WORKSHOP
Direct and Large-Eddy Simulation 14
April 10-12 2024, Erlangen, Germany

ANALYSIS OF ENTROPY STABLE DG SCHEMES BASED ON ENERGY BALANCES FOR SCALE-RESOLVED SIMULATIONS.

Amaury Bilocq[1], Nayan Levaux[1], Vincent E. Terrapon[1], Koen Hillewaert[1]
[1]Aerospace and Mechanics Department, Université de Liège
Allée de la découverte 9, 4000 Liège, Belgium
amaury.bilocq@uliege.be

INTRODUCTION

High-order simulation methods like Discontinuous Galerkin (DG) have proven suitability for Direct Numerical Simulation (DNS) and (implicit) Large Eddy Simulations (LES) of subsonic flows [1]. In supersonic conditions, shock waves may develop. The discontinuity over the shock cannot be captured by polynomial interpolation and, therefore, both convergence and stability of the simulation deteriorate as Gibbs oscillations develop. In the extreme case, these oscillations lead to unphysical solutions and the failure of the computation. Shock capturing methods usually add artificial viscosity to smooth the shock such that it can be safely represented. However, this reduces accuracy and negatively impacts the turbulent kinetic energy budget. It is therefore desirable to reduce its action to a minimum.

The instability of high-order methods, caused by under-interpolation and integration, can be mitigated by leveraging a discrete equivalent of the entropy and its evolution equation, which introduces a discrete bound for the solution. Most of these entropy-consistent schemes are based on the use of entropy variables and the "summation-by-parts" (SBP) theorem, leading to the analogue to the integration-by-parts theorem at the discrete level. The vast majority rely on nodes that coincide with the Gauss-Legendre-Lobatto (GLL) quadrature points [2]. The presence of such nodes on the boundary greatly helps constructing SBP operators between two elements. However, the GLL quadrature has lower accuracy leading to the build-up of error. More recently, entropy stable schemes based on Gauss quadrature nodes, without points on the boundary, have been developed. While these methods improve greatly the accuracy of the solution, such SBP operators are numerically very costly since they introduce an "all-to-all" flux coupling between all degrees of freedom (Dofs) in the element and the need of the so-called *entropy projection* [3].

In the literature, the entropy stable Discontinuous Galerkin spectral element method (ESDGSEM) based on the GLL quadrature nodes and the entropy stable Discontinuous Galerkin method (ESDG) based on Gauss quadrature nodes have been compared based on their robustness and performance for compressible Euler and Navier-Stokes equations [4, 5] but not with a deep focus on turbulence. In the present work, a novel approach is introduced for the analysis of these schemes based on energy balances such as the budget of kinetic energy and moments of velocity weighted by the density. This work also compares a new hybrid DG solver for shock capturing, the so-called sensor-based scheme. To alleviate the computational cost associated with the ESDG scheme, the entropy stability is only activated in cells where shock stabilization is necessary or where the turbulence is under-resolved. Everywhere else, a standard DG formulation is applied.

NUMERICAL EXPERIMENTS

The comparison of the schemes is performed on the 3D Taylor-Green Vortex, a well-known test case consisting of a laminar-turbulent transition of a decaying vortex, at Re=1600 and M=1. In this configuration, shocklets interact with the small turbulent structures, making this test case a challenge for stabilization techniques.

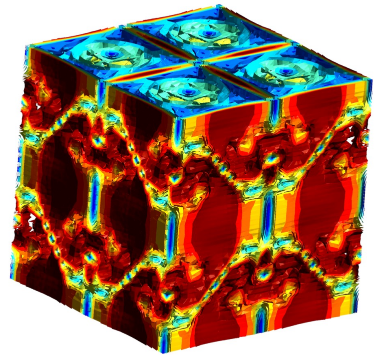

Figure 1: Magnitude of the velocity for the 3D Taylor-Green test case at convective time $T_c = 8$. The domain is a periodic box of length 2π.

First of all, the spatial convergence of each scheme has been computed based on the maximum error between the enstrophy obtained at different Dofs and the reference enstrophy from [6] (DNS of 512 Dofs per direction). Figure 2 shows that the convergence is better for the more complex ESDG scheme than for the ESDGSEM scheme but worse than for the sensor-based approach, which is an additional reason to introduce this method.

Then, the budget of kinetic energy as been investigated. In the case of a periodic domain, the conservative equation for the kinetic energy is given in its integral form by

$$\frac{d}{dt}\int_V \rho E_k dV = \int_V p\nabla\cdot\mathbf{v}dV - \int_V \tau : \nabla\mathbf{v}dV, \qquad (1)$$

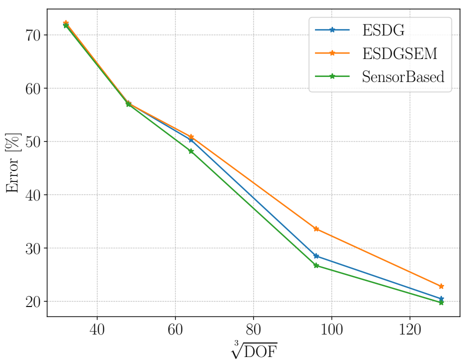

Figure 2: Convergence analysis based on the enstrophy from the reference solution [6] for the 3D Taylor-Green test case with respect to the number of degree of freedom for the two schemes.

where the first term is the variation of the kinetic energy, the second term is the pressure work and the third term is the viscous dissipation. The budget of kinetic energy is the difference between the measured variation (LHS of Eq. 1) and the theoretical variation (RHS of Eq. 1) of the kinetic energy. Figure 3 shows that, in this case, the convergence of the budget closure is the slowest for the ESDG scheme for a reduced number of Dofs. By looking to the terms constituting the budget of kinetic energy in Fig. 4 for the ESDG scheme and in Fig. 5 for the ESDGSEM, a difference in the pressure work can be spotted. This increase in the pressure work for the ESDG scheme can be explained due to presence of the *entropy projection* which improves the robustness of the method but induces a large numerical error for coarser mesh. The sensor based approach has a lower error and therefore seems interesting not only for efficiency but also for accuracy. The ESDGSEM scheme exhibits a slower convergence rate for finer mesh and tends to a constant error in the budget closure at convergence.

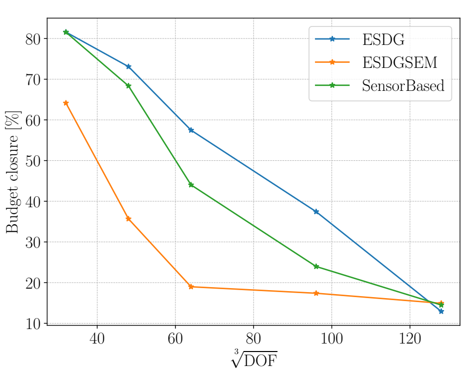

Figure 3: Budget closure of the kinetic energy for the 3D Taylor-Green test case with respect to the number of degree of freedom for the two schemes.

Therefore, the budget of kinetic energy highlights that even if the ESDG scheme is the more robust, it is adding unwanted numerical dissipation on hidden quantities such as the pressure work. Moreover, an other interest of the sensor based approach is the lower computational time compared to the

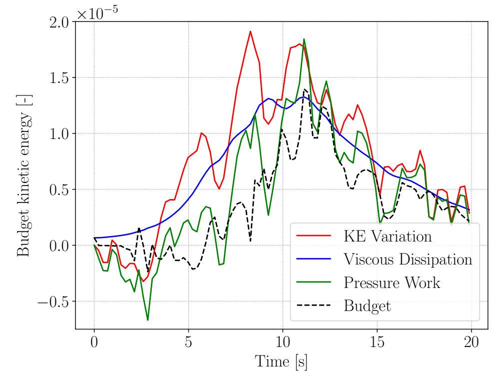

Figure 4: Variation of the terms constituting the budget of kinetic energy for the 3D Taylor-Green test case with 96 Dofs for the ESDG scheme.

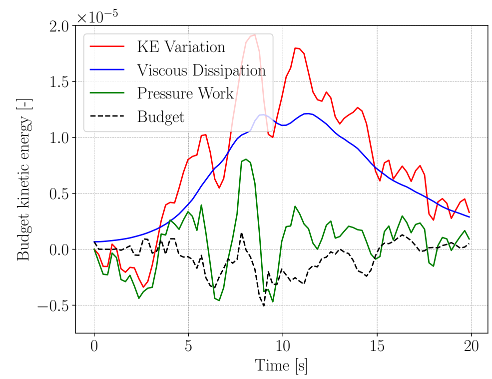

Figure 5: Variation of the terms constituting the budget of kinetic energy for the 3D Taylor-Green test case with 96 Dofs for the ESDGSEM scheme.

	ESDGSEM	ESDG	SensorBased
Cost	-	+47%	+7%

Table 1: Comparison of the cost of the method with respect to the less expensive ESDGSEM scheme to simulate the 3D Taylor-Green Vortex with 96 Dofs per direction.

ESDG scheme as shown in Tab.1.

REFERENCES

[1]Hillewaert, Koen : *Development of the discontinuous Galerkin method for high resolution, large scale CFD and acoustics in industrial geometries.* Université catholique de Louvain (2013).

[2]Gassner, Gregor : Split Form Nodal Discontinuous Galerkin Schemes with Summation-By-Parts Property for the Compressible Euler Equations, Journal of Comp. Phy. Vol. **327** (2016).

[3]Chan, Jess et al. : Efficient Entropy Stable Gauss Collocation Methods. *SIAM Journal on Scientific Computing* **41** (2019).

[4]Rojas, Diego et al. : On the Robustness and Performance of Entropy Stable Collocated Discontinuous Galerkin Methods . *Journal of Computational Physics*, **426** (2021).

[5]Chan, Jesse et al. : On the Entropy Projection and the Robustness of High Order Entropy Stable Discontinuous Galerkin Schemes for Under-Resolved FLows. *Frontiers in Physics*, **10** (2022).

[6]Lusher, David J., et Neil D. Sandham : Assessment of Low-Dissipative Shock-Capturing Schemes for the Compressible Taylor–Green Vortex. *AIAA Journal*, **59, n°2** (2021).