mathrm{d}\Omega = \delta_{ij}/3$) we find

$$\langle|v_{\mathbf{n}}|^2\rangle_{S^2} = \frac{1}{3}\mathbb{D}_{\mathfrak{su}(2)}(\rho) \equiv \frac{1}{3}\mathbb{D}(\rho), \tag{25}$$

i.e., the total variance is proportional to the square of the rotational speed of the state, averaged over the rotation axis (from now on, we simplify the notation by dropping the $\mathfrak{su}(2)$ index). As has been shown in [21] (see the discussion at the beginning of section V of that reference), rather than averaging over the direction of **n**, one may consider instead a fixed rotation axis, but work with a mixed state containing all possible reorientations of an initial spin state—this setup provides an operational definition of the averaging process used above.

This geometric interpretation of the total variance suggests an obvious generalization to mixed states. The velocity $v_{\mathbf{n}}$ acquired by such a state ρ when rotated around **n** is still given by (23), but its modulus squared $|v_{\mathbf{n}}|^2$ entails now the Bures metric [1]. Averaging $|v_{\mathbf{n}}|^2$ over **n** gives, by definition, the total variance of the mixed state ρ —we give an example of an explicit calculation below.

3.1.3. Total variance as a measure of entanglement: speed excess

Aiming at connecting total variance to entanglement, we ask now what is the addition law for the square of the rotational speed of a composite quantum system, given the square of the (rotational) speeds of its subsystems? For a curve of pure separable states, $\rho(t) = \rho_1(t)\otimes\rho_2(t)$, one finds easily

$$|v|^2 = |v_1|^2 + |v_2|^2, \tag{26}$$

where $v = \dot{\rho}$, $v_1 = \dot{\rho}_1$, $v_2 = \dot{\rho}_2$, $|v|^2 = 1/2\operatorname{Tr} v^2$, *etc.*—this result is valid for an arbitrary (separable) time evolution, *i.e.*, it is not tied to rotations. Having said that, we emphasize that, in the discussion that follows, all time evolution considered is due to a rotation around a fixed axis. Anticipating that the presence of entanglement will complicate things, in the form of additional, entanglement-dependent terms, in the r.h.s. of (26), we define the *(squared) speed excess* $F(\rho)$ of a pure bipartite state ρ as follows

$$F(\rho) = |v|^2 - |v_1|^2 - |v_2|^2, \tag{27}$$

where now $\rho_1 = \operatorname{Tr}_2\rho$, $\rho_2 = \operatorname{Tr}_1\rho$ are the reduced density matrices, corresponding, in general, to mixed states, and the moduli $|v_i|^2$ are computed, as mentioned above, with the Bures metric. $F(\rho)$ is defined for mixed states similarly, with all three terms computed using the Bures metric. Note that $F(\rho)$ depends on the rotation axis **n**—we get rid of this dependence later on by averaging over all axes. While the separable-state addition law (26) implies that the speed of a composite system is entirely due to the speed of its parts, the more general addition law in (27) implies that, in general, entanglement also contributes to the speed of the composite system, leaving open the possibility *e.g.*, that the latter may move even when its parts are 'at rest'. Consider, for example, the bipartite spin-$1/2$ symmetric state $|\Psi_t\rangle$ with Majorana constellation (see appendix B for an explanation of this parametrization of the pure spin quantum states) given by two antipodal stars on the equator, rotating around the z-axis with unit angular velocity, expressed in the $(|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle)$-basis,

$$|\Psi_t\rangle = \frac{1}{\sqrt{2}}(1,\, 0,\, 0,\, -e^{i2t}). \tag{28}$$

This composite state certainly has nonzero speed in projective space (essentially the spin-1 state space $\mathbb{C}P^2$)—one easily computes that $|v|^2 = |\dot{\rho}_{|\Psi\rangle}|^2 = 1$. On the other hand, the reduced density matrices are those of the maximally mixed state, with vanishing speeds, $|v_1|^2 = |v_2|^2 = 0$. Thus, in this case, the entire speed of the composite state is due to entanglement, and $F(\rho_{|\Psi_t\rangle}) = 1$.

Specifying (27) to the case of symmetric states of two spin-$1/2$ systems, where the reduced density matrices are equal among themselves, $\rho_1 = \rho_2$, we get

$$F(\rho) = |v|^2 - 2|v_1|^2, \tag{29}$$

with

$$\rho_1 = \frac{1}{2}(I + \mathbf{r}\cdot\boldsymbol{\sigma}), \quad \mathbf{r} = (x, y, z), \quad v_1 = \dot{\rho}_1, \quad |v_1|^2 = g_{AB}(\rho_1)v_{1\,A}v_{1\,B}, \quad v_{1\,A} = \operatorname{Tr}(v_1\sigma_A), \tag{30}$$

and the Bures metric being given by (see the metric in (9.50) of [1], which should be pulled back onto the unit-trace submanifold)

$$g(\rho_1) = \frac{1}{4(1-x^2-y^2-z^2)}\begin{pmatrix} 1-y^2-z^2 & xy & xz \\ xy & 1-x^2-z^2 & yz \\ xz & yz & 1-x^2-y^2 \end{pmatrix}. \tag{31}$$

When the time evolution of ρ is due to rotation around $\mathbf{n}$, we get (compare with (23))

$$v_{\mathbf{n}} = -\frac{i}{2}[\mathbf{n}\cdot(\boldsymbol{\sigma}\otimes I + I\otimes\boldsymbol{\sigma}),\, \rho], \qquad v_{1\,\mathbf{n}} = -\frac{i}{2}[\mathbf{n}\cdot\boldsymbol{\sigma},\, \rho_1], \tag{32}$$

from which we can calculate $F_{\mathbf{n}}(\rho) = |v_{\mathbf{n}}|^2 - 2|v_{1\,\mathbf{n}}|^2$. Averaging this over $\mathbf{n}$ we finally get the *total (rotational) speed excess*

$$\begin{aligned}\mathbb{F}(\rho) &\equiv \langle F_{\mathbf{n}}(\rho)\rangle_{S^2} \\ &= \langle |v_{\mathbf{n}}|^2\rangle_{S^2} - 2\langle |v_{1\,\mathbf{n}}|^2\rangle_{S^2}.\end{aligned} \tag{33}$$

On the other hand, a straightforward calculation, starting from the definition (33), and using (30), (31), and (32), shows that

$$\begin{aligned}\mathbb{D}(\rho) + \mathbb{D}(\rho_1) + \mathbb{D}(\rho_2) &= 3(\langle |v_{\mathbf{n}}|^2\rangle_{S^2} + 2\langle |v_{1\,\mathbf{n}}|^2\rangle_{S^2}) \\ &= |v_{\mathrm{x}}|^2 + |v_{\mathrm{y}}|^2 + |v_{\mathrm{z}}|^2 + 2(|v_{1\,\mathrm{x}}|^2 + |v_{1\,\mathrm{y}}|^2 + |v_{1\,\mathrm{z}}|^2) \\ &= 2.\end{aligned} \tag{34}$$

At first, this might look like a counterintuitive result (at least it did to the authors): it says that the bigger the total variance of the composite state ρ is, the smaller the total variances of the states of the subsystems have to be, and vice versa. A moment's thought though reveals that this result exactly encodes, in a precise quantitative manner, the fact that total variance is a measure of entanglement—we proceed to explain this statement in some detail: we start with expressing the Bures metric in spherical polar coordinates (r, θ, ϕ), where it acquires a diagonal form,

$$g^{\mathrm{polar}}(\rho_1) = \frac{1}{4}\mathrm{diag}\left(\frac{1}{1-r^2},\, r^2,\, r^2\sin^2\theta\right), \tag{35}$$

where $r = \sqrt{x^2+y^2+z^2}$, $\cos\theta = z/r$. Note that this differs from the euclidean metric $\mathrm{diag}(1,\, r^2,\, r^2\sin^2\theta)/4$, which is the trace metric (24), only in the radial direction. But $SU(2)$ transformations of ρ result in rotations of $\mathbf{r}$, the corresponding velocity having no radial component. Thus, the speed, due to rotations, of the reduced state ρ_1, calculated with the Bures metric, is just the euclidean speed of the tip of $\mathbf{r}$. After averaging over all rotation axes, the result only depends on the length r of the vector. Therefore, the higher the total variance of the bipartite state ρ is, the smaller r has to be (because of (34)), the more mixed the reduced state ρ_1 is, and, hence, the more entangled ρ turns out to be.

Having clarified this point, we return to our calculation of the total speed excess. Using (34), together with (25), (33), we arrive at

$$\mathbb{F}(\rho) = \frac{2}{3}(\mathbb{D}(\rho) - 1), \tag{36}$$

in other words, total speed excess and total variance are functions on the symmetric subspace of the 2-qubit state space (*i.e.*, on the spin-1 projective space $\mathbb{C}P^2$) that are related by a simple affine transformation. The total speed excess is minimum on the coherent states (equal to zero) and maximum on the anticoherent ones (equal to $2/3$).

3.2. Total variance for mixed states

We inquire about the extension of the total variance concept to bipartite mixed states. Restricting our analysis, as in the pure state case above, to the symmetric sector of a two-qubit system, a general density matrix can be parametrized as follows

$$\rho = \frac{1}{4}I + \mathbf{n}\cdot\boldsymbol{\Sigma} + \frac{1}{4}\sum_{A=1}^{3} t_{AA}\Sigma_{AA} + \frac{1}{8}\sum_{B>A=1}^{3} t_{AB}\Sigma_{AB}, \tag{37}$$

where

$$\boldsymbol{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3), \qquad \Sigma_A = \sigma_A\otimes I + I\otimes\sigma_A, \qquad \Sigma_{AB} = \sigma_A\otimes\sigma_B + \sigma_B\otimes\sigma_A, \tag{38}$$

with appropriate conditions on $\mathbf{n}$, t_{AB} guaranteeing non-negative eigenvalues for ρ (see, *e.g.*, [22]). Note that for vanishing $\mathbf{n}$, t_{AB} all eigenvalues of ρ are equal to $1/4$, so we can be sure that for sufficiently small values of these parameters all eigenvalues of ρ will be positive. The modulus squared, according to the Bures metric, of a tangent vector X at ρ is given by

$$|X|_{\mathrm{B}}^2 = g_{\mathrm{B}}(X,\, X)_\rho = \frac{1}{2}\mathrm{Tr}\,(GX), \tag{39}$$

where the Hermitian matrix G is uniquely determined (for positive ρ) by the relation $X = \rho\, G + G\rho$. We now map an $n \times n$ matrix A to an n^2-dimensional vector $|A\rangle = (A_{11},\, A_{12},\ldots,A_{nn})^T$—it is easily checked that, in this notation, $(A \otimes B^T)|C\rangle = |ACB\rangle$ (where $(A \otimes B^T)_{ij,rs} = A_{ir}B_{sj}$), so that the above relation for G becomes

$$|X\rangle = R|G\rangle, \quad R \equiv \rho \otimes I + I \otimes \rho^T. \tag{40}$$

It is easily seen that the eigenvalues of R are $\{\lambda_\alpha + \lambda_\beta\}$, where $\{\lambda_\alpha\}$ are those of ρ, so that R is invertible if ρ is positive, and (40) gives $|G\rangle = R^{-1}|X\rangle$, resulting in

$$\begin{aligned} g_{\mathrm{B}}(X,\, X)_\rho &= \frac{1}{2}\sum_{ij=1}^{n} G_{ij}X_{ji} \\ &= \frac{1}{2}\sum_{ijrs=1}^{n} R^{-1}_{ij,rs}X_{rs}X_{ji} \\ &= \frac{1}{2}\langle X|R^{-1}|X\rangle, \end{aligned} \tag{41}$$

where $\langle X| = |X\rangle^\dagger$ and $X^\dagger = X$ was used. The main obstruction in using (41) is the inversion of R—even for the very modest case of a bipartite spin-1/2 system, R is of dimension 16, and its inverse is, in general, difficult to compute. The only case we have managed to invert R by brute force is when $t_{ij} = 0$ in (37)—the corresponding metric is

$$g_B = \mathrm{diag}\left(\frac{2}{1 - 16r^2},\, \frac{2r^2}{1 - 4r^2},\, \frac{2r^2\sin^2\theta}{1 - 4r^2}\right), \tag{42}$$

in the coordinate (r, θ, ϕ)-basis in which $\mathbf{n} = r(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$. Since rotations transform the components of $\mathbf{n}$ and the t_{ij}'s separately, we may use (42) to compute the total variance of ρ. Under a rotation, ρ transforms according to $\rho \to (U \otimes U)\rho(U^\dagger \otimes U^\dagger)$, with $U \in SU(2)$, so that the fundamental vector field $\hat{S}_A$, corresponding to $S_A \in \mathfrak{su}(2)$, is $\hat{S}_A = -i[\Sigma_A,\, \rho]$. The total variance of the mixed state (37), with all six t_{AB} equal to zero, is found to be

$$\begin{aligned} \mathbb{D}_{\mathbf{n},t_{ij}=0}(\rho) &= \sum_{A=1}^{3}|\hat{S}_A|_{\mathrm{B}}^2 \\ &= \frac{4r^2}{1 - 4r^2}. \end{aligned} \tag{43}$$

On the other hand, the reduced state is

$$\rho_1 = \mathrm{Tr}_2\,\rho = \frac{1}{2}I + \mathbf{n}\cdot\boldsymbol{\sigma}, \tag{44}$$

with total variance $\mathbb{D}(\rho_1) = 2r^2$, so that the speed excess comes out equal to

$$\mathbb{F}_{\mathbf{n},t_{ij}=0}(\rho) = \mathbb{D}_{\mathbf{n},t_{ij}=0}(\rho) - 2\mathbb{D}(\rho_1) = \frac{16r^4}{1 - 4r^2}. \tag{45}$$

We may, alternatively, take $\mathbf{n} = 0$ in (37), keeping all six t_{AB}'s as parameters. The inverse of the resulting R takes too long to compute by brute force in Mathematica, so we have to resort to more elaborate methods that can be found in the literature [23–25]. Following the notation in [25], and specifying it to the case at hand, we define the characteristic polynomial of ρ,

$$\chi(\lambda) = \det(\lambda I - \rho) \equiv \lambda^4 + k_1\lambda^3 + k_2\lambda^2 + k_3\lambda + k_4, \tag{46}$$

and the matrices

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_4 & -k_3 & -k_2 & -k_1 \end{pmatrix}, \quad N = \begin{pmatrix} k_3 & k_2 & k_1 & 1 \\ -k_2 & -k_1 & -1 & 0 \\ k_1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{47}$$

in terms of which the following matrix A is defined

$$A = -\chi(-K^T)^{-1}N. \tag{48}$$

Proposition 2 in [25] implies, in our notation, that

$$R^{-1} = \sum_{i,j=1}^{4} A_{ij}\,\rho^{i-1} \otimes (\rho^T)^{j-1}, \tag{49}$$

giving for the Bures metric

$$
\begin{aligned}
g_B(X, X) &= \frac{1}{2}\sum_{i,j=1}^{4} A_{ij}\langle X|\rho^{i-1}\otimes(\rho^T)^{j-1}|X\rangle \\
&= \frac{1}{2}\sum_{i,j=1}^{4} A_{ij}\,\mathrm{Tr}\,(X\rho^{i-1}X\rho^{j-1}).
\end{aligned} \tag{50}
$$

Using this formula we find that

$$
\mathbb{F}_{\mathbf{n}=0,t_{AB}}(\rho) = \mathbb{D}_{\mathbf{n}=0,t_{AB}}(\rho) = \frac{-256k_2^2 + 96tk_3 - 2t^4 - 48t^2k_2 - 6t^3 + 288k_3 - 64tk_2 + 14t^2 + 240k_2 + 30t - 36}{64k_3 + 64k_2 + 4t^2 + 8t - 12}, \tag{51}
$$

where the k_i are defined in (46), $t \equiv t_{11} + t_{22} + t_{33}$, and the first equality is due to the fact that the corresponding reduced state ρ_1 is the maximally mixed one, so that its total variance vanishes. We have also been able to compute the total variance for the state ρ in (37) with both $\mathbf{n}$ and t_{AB} nonvanishing, but the corresponding expressions are too long to quote here.

4. The total acceleration of a curve in $\mathbb{P}$

Given a curve ρ_t in $\mathbb{P}$, parametrized by time, its velocity is (dropping the subscript t) $v = \dot{\rho}$, while its acceleration is $a = \rho\ddot{\rho}\tilde{\rho} + \tilde{\rho}\ddot{\rho}\rho$ (see, *e.g.*, [17]). In the case where the time evolution of ρ is generated by a Hamiltonian H via Schrödinger's equation, $\dot{\rho} = -i[H, \rho]$, it can be shown that [17]

$$
|v|^2 = h_2 - h_1^2, \qquad |a|^2 = h_4 - 4h_3h_1 - h_2^2 + 8h_2h_1^2 - 4h_1^4, \tag{52}
$$

where $h_m = \mathrm{Tr}(\rho H^m)$, and the FS metric is being used, so that, *e.g.*, $|v|^2 = \mathrm{Tr}(\dot{\rho}^2)/2$, *etc.*

The quantity $|a|^2$ is a function on $\mathbb{P}$ that depends on the Hamiltonian $H \in \mathfrak{u}$ chosen for the time evolution of the system, $|a|^2 = |a(\rho, H)|^2$. Motivated by the analysis in section 3.1.2, we specify now the Hamiltonian to be an element of $\mathfrak{su}(2) \subset \mathfrak{u}(n+1)$, $H_{\mathbf{n}} = \mathbf{n}\cdot\mathbf{S}^{(s)}$, corresponding to rotating the state around the axis $\mathbf{n}$. By averaging the squared modulus of the resulting acceleration over all rotation axes, we obtain the *total (rotational) acceleration* of ρ,

$$
\langle |a|^2\rangle_{S^2} \equiv \langle |a|^2\rangle = \int_{S^2} \mathrm{d}\Omega\, |\mathrm{a}(\rho, \mathrm{H}_{\mathbf{n}})|^2. \tag{53}
$$

4.1. Averaging by integration

Writing $H = H^A T_A$, with T_A denoting an orthonormal basis in $\mathfrak{su}(2)$, we get from the second of (52),

$$
\begin{aligned}
\langle |a|^2\rangle = (&\langle T_A T_B T_C T_D\rangle - 4\langle T_A T_B T_C\rangle\langle T_D\rangle - \langle T_A T_B\rangle\langle T_C T_D\rangle \\
&+ 8\langle T_A T_B\rangle\langle T_C\rangle\langle T_D\rangle - 4\langle T_A\rangle\langle T_B\rangle\langle T_C\rangle\langle T_D\rangle)\int_{S^2} H^A H^B H^C H^D.
\end{aligned} \tag{54}
$$

The integrals of monomials in cartesian coordinates over S^{n-1} are given by

$$
\int_{S^{n-1}} x_1^{m_1}\ldots x_n^{m_n}\mathrm{d}\Omega^{n-1} = \frac{(n-2)!!\prod_{i=1}^{n}(m_i-1)!!}{n-2+\sum_{i=1}^{n} m_i}, \tag{55}
$$

where $\int_{S^{n-1}}\mathrm{d}\Omega^{n-1} = 1$ and the above m_i are all even (if any m_i is odd, the integral is zero). In our case, we only need

$$
\int_{S^2} x_i^2 x_j^2 \mathrm{d}\Omega^2 = \frac{1}{15} \quad \text{for } i \neq j, \qquad \int_{S^2} x_i^4 \mathrm{d}\Omega^2 = \frac{1}{5}. \tag{56}
$$

For spin 1, one finds

$$
\langle |a|^2\rangle = \frac{1459 + 1344\cos(2\alpha) + 140\cos(4\alpha) + 128\cos(6\alpha) + \cos(8\alpha)}{60(3+\cos(2\alpha))^4}, \tag{57}
$$

where 2α is the angle between the two Majorana stars—a plot appears in figure 1 (left). For spin 3/2 things get considerably more complicated. We have computed $\langle |a|^2\rangle_{S^2}$ in terms of the angles α, β, γ between the three Majorana stars. The expression simplifies along the diagonal ($\alpha = \beta = \gamma$),

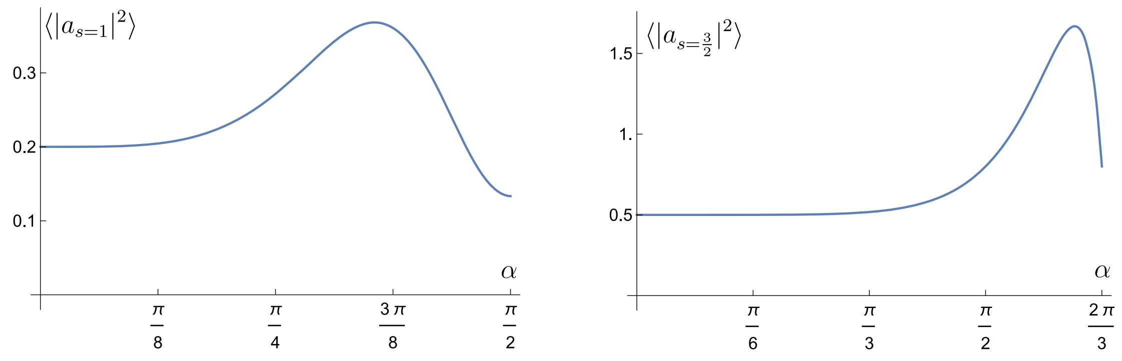


Figure 1. Left: Plot of $\langle|a|^2\rangle$ *versus* α ($s = 1$, α is the half-angle between the two Majorana stars). Right: Plot of $\langle|a|^2\rangle$ *versus* $\alpha = \beta = \gamma$ ($s = 3/2$, α is the angle between any two of the three Majorana stars).

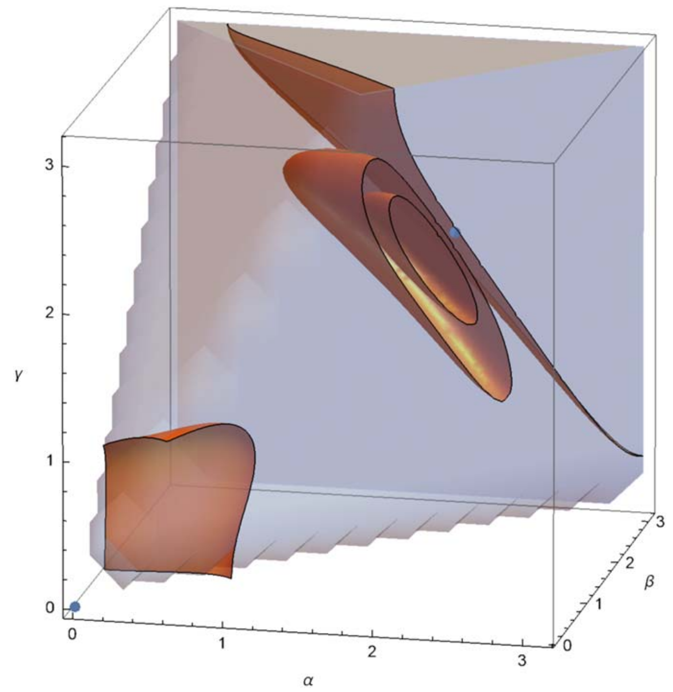


Figure 2. ContourPlot of $\langle|a|^2\rangle$ *versus* α, β, γ ($s = 3/2$, the constellation shape is parametrized in terms of the angles between the stars). Shown are the level surfaces at .51 (lower left and upper right surfaces), 1.3 (outer closed surface), 1.5 (inner closed surface). The blue dots correspond to the coherent state $\alpha = \beta = \gamma = 0$ (lower left corner) and the GHZ state $\alpha = \beta = \gamma = 2\pi/3$ (between the two uppermost surfaces).

$$\langle|a|^2\rangle_{\alpha=\beta=\gamma} = \frac{5774\cos(\alpha) + 1793\cos(2\alpha) + 1027\cos(3\alpha) + 82\cos(4\alpha) - 17\cos(5\alpha) - \cos(6\alpha) + 2862}{1440(\cos(\alpha) + 1)^4}. \tag{58}$$

A plot appears in figure 1 (right), while a contour plot of the full function is shown in figure 2.

A rather long calculation, the details of which are given in appendix C., results in the following expression for the average norm squared of the (rotational) acceleration for a general spin s,

$$\langle|a|^2\rangle = \lambda_1 + \lambda_2|\boldsymbol{\rho}_1|^2 + \lambda_3|\boldsymbol{\rho}_2|^2 + \lambda_4 c^{2N_1+N_2}_{1N_1 1N_2}\rho_{2N_1+N_2}\rho^*_{1N_1}\rho^*_{1N_2} + \lambda_5|\boldsymbol{\rho}_1|^4, \tag{59}$$

where $\boldsymbol{\rho}_L = (\rho_{LL},\ldots,\rho_{L-L})$, with $\rho_{LM} = \langle T^\dagger_{LM}\rangle$, $c^{2N_1+N_2}_{1N_1 1N_2}$ is a Clebsch-Gordan coefficient [26], and

$$\lambda_1 = \frac{s(s+1)(2s-1)(2s+3)}{45}, \tag{60}$$

Table 1. List of the λ_i values in equations (60)–(64) for $i = 1,\ldots,5$, and $s = 1/2, 1, 3/2, 2$.

s	λ_1	λ_2	λ_3	λ_4	λ_5
$\frac{1}{2}$	0	$\frac{1}{6}$	0	0	$-\frac{1}{5}$
1	$\frac{2}{9}$	$\frac{8}{9}$	$-\frac{2}{15}$	$\frac{32}{15}$	$-\frac{16}{5}$
$\frac{3}{2}$	1	3	$-\frac{4}{5}$	$16\sqrt{\frac{2}{3}}$	-20
2	$\frac{14}{5}$	8	$-\frac{14}{5}$	$32\sqrt{\frac{7}{3}}$	-80

$$\lambda_2 = \begin{cases} \frac{4}{27}s^2(s+1)^2(2s+1) & s \leqslant 1/2 \\ \frac{4}{135}s(s+1)(2s+1)(s^2+s+3) & s > 1/2 \end{cases}, \tag{61}$$

$$\lambda_3 = -\frac{1}{225}s(s+1)(2s-1)(2s+1)(2s+3), \tag{62}$$

$$\lambda_4 = \frac{8}{45}\sqrt{\frac{2}{15}}s(s+1)(2s+1)\sqrt{s(s+1)(2s-1)(2s+1)(2s+3)}, \tag{63}$$

$$\lambda_5 = -\frac{4}{45}s^2(s+1)^2(2s+1)^2. \tag{64}$$

We test our equation (59) with the spin-1 state $|\psi\rangle = (\cos^2\frac{\alpha}{2},\ 0,\ -\sin^2\frac{\alpha}{2})\Big/\sqrt{\cos^4\frac{\alpha}{2} + \sin^4\frac{\alpha}{2}}$, and recover the expression in (57). Putting $\cos A = \cos\frac{\alpha}{2}\Big/\sqrt{\cos^4\frac{\alpha}{2} + \sin^4\frac{\alpha}{2}}$, so that $|\psi\rangle = (\cos A,\ 0,\ -\sin A)$, we find

$$\langle |a|^2 \rangle = \frac{1}{30}(8 + \cos(4A) - 3\cos(8A)). \tag{65}$$

Table 1 contains the values of λ_i, $i = 1,\ldots,5$, for $s = 1/2,\ldots,2$.

Note that (59) implies that all 2-anticoherent spin states, for which $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2 = 0$, have the same total acceleration $\langle |a|^2 \rangle = \lambda_1$. Also, for a spin-$s$ coherent state $|n\rangle$ we find

$$\langle |a|^2 \rangle_{|n\rangle} = \frac{1}{45}s(8s^2(s+1) - 4s - 3), \tag{66}$$

which is seen to grow asymptotically, for large s, like s^4. All of the above integrals, involving polynomial functions, were also calculated using spherical designs (see appendix D for more details).

5. Acceleration addition law

5.1. Acceleration addition law for pure bipartite separable states

We look here for the addition law for the norm squared of the acceleration of a separable state $\rho = \rho_1 \otimes \rho_2$—the corresponding result for the norm squared of the velocity is given in (26). With the acceleration given by (9), where

$$\begin{aligned} \dot\rho &= \dot\rho_1 \otimes \rho_2 + \rho_1 \otimes \dot\rho_2, \\ \ddot\rho &= \ddot\rho_1 \otimes \rho_2 + 2\dot\rho_1 \otimes \dot\rho_2 + \rho_1 \otimes \ddot\rho_2, \end{aligned} \tag{67}$$

we get

$$|a|^2 = \frac{1}{2}\mathrm{Tr}(a^2) = \ldots = \mathrm{Tr}(\rho\ddot\rho^2 - \rho\ddot\rho\rho\ddot\rho), \tag{68}$$

where we used the cyclicity of the trace and the fact that $\rho\tilde\rho = \tilde\rho\rho = 0$. Some further gymnastics reveal that

$$\mathrm{Tr}(\rho\ddot\rho^2) = \mathrm{Tr}(\rho_1\ddot\rho_1^2 + \rho_2\ddot\rho_2^2) + 2\,\mathrm{Tr}(\rho_1\ddot\rho_1)\,\mathrm{Tr}(\rho_2\ddot\rho_2^2) + 4\,\mathrm{Tr}(\rho_1\dot\rho_1^2)\,\mathrm{Tr}(\rho_2\dot\rho_2^2), \tag{69}$$

$$\mathrm{Tr}(\rho\ddot\rho\rho\ddot\rho) = \mathrm{Tr}(\rho_1\ddot\rho_1\rho_1\ddot\rho_1 + \rho_2\ddot\rho_2\rho_2\ddot\rho_2) + 2\,\mathrm{Tr}(\rho_1\ddot\rho_1)\,\mathrm{Tr}(\rho_2\ddot\rho_2) + 4\,\mathrm{Tr}(\rho_1\dot\rho_1\rho_1\dot\rho_1)\,\mathrm{Tr}(\rho_2\dot\rho_2\rho_2\dot\rho_2). \tag{70}$$

Noting that

$$\mathrm{Tr}(\rho_1\dot\rho_1\rho_1\dot\rho_1) = \mathrm{Tr}(\rho_1\dot\rho_1)^2 = 0, \qquad \mathrm{Tr}(\rho\dot\rho^2) = \frac{1}{2}\mathrm{Tr}(\dot\rho^2), \tag{71}$$

we finally get

$$|a|^2 = |a_1|^2 + |a_2|^2 + 4|v_1|^2|v_2|^2, \tag{72}$$

showing that the speed of the subparts contributes to the acceleration of the whole—in particular, a bipartite system can have acceleration even though its subparts do not. For example, consider a two spin-1/2 system in the coherent state in the x-direction,

$$|\psi\rangle = |+, \hat{x}\rangle \otimes |+, \hat{x}\rangle, \tag{73}$$

where $|+, \hat{x}\rangle = \frac{1}{\sqrt{2}}(1,\ 1)^T$ in the eigenbasis of S_z. If $|\psi\rangle$ is evolved by rotating about the z axis, one easily finds that $|a_1|^2 = |a_2|^2 = 0$, since the subsystem stars move along great circles on the Bloch sphere, while

$$|\psi(t)\rangle = \left(\frac{e^{-it}}{2}, \frac{1}{2}, \frac{1}{2}, \frac{e^{it}}{2}\right)^T \quad \Rightarrow \quad a = [\rho, [\rho, \ddot{\rho}]] = \frac{1}{4}\begin{pmatrix} -1 & 0 & 0 & -e^{-2it} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -e^{2it} & 0 & 0 & -1 \end{pmatrix}, \tag{74}$$

so that $|a|^2 = \frac{1}{2}\mathrm{Tr}(a^2) = \frac{1}{4}$, *i.e.*, the entire acceleration of the composite system is due to the speed of its subsystems, each of which moves along geodesics of the FS metric.

Another interesting consequence of (72) is that the only separable state curves that are geodesics (characterized by $|a| = 0$) are those for which one of the parts does not evolve at all, while the other follows a geodesic.

5.2. Acceleration of mixed bipartite states

5.2.1. Parallel transport in the space of density matrices

Consider two vector fields X and Y defined in a neighborhood of a density matrix ρ. Denote by $\overline{X}$ and $\overline{Y}$ their horizontal lifts in S. Then, as we prove for completeness below,

$$\nabla_X Y = \pi_*(\overline{\nabla}_{\overline{X}}\overline{Y}), \tag{75}$$

where ∇ denotes the Levi-Civita connection of the space of density matrices, and $\overline{\nabla}$, the one of S. Geometrically, (75) states that the parallel transport of Y along X can be obtained by effecting the corresponding parallel transport in S using horizontal lifts, and then projecting back to the space of density matrices.

By definition, proving that (75) gives the Levi-Civita connection reduces to showing that ∇ is compatible with the metric, and that it is torsion free.

To verify compatibility with the metric, it is enough to prove that, for vector fields X, Y, Z, the equality $g(Y + \epsilon\nabla_X Y, Z + \epsilon\nabla_X Z) = g(Y, Z) + \mathcal{O}(\epsilon^2)$ holds. For (75), this requires proving that

$$g(Y, Z) = g(\pi_*[\overline{Y} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Y}], \pi_*[\overline{Z} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Z}]) + \mathcal{O}(\epsilon^2). \tag{76}$$

Recall that g can be obtained from $\overline{g}$ by considering the horizontal components of the involved vectors. Therefore,

$$\begin{aligned}\overline{g}(\overline{Y} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Z}) &= g(\pi_*[\overline{Y} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Y}], \pi_*[\overline{Z} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Z}]) \\ &\quad + \overline{g}(\overline{Y}^V + \epsilon(\overline{\nabla}_{\overline{X}}\overline{Y})^V, \overline{Z}^V + \epsilon(\overline{\nabla}_{\overline{X}}\overline{Z})^V) \\ &= g(\pi_*[\overline{Y} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Y}], \pi_*[\overline{Z} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Z}]) + \epsilon^2\overline{g}([\overline{\nabla}_{\overline{X}}\overline{Y}]^V, [\overline{\nabla}_{\overline{X}}\overline{Z}]^V)\end{aligned} \tag{77}$$

where W^V denotes the vertical component of W, and we used the fact that $\overline{Y}^V = \overline{Z}^V = 0$ holds by definition to obtain the last equality. Finally, by writing $g(Y, Z) = \overline{g}(\overline{Y}, \overline{Z})$ and noting that $\overline{\nabla}$ is compatible with $\overline{g}$ we conclude,

$$g(Y, Z) = \overline{g}(\overline{Y}, \overline{Z}) = \overline{g}(\overline{Y} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Z}) = g(\pi_*[\overline{Y} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Y}], \pi_*[\overline{Z} + \epsilon\overline{\nabla}_{\overline{X}}\overline{Z}]) + \mathcal{O}(\epsilon^2), \tag{78}$$

where we used (77) for the last line. This proves that (75) is compatible with g.

The second requirement is that (75) be torsion free. This is equivalent to the equality $\pi_*(\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X}) = [X,\ Y]$ holding for all vector fields X, Y. As it is well-known from the theory of fiber bundles, the projection π_* is compatible with the commutator of horizontal fields, so $\pi_*[\overline{X},\ \overline{Y}] = [\pi_*\overline{X},\ \pi_*\overline{Y}]$ holds. Therefore,

$$[X,\ Y] = [\pi_*\overline{X},\ \pi_*\overline{Y}] = \pi_*[\overline{X},\ \overline{Y}] = \pi_*(\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X}), \tag{79}$$

where we used that $\overline{\nabla}$ is torsion free. This proves that (75) does indeed give the Levi-Civita connection ∇.

As an immediate application of the previous result, we can compute the intrinsic acceleration of a curve of density matrices $\rho(t)$ with tangent vectors $X(t)$,

$$a = \nabla_X X = \pi_*\overline{\nabla}_{\overline{X}}\overline{X} = \pi_*\overline{a}. \tag{80}$$

This means that, to obtain a, we can compute the corresponding acceleration in S and then project it with π_*. For the Bures metric, this procedure becomes particularly simple, since S is embedded in $\mathcal{HS}^*$, and the latter is euclidean—we work out the details in what follows.

Consider an horizontal curve $A(t)$ in S. Horizontality implies that we can write $\dot{A} = GA$, with G Hermitian. Its intrinsic acceleration is simply the orthogonal projection of $\ddot{A}$ to the sphere S, as shown in (18). This result, together with (80), implies that the intrinsic acceleration of $\rho = AA^\dagger$ is given by

$$a = \pi_*(\ddot{A} - (1/2)\mathrm{Tr}(\ddot{A}A^\dagger + A\ddot{A}^\dagger)A) = \ddot{A}A^\dagger + A\ddot{A}^\dagger - \mathrm{Tr}(\ddot{A}A^\dagger + A\ddot{A}^\dagger)\rho. \tag{81}$$

We proceed to write everything in terms of ρ and G. Since $\dot{A} = GA$,

$$\ddot{A} = \dot{G}A + G\dot{A} = \dot{G}A + G^2A, \tag{82}$$

so,

$$\begin{aligned} a &= (\dot{G}A + G^2A)A^\dagger + A(A^\dagger\dot{G} + A^\dagger G^2) - \mathrm{Tr}(\ddot{A}A^\dagger + A\ddot{A}^\dagger)\rho \\ &= (\dot{G} + G^2)\rho + \rho(\dot{G} + G^2) - \mathrm{Tr}(\ddot{A}A^\dagger + A\ddot{A}^\dagger)\rho \end{aligned} \tag{83}$$

On the other hand, by (15), $\dot{\rho} = G\rho + \rho G$, so

$$\begin{aligned} \ddot{\rho} &= \dot{G}\rho + \rho\dot{G} + G\dot{\rho} + \dot{\rho}G \\ &= \dot{G}\rho + \rho\dot{G} + G(G\rho + \rho G) + (G\rho + \rho G)G \\ &= \dot{G}\rho + \rho\dot{G} + G^2\rho + \rho G^2 + 2G\rho G \\ &= a + 2G\rho G + \mathrm{Tr}(\ddot{A}A^\dagger + A\ddot{A}^\dagger)\rho. \end{aligned} \tag{84}$$

Therefore,

$$a = \ddot{\rho} - 2G\rho G - \mathrm{Tr}(\ddot{A}A^\dagger + A\ddot{A}^\dagger)\rho. \tag{85}$$

By noting that $\mathrm{Tr}\, a = \mathrm{Tr}\, \ddot{\rho} = 0$, and taking the trace in the previous expression, we obtain the equality $\mathrm{Tr}(\ddot{A}A^\dagger + A\ddot{A}^\dagger) = -2\,\mathrm{Tr}(G\rho G)$, so,

$$a = \ddot{\rho} - 2G\rho G + 2\,\mathrm{Tr}(G\rho G)\rho. \tag{86}$$

Equation (86) is the main result of this section. For the particular case of Hamiltonian evolution,

$$\ddot{\rho} = -[H, [H, \rho]], \tag{87}$$

so we have,

$$a = -[H, [H, \rho]] - 2G\rho G + 2\,\mathrm{Tr}(G\rho G)\rho, \tag{88}$$

where G is uniquely determined by,

$$-i[H, \rho] = G\rho + \rho G, \tag{89}$$

provided all eigenvalues of ρ are positive [17]. Equation (86) is also derived in [27].

5.2.2. Working with the projection $\tilde{\pi}(A) = AA^\dagger/\mathrm{Tr}(AA^\dagger)$

When working with a curve $A(t)$ in $\mathcal{HS}^*$ with the idea of projecting it to a curve $\rho(t)$, it is often useful to consider the projection $\tilde{\pi}(A) = AA^\dagger/\mathrm{Tr}(AA^\dagger)$ instead of π, so that we are not required to work with matrices A such that $\mathrm{Tr}(AA^\dagger) = 1$. Below we provide the necessary details to work with $\tilde{\pi}$.

Consider $A(t)$ a π-horizontal curve (so, by (15), $\dot{A} = \tilde{G}A$ with $\tilde{G}$ Hermitian) and considered the projected curve $\rho = \tilde{\pi}(A)$. Note that,

$$\begin{aligned} \dot{\rho} &= \frac{d}{dt}\left(\frac{AA^\dagger}{\mathrm{Tr}(AA^\dagger)}\right) = (\tilde{G}\rho + \rho\tilde{G}) - 2\frac{\mathrm{Tr}(\tilde{G}\rho)}{\mathrm{Tr}(AA^\dagger)}AA^\dagger \\ &= (\tilde{G}\rho + \rho\tilde{G}) - 2\,\mathrm{Tr}(\tilde{G}\rho)\rho = (\tilde{G} - \mathrm{Tr}(\tilde{G}\rho)I)\rho + \rho(\tilde{G} - \mathrm{Tr}(\tilde{G}\rho)I) \end{aligned} \tag{90}$$

where I denotes the identity matrix and we used the expressions $(d/dt)(AA^\dagger) = \mathrm{Tr}(AA^\dagger)(\tilde{G}\rho + \rho\tilde{G})$ and $(d/dt)\mathrm{Tr}(AA^\dagger) = 2\,\mathrm{Tr}(AA^\dagger)\mathrm{Tr}(\tilde{G}\rho)$ for the first equality. If we define $G = \tilde{G} - \mathrm{Tr}(\tilde{G}\rho)I$, we recover (15), and we can compute the Bures metric using (20).

Note that in this case, neither G satisfies the relation $\dot{A} = GA$, nor does the size of $\dot{\rho}$ coincides with the one of $\dot{A}$ (not even after normalizing $\dot{A}$ by dividing it by $[\mathrm{Tr}(AA^\dagger)]^{1/2}$),

$$\frac{\underline{g}(\dot{A}, \dot{A})}{\mathrm{Tr}(AA^\dagger)} = \frac{\mathrm{Tr}(\dot{A}\dot{A}^\dagger)}{\mathrm{Tr}(AA^\dagger)} = \mathrm{Tr}(\tilde{G}\rho\tilde{G}) = \mathrm{Tr}(\rho\tilde{G}^2), \tag{91}$$

where we used $\dot{A} = \tilde{G}A$ to obtain the second equality. By writing $\tilde{G}$ in terms of G and using $\mathrm{Tr}(G\rho) = 0$ we conclude,

$$\frac{\bar{g}(\dot{A}, \dot{A})}{\mathrm{Tr}(AA^\dagger)} = g(\dot{\rho}, \dot{\rho}) + [\mathrm{Tr}(\rho\tilde{G})]^2 \geqslant g(\dot{\rho}, \dot{\rho}). \tag{92}$$

The reason for the difference between the sizes of $\dot{\rho}$ and that of $\dot{A}/[\mathrm{Tr}(AA^\dagger)]^{1/2}$ is that the latter has a vertical component with respect to $\tilde{\pi}$. Recall that we assumed that $A(t)$ was an horizontal vector for π, not for $\tilde{\pi}$. For the projection $\tilde{\pi}$, the horizontal vectors V, besides having to satisfy the condition $V = \tilde{G}A$ (with $\tilde{G}$ Hermitian), they also have to satisfy the condition $\mathrm{Tr}(VA^\dagger) = \mathrm{Tr}(\tilde{G}AA^\dagger) = 0$. The reason for this is that curves of the form $\gamma(t) = tA$ are horizontal for π, but vertical for $\tilde{\pi}$—indeed they get projected by $\tilde{\pi}$ to a single point. If we assume that V is orthogonal to this kind of curves, we conclude that $\tilde{\pi}$-horizontal vectors V satisfy additionally the condition $\mathrm{Tr}(VA^\dagger) = 0$.

Working with $\tilde{\pi}$-horizontal curves, by the definition of G, we immediately have $\tilde{G} = G$ and the equality in (92) holds, as expected.

Finally, note that $\tilde{\pi}$-horizontal curves live in the sphere where the term $\mathrm{Tr}(AA^\dagger)$ is constant,

$$\frac{d}{dt}\mathrm{Tr}(AA^\dagger) = \mathrm{Tr}(\dot{A}A^\dagger + A\dot{A}^\dagger) = 2\,\mathrm{Tr}(\tilde{G}AA^\dagger) = 0. \tag{93}$$

In particular, if $\mathrm{Tr}(AA^\dagger) = 1$ at the initial time, we are automatically in the usual case of the Bures metric.

So, in conclusion, working with $\tilde{\pi}$-horizontal curves, we can compute the Bures metric like we did in the previous section, since $\tilde{G} = G$ in this case. Otherwise, we have to compute it using the operator G defined as $G = \tilde{G} - \mathrm{Tr}(\tilde{G}\rho)I$, where we remind the reader that $\tilde{G}$ is given by the equality $\dot{A} = \tilde{G}A$.

5.2.3. Explicit expression for the Christoffel symbols of the Bures metric

As we show below, equation (86) allows to find an expression for the Christoffel symbols that does not involve the derivatives of the operator G. Since computing G is generally a numerically demanding task, it is useful to avoid computing its derivatives, as it would be done in the usual approach to compute the Christoffel symbols.

Suppose we parametrize the space of density matrices using coordinates μ. The expression for the acceleration of a curve is,

$$a = \ddot{\mu}\rho_\mu + \Gamma^\alpha{}_{\mu\nu}\dot{\mu}\dot{\nu}\rho_\alpha, \tag{94}$$

where we defined $\rho_\mu = \partial_\mu\rho$. On the other hand,

$$\ddot{\rho} = \frac{d}{dt}(\dot{\mu}\rho_\mu) = \ddot{\mu}\rho_\mu + \dot{\mu}\dot{\nu}\rho_{\mu\nu}. \tag{95}$$

where $\rho_{\mu\nu} = \partial_{\mu\nu}\rho$. By comparing (94) with (86), we conclude that

$$\dot{\mu}\dot{\nu}\rho_{\mu\nu} - 2G\rho G + 2\,\mathrm{Tr}(G\rho G)\rho = \Gamma^\alpha{}_{\mu\nu}\dot{\mu}\dot{\nu}\rho_\alpha. \tag{96}$$

Denoting by G_μ the operator defined analogously to G in (15),

$$\rho_\mu = G_\mu\rho + \rho G_\mu, \tag{97}$$

we find

$$\begin{aligned} 2G\rho G &= 2\dot{\mu}\dot{\nu}G_\mu\rho G_\nu = \dot{\mu}\dot{\nu}(G_\mu\rho G_\nu + G_\nu\rho G_\mu), \\ 2\,\mathrm{Tr}(G\rho G) &= \dot{\mu}\dot{\nu}\,\mathrm{Tr}(G_\mu\rho G_\nu + G_\nu\rho G_\mu), \end{aligned} \tag{98}$$

so that (96) becomes,

$$\dot{\mu}\dot{\nu}(\rho_{\mu\nu} - G_\mu\rho G_\nu - G_\nu\rho G_\mu + \mathrm{Tr}[G_\mu\rho G_\nu + G_\nu\rho G_\mu]\rho) = \dot{\mu}\dot{\nu}\Gamma^\alpha{}_{\mu\nu}\rho_\alpha. \tag{99}$$

Since this expression holds for arbitrary $\dot{\mu}$, $\dot{\nu}$, we get

$$\rho_{\mu\nu} - G_\mu\rho G_\nu - G_\nu\rho G_\mu + \mathrm{Tr}(G_\mu\rho G_\nu + G_\nu\rho G_\mu)\rho = \Gamma^\alpha{}_{\mu\nu}\rho_\alpha. \tag{100}$$

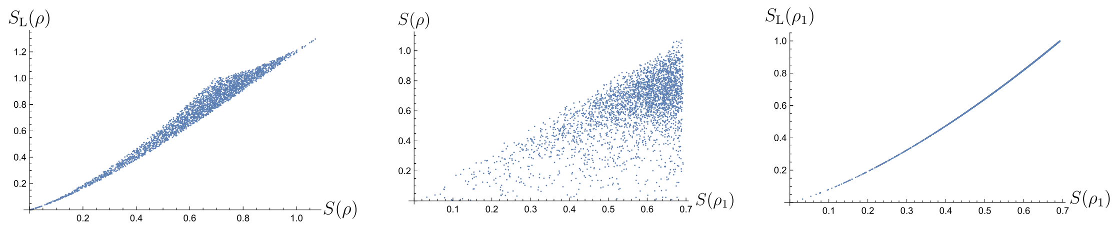
To solve for Γ, we compute the inner product of both sides of the equation with ρ_β,

$$g(\rho_{\mu\nu} - G_\mu\rho G_\nu - G_\nu\rho G_\mu + \mathrm{Tr}[G_\mu\rho G_\nu + G_\nu\rho G_\mu]\rho, \rho_\beta) = \Gamma^\alpha{}_{\mu\nu}g(\rho_\alpha, \rho_\beta) = \Gamma_{\beta\mu\nu}. \tag{101}$$

We can compute the l.h.s. using a direct approach—by writing g in terms of traces with the help of (19),

$$g(\rho_{\mu\nu} - G_\mu\rho G_\nu - G_\nu\rho G_\mu + \mathrm{Tr}[G_\mu\rho G_\nu + G_\nu\rho G_\mu]\rho, \rho_\beta) = \frac{1}{2}\mathrm{Tr}(G_\beta(\rho_{\mu\nu} - G_\mu\rho G_\nu - G_\nu\rho G_\mu)), \tag{102}$$

where we used that $\mathrm{Tr}(G_\beta\rho) = \mathrm{Tr}(\dot{\rho})/2 = 0$ to get rid of the last term. By comparing these equations we conclude finally



(a) $S_{\mathrm{L}}(\rho)$ *vs* $S(\rho)$.

(b) $S(\rho)$ *vs* $S(\rho_1)$.

(c) $S_{\mathrm{L}}(\rho_1)$ *vs* $S(\rho_1)$.

Figure 3. Comparison between purity measures. Plots (a) and (c) show the linear entropy versus von Neumann entropy of the full state, and of the reduced density matrix, respectively. Notice that in the latter case, the relation is functional since both are measures of purity, in the case of (a) only for extreme values of purity the relation tends to a bijective one. The von Neumann entropy of the full system and of the reduced state are compared in (b), where no functional relation is found, nonetheless, the numerical exploration suggests that there exists an upper bound: the purity of ρ is constrained by the purity of the reduced states, $S(\rho_1) \geqslant \lambda S(\rho)$.

$$\Gamma_{\beta\mu\nu} = \frac{1}{2}\mathrm{Tr}(G_\beta[\rho_{\mu\nu} - G_\mu \rho G_\nu - G_\nu \rho G_\mu]). \tag{103}$$

The correctness of the previous result was verified numerically for a spin 1 system at several random points.

6. Kinematical quantities, purity and quantum correlations

In this section we explore numerically the relation between averaged velocities and accelerations (both of the whole system and of the corresponding subsystems), different measures of quantum correlations (concurrence, negativity and geometric quantum discord) and the purity of the states (von Neumann entropy and linear entropy, also known as 1-anticoherence measure). We restrict our analysis to the case of bipartite systems of qubits in a symmetric state evolving through rotations. Even though far from comprehensive, this numerical approach reveals interesting relations that we plan to explore further in future work.

Let us recall some definitions of quantities explored in this section. Concurrence and negativity are two well-known measures of entanglement, while von Neumann entropy and linear entropy quantify how mixed a quantum state is. Geometric quantum discord captures information about the state when a measurement is performed in one of the subsystems that compose the entire system. Consider an arbitrary two-qubit state,

$$\rho = \frac{1}{4}\left(I_2 \otimes I_2 + x \cdot \sigma \otimes I_2 + I_2 \otimes y \cdot \sigma + \sum_{i,j=1}^{3} T_{ij}\sigma_i \otimes \sigma_j\right), \tag{104}$$

where $x_i = \mathrm{tr}[\rho(\sigma_i \otimes I_2)]$, $y_i = \mathrm{tr}[\rho(I_2 \otimes \sigma_i)]$ are the Bloch vectors of the reduced states and $T_{ij} = \mathrm{tr}(\rho\sigma_i \otimes \sigma_j)$ are the entries of its correlation matrix. We start with the entropies: the von Neumann entropy

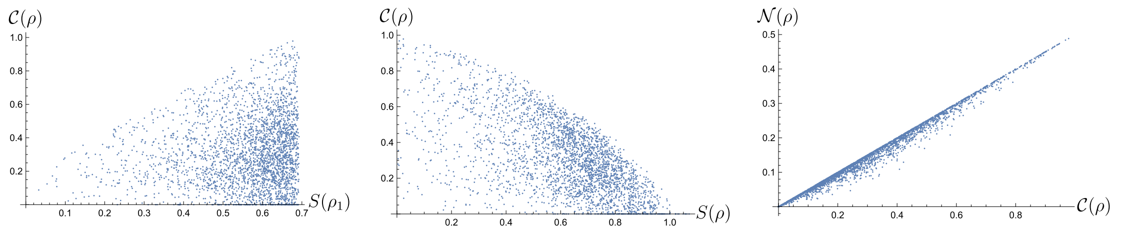
$$S(\rho) = -\sum_{i=1}^{3} \lambda_i \log \lambda_i, \tag{105}$$

captures how mixed a quantum state is, where λ_i are the eigenvalues of ρ. Similarly, we can calculate the linear entropy

$$S_{\mathrm{L}}(\rho) = 2(1 - \mathrm{Tr}(\rho^2)). \tag{106}$$

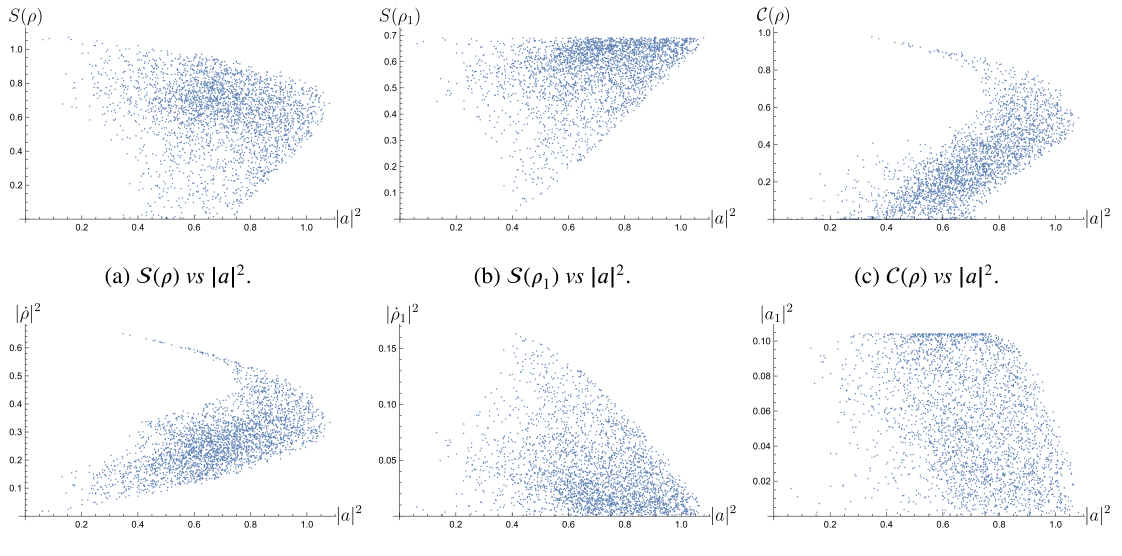
Linear entropy measures the *mixedness* (or equivalently, the purity) of the quantum system [28]. Similarly, we can calculate both entropies for the reduced density states, $S(\rho_1)$, and $S_L(\rho_1)$. The latter is also called a measure of 1-anticoherence based on the purity [29]. For pure states, both quantities can be written in terms of the Schmidt numbers, which at the same time are associated to the entropy of entanglement and the linear entropy of entanglement (see appendix A). We plot in figure 3 the points $(S(\rho),\ S_{\mathrm{L}}(\rho))$, $(S(\rho_1),\ S(\rho))$, and $(S(\rho_1),\ S_{\mathrm{L}}(\rho_1))$ for 3000 randomly generated mixed states. The first two plots result in 2-dimensional 'clouds', which makes sense because both quantities involved depend on the eigenvalues of the mixed states (the eigenspectrum of the spin-1 mixed states can be parametrized by two variables λ_1, λ_2, with $\lambda_3 = 1 - \lambda_1 - \lambda_2$). On the other hand, the reduced states, which are two dimensional matrices, have two eigenvalues and therefore their eigenspectrum is parametrized by only one variable. Thus, the plot $(S_L(\rho_1),\ S(\rho_1))$ (in the last frame in figure 3) produces a curve. We also point out, in the second figure, that for pure states, with $S(\rho) = 0$, $S(\rho_1)$ can take all the admissible values —this is so because, in this case, $S(\rho_1)$ is a measure of the entanglement of the pure state.

We introduce now some measures of entanglement for mixed states. We start with the concurrence defined as [30]



(a) $\mathcal{C}(\rho)$ *vs* $S(\rho_1)$. (b) $\mathcal{C}(\rho)$ *vs* $S(\rho)$. (c) $\mathcal{N}(\rho)$ *vs* $\mathcal{C}(\rho)$.

Figure 4. Comparison between purity measures and entanglement measures. Plots (a) and (b) show the concurrence versus von Neumann entropy of the reduced density matrix, and the full density matrix, respectively. Plot (a) seems to suggest a relation of the form $S(\rho_1) \geqslant \lambda\mathcal{C}(\rho) + \mu$, valid for $S(\rho_1) \geqslant .2$. In (c) two measures of entanglement, negativity and concurrence, are compared, showing an approximate proportionality relation, valid for most of the states.



(a) $S(\rho)$ *vs* $|a|^2$. (b) $S(\rho_1)$ *vs* $|a|^2$. (c) $C(\rho)$ *vs* $|a|^2$.

(d) $|\dot{\rho}|^2$ *vs* $|a|^2$. (e) $|\dot{\rho}_1|^2$ *vs* $|a|^2$. (f) $|a_1|^2$ *vs* $|a|^2$.

Figure 5. Comparison between the squared norm of the acceleration $|a|^2$ and other quantities. In subfigures (a) and (c), we can identify two boundaries enclosing the region. In (c) we can also note the definition of potential borders. Note in figure (b) that for $|a|^2 > 0.4$, the measure of purity should exceed certain values defined by the boundary of the region.

$$\mathcal{C}(\rho) = \max(0,\ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{107}$$

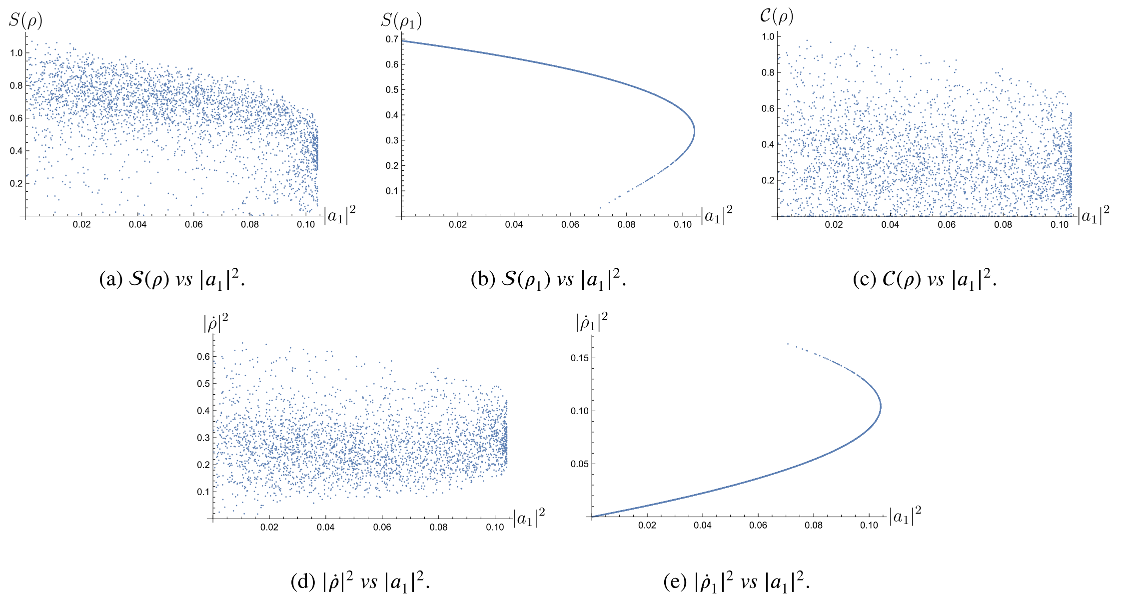
where λ_i are the eigenvalues, in decreasing order, of the matrix $M = \sqrt{\sqrt{\rho}\,\mu(\rho)\sqrt{\rho}}$, being $\mu(\rho) := (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. In the case of pure states, it reduces to the simpler expression $\mathcal{C}(\rho) = \sqrt{2(1 - |x|^2)}$. Negativity, denoted by $\mathcal{N}(\rho)$, is defined in terms of the trace norm of the partial transpose of the density matrix [31], namely,

$$\mathcal{N}(\rho) = \frac{|\rho^{T_A}|_1 - 1}{2} \tag{108}$$

where $|X|_1 := \mathrm{tr}\sqrt{X^\dagger X}$ and ρ^{T_A} denotes the partial transpose of the density matrix ρ with respect to the subsystem A. We plot in figure 4 the points $(S(\rho_1),\ \mathcal{C}(\rho))$, $(S(\rho),\ \mathcal{C}(\rho))$, and $(\mathcal{C}(\rho),\ \mathcal{N}(\rho))$, for the above set of states. While some inequalities can be inferred, there is no functional relation observed.

Geometric quantum discord [32] is a measure of quantum correlations related to quantum discord, but easier to compute, both capturing nonclassical properties not necessarily encoded in quantum entanglement. The geometric discord of a state ρ of the above form (104) is given by [32]

$$D_G(\rho) = \frac{1}{4}(||yy^T||_2 + ||T||_2^2 - k), \tag{109}$$



(a) $\mathcal{S}(\rho)$ *vs* $|a_1|^2$.

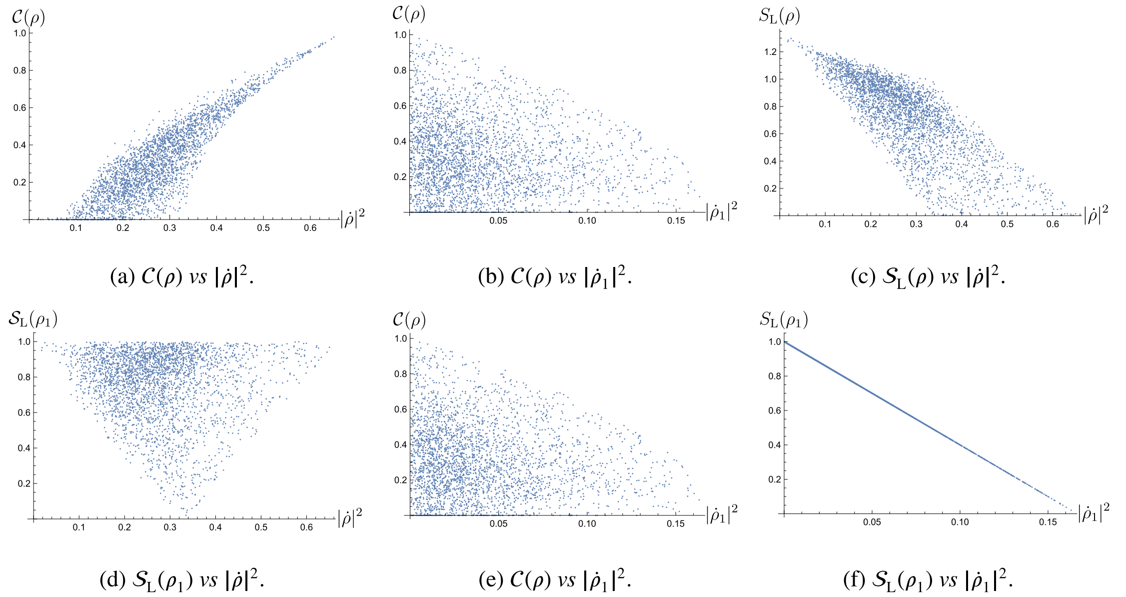
(b) $\mathcal{S}(\rho_1)$ *vs* $|a_1|^2$.

(c) $\mathcal{C}(\rho)$ *vs* $|a_1|^2$.

(d) $|\dot{\rho}|^2$ *vs* $|a_1|^2$.

(e) $|\dot{\rho}_1|^2$ *vs* $|a_1|^2$.

Figure 6. Comparison between the squared norm of the acceleration $|a_1|^2$ (corresponding to the reduced density matrices) and other quantities. In subfigures (b), (e) we note a clear functional relationship between the quantities involved.



(a) $\mathcal{C}(\rho)$ *vs* $|\dot{\rho}|^2$.

(b) $\mathcal{C}(\rho)$ *vs* $|\dot{\rho}_1|^2$.

(c) $\mathcal{S}_L(\rho)$ *vs* $|\dot{\rho}|^2$.

(d) $\mathcal{S}_L(\rho_1)$ *vs* $|\dot{\rho}|^2$.

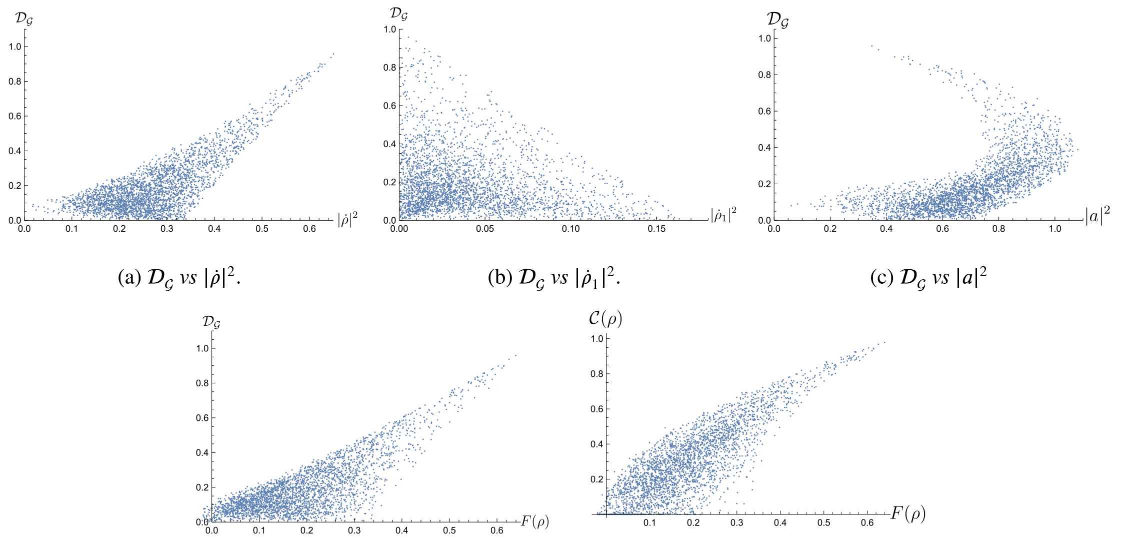
(e) $\mathcal{C}(\rho)$ *vs* $|\dot{\rho}_1|^2$.

(f) $\mathcal{S}_L(\rho_1)$ *vs* $|\dot{\rho}_1|^2$.

Figure 7. Comparison between the squared norm of the velocities $|\dot{\rho}|^2$, $|\dot{\rho}_1|^2$ and other quantities. Note in subfigure (a) that, apparently, for each degree of entanglement, there exists an allowable region of $|\dot{\rho}|^2$, and for high degrees of entanglement ($\mathcal{C} > 0.6$), this region becomes narrower. We observe that maximally entangled states also exhibit the highest value of $|\dot{\rho}|^2$. In subfigures (b)-(e) we can note that it seems that there are definite regions for the possible simultaneous values of both variables. In subfigure (f), we identify a linear relationship between the purity and the speed of the reduced states.

where k is the largest eigenvalue of the matrix $yy^T + T^T T$. We will work with $\mathcal{D}_G := 2D_G$ in our analysis, instead of the geometric quantum discord D_G, to have a quantity normalized to one. We study now the relation between these quantities and the kinematical ones introduced in the previous sections. Due to the complexity of the analytical expressions for the squared norm of the acceleration and velocity when using the Bures metric, we opted for a numerical analysis using Mathematica. We started by randomly generating pure spin-1 states, *i.e.*, Hermitian matrices ρ of dimension 3, with trace equal to 1, and satisfying the condition $\rho^2 = \rho$. Each of these states corresponds to the symmetric part of a bipartite state composed of two qubits. Subsequently, we created 3000 mixtures of these pure states with random weights, *i.e.*, 3000 random spin-1 mixed symmetric states.

In figures 5 through 8, where, again, each dot represents a randomly generated state, we compare the above presented quantities with average (squared) speed and acceleration, at both the full system and the subsystem level. In



(a) $\mathcal{D}_{\mathcal{G}}$ *vs* $|\dot{\rho}|^2$. (b) $\mathcal{D}_{\mathcal{G}}$ *vs* $|\dot{\rho}_1|^2$. (c) $\mathcal{D}_{\mathcal{G}}$ *vs* $|a|^2$

(d) $\mathcal{D}_{\mathcal{G}}$ *vs* $F(\rho)$ (e) $\mathcal{C}(\rho)$ *vs* $F(\rho)$.

Figure 8. Comparison between kinematical quantities and geometric quantum discord $\mathcal{D}_{\mathcal{G}}$ is shown in subfigures (a)-(d). Subfigure (e) compares the speed excess with the concurrence—note that there is no functional relationship between them unlike in the case of pure states. This implies that, for mixed states, speed excess is not an entanglement measure. Note also that in (d) and (e) a very small percentage of states has negative *F*.

plots figures 6(b), (e), and 7(f) below, the numerical exploration suggests a direct functional relation between the two compared variables. In contrast, in the rest of the plots, a 'cloud' of points appears, rather than a well-defined curve—in those cases, it would be of interest to try and identify the boundary curves delimiting the 'clouds'.

7. Concluding remarks

We summarize the main points of this work: (i) The total variance of a pure spin state is a measure of its squared rotational speed, averaged over all rotation axes. The concept is generalized for mixed states using the Bures metric. (ii) Entanglement increases rotational speed, hence the relevance of total variance as an entanglement measure. The addition law for total variance is pythagorean for pure separable states, $|v|^2 = |v_1|^2 + |v_2|^2$, and receives additional positive contributions for pure entangled states. Speed excess, defined as $|v|^2 - |v_1|^2 - |v_2|^2$, may thus be used to quantify entanglement. (iii) Total (average, squared, rotational) acceleration may be similarly defined for both pure and mixed states. For separable states we find $|a|^2 = |a_1|^2 + |a_2|^2 + 4|v_1|^2|v_2|^2$, which, incidentally, means that the quantity $|a|^2 - 2|v|^4$ is additive under system composition. We found a simple analytical formula for the acceleration of a mixed state according to the Bures metric (equation (86)). The numerical results of figures 5 and 6, exploring the correlation of total acceleration with other physical characteristics of the state, display the full gamut of possibilities. As shown in (d), (e) of figure 8, speed excess can be negative for a very small percentage of state space volume.

Some directions for further work along similar lines include: (i) The additivity of $|a|^2 - 2|v|^4$ under separable system composition suggests exploring the physical significance of this quantity. (ii) The correlation of total acceleration with other relevant physical quantities, which we started in section 6, should also be pursued analytically. It has been established here that all 2-anticoherent states, of a certain spin *s*, have the same value of total acceleration. Our conjecture at this point is that there exists a similar connection between *t*-anticoherent states and *t*-th order covariant derivatives of the curve in state space describing rotations. (iii) Higher-order Bures metric covariant derivative formulas like (86) would be desirable. The physical significance of the average of the modulus squared of these quantities should be explored. Do the states that extremize these quantities have any desirable physical properties? (iv) What are the physical characteristics of the states that have negative speed excess?

Acknowledgments

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Total variance and entanglement for spin states

Here, we prove that the total variance is a function of the linear entropy of entanglement for a spin-s system. A spin-s state is the fully symmetric state of $2s$ spin-$1/2$ (qubit) constituents. Consider the bipartition $1|2s-1$ with the kth-qubit $A=\{k\}$ and its complement $B=A^c$. The Schmidt decomposition of the considered bipartition guarantees that a state $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{\alpha=1}^{2} \sqrt{\Gamma_{k,\alpha}}\,|\phi_{k,\alpha}^{A}\rangle\,|\phi_{k,\alpha}^{B}\rangle, \tag{A1}$$

where $\{\phi_{k,\alpha}^{A}\}$ and $\{\phi_{k,\alpha}^{B}\}$ are orthogonal sets in their respective Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$, and the Schmidt numbers $\sqrt{\Gamma_{k,\alpha}}$ are normalized to one, $\Gamma_{k,1}+\Gamma_{k,2}=1$. For a generic state, the Schmidt decomposition depends on the chosen kth-qubit. However, since the state is fully symmetric, $\Gamma_{k,\alpha}=\Gamma_{k',\alpha}\equiv\Gamma_\alpha$ for all $k,\ k'$, so that the subindex k can be omitted. The entropy S_E and linear entropy S_L of entanglement of a state, with respect to any partition $1|2s-1$, is defined via the Schmidt numbers as

$$S_E(|\psi\rangle) = -\sum_{\alpha=1}^{2}\Gamma_\alpha \ln \Gamma_\alpha, \quad S_L(|\psi\rangle) = 2(1-\mathrm{Tr}(\rho_1^2)) = 2\left(1-\sum_{\alpha=1}^{2}\Gamma_\alpha^2\right). \tag{A2}$$

On the other hand, it is known that the angular momentum operators S_a, with $a=1,2,3$, can be written as

$$S_a = \frac{1}{2}(\sigma_{a0\ldots0}+\sigma_{0a0\ldots0}+\ldots+\sigma_{0\ldots0a}), \quad \text{with} \quad \sigma_{\mu_1\ldots\mu_{2s}} \equiv \sigma_{\mu_1}\otimes\sigma_{\mu_2}\otimes\ldots\otimes\sigma_{\mu_{2s}}, \tag{A3}$$

and σ_μ are the Pauli matrices, with $\mu=0,1,2,3$ and $\sigma_0=1_2$ is the identity matrix. In particular, for a spin-s state, we obtain that

$$\langle\psi|\sigma_{a0\ldots0}|\psi\rangle = \langle\psi|\sigma_{0\ldots0a0\ldots0}|\psi\rangle = \sum_{\alpha=1}^{2}\Gamma_\alpha\langle\phi_\alpha^A|\sigma_a|\phi_\alpha^A\rangle, \tag{A4}$$

so that

$$\sum_{a=1}^{3}\langle\psi|S_a|\psi\rangle^2 = s^2\sum_{a=1}^{3}\left(\sum_{\alpha=1}^{2}\Gamma_\alpha\langle\phi_\alpha^A|\sigma_a|\phi_\alpha^A\rangle\right)^2. \tag{A5}$$

For a spin-s state the total variance can be written in terms of just one of the Schmidt coefficients,

$$\begin{aligned}\mathbb{D}(\rho) &= s(s+1) - \sum_{a=1}^{3}\langle\psi|S_a|\psi\rangle^2 \\ &= s(s+1) - s^2(\Gamma_1-\Gamma_2)^2\sum_{a=1}^{3}\langle\phi_1^A|\sigma_a|\phi_1^A\rangle^2 \\ &= s(s+1) - s^2(1-2\Gamma_1)^2,\end{aligned}$$

where we used the identity

$$\begin{aligned}\sum_{\alpha=1}^{2}\Gamma_\alpha\langle\phi_\alpha^A|\sigma_a|\phi_\alpha^A\rangle &= \mathrm{Tr}\left[\left(\sum_{\alpha=1}^{2}\Gamma_\alpha|\phi_\alpha^A\rangle\langle\phi_\alpha^A|\right)\sigma_a\right] \\ &= \mathrm{Tr}\left[((\Gamma_1-\Gamma_2)|\phi_1^A\rangle\langle\phi_1^A| + \Gamma_2\sigma_0)\sigma_a\right] \\ &= (\Gamma_1-\Gamma_2)\langle\phi_1^A|\sigma_a|\phi_1^A\rangle.\end{aligned}$$

Therefore, the total variance of a pure bipartite symmetric state is equal to

$$\mathbb{D}(\rho) = s(s+1) - s^2(1-2\Gamma_1)^2. \tag{A6}$$

We note that S_L, S_E and $\mathbb{D}$ only depend on Γ_1—all three quantities monotonically increase as Γ_1 ranges from 0 to $1/2$. Thus, the properties of entanglement of S_L and S_E are passed on to $\mathbb{D}$ for pure states. In particular, $\mathbb{D}$ is a measure of entanglement for pure states [7].

[7] We use the notion of bipartite measure of entanglement for pure states [1], only demanding that the measure be zero for separable states, and not increasing under LOCC transformations.

Appendix B. Majorana representation of spin states

Ettore Majorana showed in 1932 [33] that there is a bijective correspondence between pure spin-s states and an unordered set of $2s$ points $\{\boldsymbol{n}_k\}_{k=1}^{2s}$ (*a constellation*) on the sphere. In particular, the state corresponding to a constellation $\{\boldsymbol{n}_k\}_{k=1}^{2s}$ is given by the fully symmetric state formed by the tensor product of $2s$ spin-$1/2$ states pointing in the $\boldsymbol{n}_k$ directions

$$|\psi\rangle = N \sum_{\sigma \in S_{2s}} \sigma(|\boldsymbol{n}_1\rangle \otimes \ldots \otimes |\boldsymbol{n}_{2s}\rangle), \tag{B1}$$

where N is a normalization factor, and σ ranges over the permutation group S_{2s}. A spin-$1/2$ state pointing in the direction $\boldsymbol{n}$ with spherical angles (θ, ϕ) can be parametrized as $|\boldsymbol{n}\rangle = \cos(\theta/2)|1/2,\ 1/2\rangle + e^{i\phi}\sin(\theta/2)|1/2,\ -1/2\rangle$. We use this representation in the main text to make intuitive parametrizations for spin equal to 1 and $3/2$. See [34] for more details regarding Majorana's stellar representation.

Appendix C. Calculating the average of the norm squared of the acceleration

The average norm of the acceleration is given by

$$\begin{aligned}\langle |\dot{v}|^2\rangle = (&\langle T_A T_B T_C T_D\rangle - 4\langle T_A T_B T_C\rangle\langle T_D\rangle - \langle T_A T_B\rangle\langle T_C T_D\rangle \\ &+ 8\langle T_A T_B\rangle\langle T_C\rangle\langle T_D\rangle - 4\langle T_A\rangle\langle T_B\rangle\langle T_C\rangle\langle T_D\rangle)\int_{S^2} H^A H^B H^C H^D d\Omega,\end{aligned} \tag{C1}$$

where $H = H^A T_A$. The previous equation can be written in terms of the cartesian angular momentum operators S_α, or of the tensor operators $\{T_{1m}\}_{m=-1}^{1}$,

$$H = \sum_{\alpha=x,y,z} n_\alpha S_\alpha = \sum_{m=-1}^{1} r_m T_{1m}^{(s)}, \tag{C2}$$

with

$$n_x = \frac{A}{\sqrt{2}}(r_{-1} - r_1) \quad n_y = -i\frac{A}{\sqrt{2}}(r_{-1} + r_1), \quad n_z = A r_0, \tag{C3}$$

$$r_1 = -\frac{1}{\sqrt{2}A}(n_x - i n_y), \quad r_{-1} = \frac{1}{\sqrt{2}A}(n_x + i n_y), \quad r_0 = \frac{n_z}{A}, \tag{C4}$$

where $A = A(s) = \sqrt{\frac{3}{s(s+1)(2s+1)}}$. The integrals of the product of components r_m are equal to

$$\int_{S^2} r_0^4 d\Omega = \frac{1}{5A^4}, \quad \int_{S^2} \mathrm{r}_0^2 \mathrm{r}_1 \mathrm{r}_{-1} d\Omega = -\frac{1}{15A^4}, \quad \int_{S^2} \mathrm{r}_1^2 \mathrm{r}_{-1}^2 = \frac{2}{15A^4} d\Omega, \tag{C5}$$

and the rest are equal to zero. As a concrete application of the above scheme, consider the term $\int_{S^2}\langle H^4\rangle d\Omega$, expressed in terms of Wigner-D matrices, $D_{mm'}^{(s)} = D_{mm'}^{(s)}(\alpha,\ \beta,\ \gamma)$

$$\int_{S^2}\langle H^4\rangle d\Omega = \mathrm{r}_0^4 \int_{S^2}\langle (\mathrm{D}^{(s)} \mathrm{T}_{10} \mathrm{D}^{(s)\dagger})^4\rangle d\Omega = \mathrm{A}^{-4}\int_{S^2}\langle \mathrm{D}^{(s)} \mathrm{T}_{10}^4 \mathrm{D}^{(s)\dagger}\rangle d\Omega \tag{C6}$$

taking n_z=1. Now, we make use of the fact that the polarization operators $\{T_{lm}\}$ form a basis. In particular, it holds that

$$T_{l_1 m_1} T_{l_2 m_2} = (-1)^{2s+l}\sqrt{(2l_1+1)(2l_2+1)}\begin{Bmatrix} l_1 & l_2 & l \\ s & s & s \end{Bmatrix} c_{l_1 m_1, l_2 m_2}^{lm} T_{lm} \equiv \chi(l_1,\ l_2,\ l;\ s) c_{l_1 m_1, l_2 m_2}^{lm} T_{lm}, \tag{C7}$$

where a summation (in this case, over l and m) is implied, and the curly bracket denotes a Wigner 6-j symbol [26]. Hence, we can calculate the expansion of T_{10}^4 in the T-basis,

$$T_{10}^4 = \chi(1,\ 1,\ l;\ s)\chi(1,\ 1,\ l';\ s) c_{10,10}^{lm} c_{10,10}^{l'm'} T_{lm} T_{l'm'} = \chi(1,\ 1,\ l;\ s)\chi(1,\ 1,\ l';\ s)\chi(l,\ l',\ L;\ s) c_{10,10}^{l0} c_{10,10}^{l'0} c_{l0,l'0}^{LM} T_{LM}. \tag{C8}$$

Using

$$\int_{S^2} D^{(s)} T_{LM} D^{(s)\dagger} d\Omega = \int_{S^2} \mathrm{D}_{M'M}^{(L)}(\phi,\ \theta,\ 0) \mathrm{T}_{LM'} d\Omega = \delta_{M0}\delta_{L0} \mathrm{T}_{00}, \tag{C9}$$

we get

$$\int_{S^2}\langle H^4\rangle \mathrm{d}\Omega = A^4\chi(1,1,l;s)\chi(1,1,l';s)\chi(l,l',0;s)c^{l0}_{10,10}c^{l'0}_{10,10}c^{00}_{l0,l'0}\langle T_{00}\rangle$$
$$= \sum_{L=0}^{2}\frac{A^{-4}}{\sqrt{2s+1}}\chi(1,1,L;s)^2\chi(L,L,0;s)c^{L0}_{10,10}c^{L0}_{10,10}c^{00}_{L0,L0}. \quad \text{(C10)}$$

In the same way, we calculate the next two terms

$$\langle D^{(s)}T_{10}T_{10}T_{10}D^{(s)\dagger}\rangle\langle D^{(s)}T_{10}D^{(s)\dagger}\rangle = \chi(1,1,l;s)c^{l0}_{10,10}D^{(1)}_{M0}\langle D^{(s)}T_{l0}T_{10}D^{(s)\dagger}\rangle\langle T_{1M}\rangle$$
$$= \chi(1,1,l;s)\chi(l,1,L;s)c^{l0}_{10,10}c^{L0}_{l0,10}D^{(1)}_{M0}\langle D^{(s)}T_{L0}D^{(s)\dagger}\rangle\langle T_{1M}\rangle$$
$$= \chi(1,1,l;s)\chi(l,1,L;s)c^{l0}_{10,10}c^{L0}_{l0,10}D^{(1)}_{M0}D^{(L)}_{M'0}\langle T_{LM'}\rangle\langle T_{1M}\rangle. \quad \text{(C11)}$$

We now use the integral over two Wigner-D matrices

$$D^{(L)}_{MN} = (-1)^{N-M}D^{(L)\dagger}_{-M-N}, \quad \text{(C12)}$$

$$\int_{S^2} D^{(L)}_{MN}(\phi,\theta,0)D^{(L')}_{M'N'}(\phi,\theta,0)\mathrm{d}\Omega = (-1)^{N'-M'}\int_{S^2}\mathrm{D}^{(L)}_{MN}\mathrm{D}^{(L')\dagger}_{-M'-N'}\mathrm{d}\Omega = \frac{(-1)^{M-N}}{2\mathrm{L}+1}\delta_{LL'}\delta_{M-M'}\delta_{N-N'}. \quad \text{(C13)}$$

Then, we obtain that

$$\int_{S^2}\langle D^{(s)}T_{10}T_{10}T_{10}D^{(s)\dagger}\rangle\langle D^{(s)}T_{10}D^{(s)\dagger}\rangle \mathrm{d}\Omega$$
$$= \frac{1}{2L+1}\chi(1,1,l;s)\chi(l,1,L;s)c^{l0}_{10,10}c^{L0}_{l0,10}(-1)^M\delta_{L1}\delta_{M-M'}\langle T_{LM'}\rangle\langle T_{1M}\rangle$$
$$= \frac{1}{3}\chi(1,1,l;s)\chi(l,1,1;s)c^{l0}_{10,10}c^{10}_{l0,10}(-1)^M\langle T_{1-M}\rangle\langle T_{1M}\rangle$$
$$= \frac{1}{3}\chi(1,1,l;s)\chi(l,1,1;s)c^{l0}_{10,10}c^{10}_{l0,10}\langle T^{\dagger}_{1M}\rangle\langle T_{1M}\rangle. \quad \text{(C14)}$$

Therefore

$$\int_{S^2}\langle H^3\rangle\langle H\rangle \mathrm{d}\Omega = \frac{1}{3A^4}\chi(1,1,\mathrm{L};s)\chi(\mathrm{L},1,1;s)c^{L0}_{10,10}c^{10}_{L0,10}\langle \mathrm{T}^{\dagger}_{1M}\rangle\langle \mathrm{T}_{1M}\rangle. \quad \text{(C15)}$$

Similarly, we calculate

$$\int_{S^2}\langle H^2\rangle\langle H^2\rangle \mathrm{d}\Omega = \frac{[\chi(1,1,\mathrm{L};s)c^{L0}_{10,10}]^2}{\mathrm{A}^4(2\mathrm{L}+1)}\langle \mathrm{T}^{\dagger}_{LM}\rangle\langle \mathrm{T}_{LM}\rangle \quad \text{(C16)}$$

For the next term, we use the following formula (equation (4), p.96 of [26])

$$\int_{S^2} D^{(J_3)}_{M_3M_3'}D^{(J_2)}_{M_2M_2'}D^{(J_1)}_{M_1M_1'}\mathrm{d}\Omega = \frac{(-1)^{M_3-M_3'}}{2\mathrm{J}_3+1}\mathrm{c}^{J_3-M_3}_{J_1M_1J_2M_2}\mathrm{c}^{J_3-M_3'}_{J_1M_1'J_2M_2'}, \quad \text{(C17)}$$

$$\int_{S^2}\langle H^2\rangle\langle H\rangle^2 \mathrm{d}\Omega = A^{-4}\int_{S^2}\langle DT_{10}T_{10}D^{\dagger}\rangle\langle DT_{10}D^{\dagger}\rangle\langle DT_{10}D^{\dagger}\rangle \mathrm{d}\Omega$$
$$= A^{-4}c^{L0}_{1010}\chi(1,1,L;s)\langle T_{LM}\rangle\langle T_{1N_1}\rangle\langle T_{1N_2}\rangle\int_{S^2} D^{(L)}_{M0}D^{(1)}_{N_10}D^{(1)}_{N_20}\mathrm{d}\Omega$$
$$= A^{-4}\langle T_{LM}\rangle\langle T_{1N_1}\rangle\langle T_{1N_2}\rangle\frac{(-1)^M}{2L+1}c^{L-M}_{1N_21N_1}c^{L0}_{1010}c^{L0}_{1010}\chi(1,1,L;s)$$
$$= \frac{c^{LM}_{1N_21N_1}(c^{L0}_{1010})^2}{A^4(2L+1)}\chi(1,1,L;s)\langle T^{\dagger}_{LM}\rangle\langle T_{1N_1}\rangle\langle T_{1N_2}\rangle \quad \text{(C18)}$$

Finally, we calculate the last term with the integrals of equation (C5)

$$\int_{S^2}\langle H\rangle^4 \mathrm{d}\Omega = \frac{1}{15A^4}[3\langle T_{10}\rangle^4 - 12\langle T_{10}\rangle^2\langle T_{11}\rangle\langle T_{1-1}\rangle + 12\langle T_{11}\rangle^2\langle T_{1-1}\rangle^2]$$
$$= \frac{1}{5A^4}[\langle T_{10}\rangle^2 - 2\langle T_{11}\rangle\langle T_{1-1}\rangle]^2$$
$$= \frac{1}{5A^4}[\langle T_{1M}\rangle\langle T^{\dagger}_{1M}\rangle]^2 \quad \text{(C19)}$$

The final result is given by equations (C10), (C15), (C16), (C18) and (C19),

$$\begin{aligned}\langle|\dot{v}|^2\rangle = A^{-4}\Bigg(\sum_{L=0}^{2}\Bigg\{&\frac{1}{\sqrt{2s+1}}[\chi(1,1,L;s)c_{10,10}^{L0}]^2\chi(L,L,0;s)c_{L0,L0}^{00}\\ &-\frac{4}{3}\chi(1,1,L;s)\chi(L,1,1;s)c_{10,10}^{L0}c_{L0,10}^{10}\langle T_{1M}^{\dagger}\rangle\langle T_{1M}\rangle\\ &-\frac{[\chi(1,1,L;s)c_{10,10}^{L0}]^2}{2L+1}\langle T_{LM}^{\dagger}\rangle\langle T_{LM}\rangle+\frac{8\chi(1,1,L;s)c_{1N_2 1N_1}^{LM}[c_{1010}^{L0}]^2}{2L+1}\langle T_{LM}^{\dagger}\rangle\langle T_{1N_1}\rangle\langle T_{1N_2}\rangle\Bigg\}\\ &-\frac{4}{5}[\langle T_{1M}\rangle\langle T_{1M}^{\dagger}\rangle]^2\Bigg).\end{aligned} \tag{C20}$$

Some quantities in the previous equations can be simplified by using formulas for the 6j-symbols and the CG coefficients. For example, some quantities of equation (C20) are simplified to

$$A^{-4}\sum_{L=0}^{2}\frac{1}{\sqrt{2s+1}}[\chi(1,1,L;s)c_{10,10}^{L0}]^2\chi(L,L,0;s)c_{L0,L0}^{00}=\begin{cases}\dfrac{s^2(s+1)^2}{9} & s\leqslant 1/2\\ \dfrac{s(s+1)(3s^2+3s-1)}{15} & s\geqslant 1\end{cases}. \tag{C21}$$

$$\frac{4}{3}A^{-4}\sum_{L=0}^{2}\chi(1,1,L;s)\chi(L,1,1;s)c_{10,10}^{L0}c_{L0,10}^{10}=\begin{cases}\dfrac{4}{27}s^2(s+1)^2(2s+1) & s\leqslant 1/2\\ \dfrac{4}{45}s(s+1)(2s+1)(3s^2+3s-1) & s\geqslant 1\end{cases}. \tag{C22}$$

$$\frac{[\chi(1,1,L;s)c_{10,10}^{L0}]^2}{A^4(2L+1)}=\begin{cases}\dfrac{s^2(s+1)^2(2s+1)}{9} & L=0\\ 0 & L=1.\\ \dfrac{s(s+1)(2s-1)(2s+1)(2s+3)}{225} & L=2\end{cases} \tag{C23}$$

$$\frac{8\chi(1,1,L;s)[c_{1010}^{L0}]^2}{A^4(2L+1)}=\begin{cases}-\dfrac{8}{9\sqrt{3}}s^2(s+1)^2(2s+1)^{3/2} & L=0\\ 0 & L=1,\\ \dfrac{8}{45\sqrt{30}}s(s+1)(2s+1)\sqrt{(2s+3)(2s+2)(2s+1)2s(2s-1)} & L=2\end{cases} \tag{C24}$$

leading to equation (59) in the main text.

Appendix D. Averaging using spherical designs

While for a general function $f: S^2 \to \mathbb{R}$ integration is necessary in order to compute its average, the polynomial functions that show up in, *e.g.*, (54), can in fact be averaged by sampling them over an appropriate finite set of points. Our first relevant concept is that of a *spherical t-design* in dimension d, defined as a set of points $\{p_i\}$, $i = 1,\ldots,N$, on S^d such that the average of any polynomial f of degree t or less (in the cartesian coordinates $\{x^1,\ldots,x^{d+1}\}$ of the ambient $\mathbb{R}^{d+1}$) over the set coincides with the average over S^d, *i.e.*,

$$\frac{1}{N}\sum_{i=1}^{N}f(p_i)=\frac{1}{|S^d|}\int_{S^d}f\,\mathrm{d}\Omega, \tag{D1}$$

where $|S^d| = \int_{S^d}\mathrm{d}\Omega$, and $\mathrm{d}\Omega$ is the euclidean measure on S^d [35, 36]. There are two features of the integrand in, *e.g.*, (54), that call for a refinement of the above concept: the polynomial in question is homogeneous, and of an even degree, as fit for a squared modulus. The appropriate concept then is that of a *spherical (t,t)-design* in dimension d, which is the specification of the previous definition to the case of a homogeneous polynomial of degree $2t$ [37]. An obvious property of spherical designs (of either of the above types) is that it can be rotated arbitrarily, remaining a spherical design. We have already seen a spherical design entering the discussion above: the average of the modulus squared of the rotational velocity, which is homogeneous quadratic in the components of the Hamiltonian H, came out as the average of its value for rotations around the three coordinate axes. In the parlance introduced above, this follows from the statement that any three mutually orthogonal directions on S^2 furnish a spherical (1, 1)-design. The squared modulus of the acceleration in (54) is homogeneous quartic in the components of the Hamiltonian, so the integral over S^2 in that expression may also be computed by averaging over a spherical (2, 2)-design. A minimal such design (*i.e.*, with the minimum possible number of points) is given in [37]—it consists of the six equiangular lines which go through the vertices of an icosahedron (for each such line, any of the two half-lines emanating from the origin may be chosen, since the functions being averaged are symmetric on antipodal points of S^2). Choosing the vertices with positive z-coordinate, among the icosahedron vertices provided by Mathematica 13.2, we find the six-point spherical (2, 2)-design

$$p_1 = (0,\, 0,\, 1), \quad p_2 = \left(\frac{2}{\sqrt{5}},\, 0,\, \sigma\right), \quad p_3 = (-\mu^2,\, -\nu,\, \sigma), \quad p_4 = (-\mu^2,\, \nu,\, \sigma),$$
$$p_5 = (\nu^2,\, -\mu,\, \sigma), \quad p_6 = (\nu^2,\, \mu,\, \sigma), \tag{D2}$$

where $\mu \equiv \sqrt{(5+\sqrt{5})/10}$, $\nu \equiv \sqrt{(5-\sqrt{5})/10}$, and $\sigma = 1/\sqrt{5}$. The spherical design consists of one point at the north pole, and the other five on the vertices of a regular pentagon at $z = 1/\sqrt{5}$. For any value of spin s, the Hamiltonians $H_i = p_i \cdot \mathbf{S}^{(s)}$ are to be used in the second of (52) to compute the corresponding squared moduli $|a_i|^2$, the average of which gives $\langle |a|^2 \rangle$. Using this procedure, we have verified (57) for $s = 1$.

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