Chapter VII
Conclusions and perspectives

A Natural Neighbours Method (NEM) based on the FRAEIJS de VEUBEKE (FdV) variational principle is developed in the present thesis.

All the developments are made in the domain of 2D infinitesimal transformations for simplicity since the goal of the present thesis is to explore the possibilities offered by the FdV variational principle in the frame of the NEM.

Chapter I and II are dedicated to introduce the backgrounds and some basic information of this work. This method firstly applies to treat linear elastic problems in chapter III and then is extended to materially nonlinear problems in chapter IV. Chapter V presents the development of this method in linear fracture mechanics domain. Afterwards, in chapter VI, an eXtended Natural nEighbours Method (XNEM) is developed to solve problems of linear fracture mechanics which is inspired by the eXtended Finite Elements Method (XFEM).

Thanks to the FdV variational principle, we can discretize the displacement field, the stress field, the strain field and the support reactions field independently in all these developments, which provides a large flexibility for the development of numerical methods.

In linear elastic problems (chapter III), the following discretization hypotheses are used:

1. The assumed displacements are interpolated between the nodes with the Laplace interpolation function.
2. The assumed support reactions are constant over each edge of Voronoi cells on which displacements are imposed.
3. The assumed stresses are constant over each Voronoi cell.
4. The assumed strains are constant over each Voronoi cell.

The additional degrees of freedom linked with the assumed stresses and strains can be eliminated at the level of the Voronoi cells so that the final system of equations only involves the nodal displacements and the assumed support reactions.

The support reactions can be further eliminated from the equation system if the imposed support conditions only involve displacements imposed as constant (in particular displacements imposed to zero) on a part of the solid contour, finally leading to a system of equations of the same size as in the classical displacement-based method.

The main properties of this approach are:

- In the absence of body forces, the calculation of integrals over the area of the domain is avoided: only integrations on the edges of the Voronoi cells are required, for which classical Gauss numerical integration with 2 integration points is sufficient to pass the patch test.

- The derivatives of the nodal shape functions are not required in the resulting formulation.

In chapter IV, the hypotheses used in chapter III are also introduced to the non linear problems. But instead of using the hypotheses on the assumed displacements and the assumed strains, similar hypotheses on the assumed velocities (interpolated between the nodes with the Laplace interpolation function) and on the assumed strain rates (constant over each Voronoi cell) are used.

As in linear elastic case, the final equations system in non linear case only involves the nodal velocities. It can be solved step by step by integration on time and Newton-Raphson iterations at the level of the different time steps.

The advantages of this method applied in linear elasticity remain valid in the extension to the elasto-plasticity problems.

In the development of this method for linear fracture mechanics in chapter V, a node is located on each crack tip. The Voronoi cells containing the crack tip are called Linear Fracture Mechanics Voronoi Cells (LFMVC). All the other cells are called Ordinary Voronoi cells (OVC). Based on the hypotheses used in linear elastic case, the assumed stresses and the assumed strains are constant in the OVC. In each LFMVC, the stress and the strain discretization includes not only a constant term but also additional terms corresponding to the solutions of LEFM for modes 1 and 2.

In this approach, the stress intensity coefficients are obtained as primary variables of the solution. The final equations system only involves the nodal displacements and the stress intensity coefficients.

The present approach also retains the properties of the linear elastic case. Indeed, in the LFMVCs, some integrations on the area of the LFMVCs are required but they can be calculated analytically. The other integrals are integrals on the edges of the LFMVCs

In chapter VI, an eXtended Natural nEighbours Method (XNEM) is proposed in which the crack is represented by a line that does not conform to the nodes or the edges of the cells. Based on the hypotheses used in linear elastic domain, the discretization of the displacement field is enriched with Heaviside functions allowing a displacement discontinuity at the level of the crack. The additional degrees of freedom associated with this discontinuity are called the displacement jump parameters.

There are three types of cells in chapter VI:

- cells of type *O* that do not contain a crack;
- cells of type *H* that are divided into 2 parts by a crack;
- cells of type *C* that contain a crack tip.

The stresses and strains are assumed to be constant over each cell of type O and over each part of the cells of type H respectively. For cells of type C, , the stress and strain fields are enriched in a similar way as the LFMVCs of chapter V, that is, by introducing the solutions of LEFM for mode 1 and mode 2 in the stress and strain discretization.

As in the method developed in chapter V, the stress intensity coefficients are obtained as primary variables of the solution in chapter VI.

The final equation system only involves the nodal displacements and the displacement jump parameters. After solving the equations system, the stress intensity coefficients can be calculated from the displacements and the jump parameters.

This approach also retains the properties of the linear elastic case. Indeed, in the cells of type C, some integrations on the area are required but they can be calculated analytically. The other integrals are integrals on the edges of the cells.

Furthermore, in all these developments in linear elastic domain, non linear elastic domain and linear fracture mechanics domain, the displacements can be imposed in two ways:

- In the spirit of the FdV variational principle, boundary conditions of the type $u_i = \tilde{u}_i$ on S_u can be imposed in the average sense; hence, any function $\tilde{u}_i = \tilde{u}_i(s)$ can be accommodated by the method;
- However, since the natural neighbours method is used, the interpolation of displacements on the solid boundary is linear between 2 adjacent nodes. So, if the imposed displacements $\tilde{u}_i$ are linear between 2 adjacent nodes, they can be imposed exactly. This is obviously the case with $\tilde{u}_i = 0$. In such a case, it is equivalent to impose the displacements of these 2 adjacent nodes to zero.

A set of applications are performed to evaluate the method and lead to the following general conclusions.

a. The patch test can be successfully passed in all cases (linear elastic, nonlinear, linear elastic fracture mechanics).
b. Problems involving nearly incompressible materials can be solved without incompressibility locking in all cases.
c. The solutions provided by the present approach converge to the exact solution, in particular:
    - Calculations for the elasto-plastic bending case show a good convergence to the solution based on the direct integration of the equations.
    - The example of the square membrane with circular hole shows that the present approach compares favourably with the classical FEM.

Globally, in the linear elastic domain and in the non linear domain, the results provided by the approach developed in this thesis are perfectly satisfactory and it can be concluded that, at least in 2D Solid Mechanics, its qualities are equivalent or superior to the classical FEM.

However, some questions remain about the development of this method in linear elastic fracture mechanics.

- In chapter V, an attempt has been made to enrich the discretization of the displacements near the crack tip nodes by the solution of LEFM in the vicinity of the crack tip. Unfortunately, this enrichment does not provide a significant improvement of the results while the development of the corresponding equations becomes much more complex. Consequently, this approach was abandoned, but this is disappointing because a similar enrichment is satisfactorily used in XFEM.

- The development of the XNEM in chapter VI was considered as a first step to explore the possibility of propagating the crack without re-meshing. But the results show that it is necessary to increase the node density near the crack, which means that local re-meshing can not be avoided if the crack propagates.

These consideration indicate some directions for future research on the use of the NEM and XNEM based on the FdV variational principle

- Continue to study the enrichment on the displacement field which has been tested in chapter V.
- Instead of using the approximate near-crack-tip solution for stresses and strains as enrichment in the LFMVC (chapter V) and in the cells of type C (chapter VI), it is worth trying to use the exact solution. But in this situation, it is not sure that the analytical calculation of the integrals over the area of these cells can be obtained. This point should be considered.
- The extension of the method to non linear fracture mechanics could be studied. This requires a different enrichment of the stresses and strains near the crack tip.
- The use of the XNEM for crack propagation should be considered. Considering that the direction and the extend of the crack propagation have already been studied in literature and applied in XFEM, the main point would be to find an efficient algorithm to increase the node density near the crack tip.