Annex 1
Laplace function for a regular grid of nodes

Content

A1.1. Laplace function

Let us consider the case of a regular grid of nodes with spacing $a_1 = a_2 = a$ (figure A1.1).

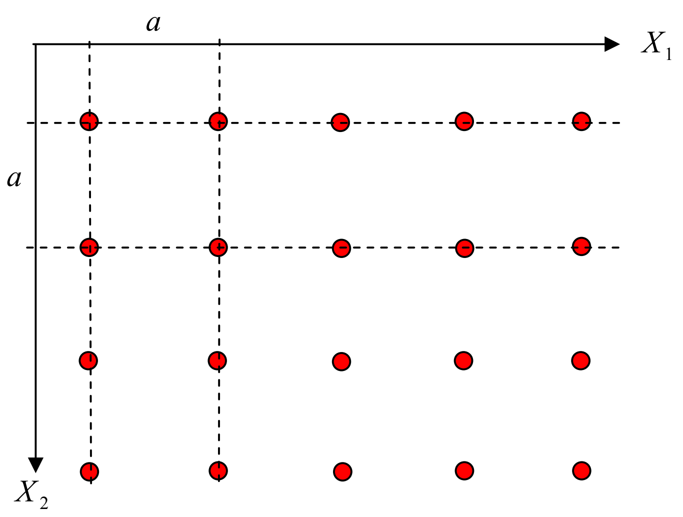


Figure A1.1. A regular grid of nodes.

It is easy to show that the Voronoi polygons associated with these nodes are the squares indicated in figures A1.2.

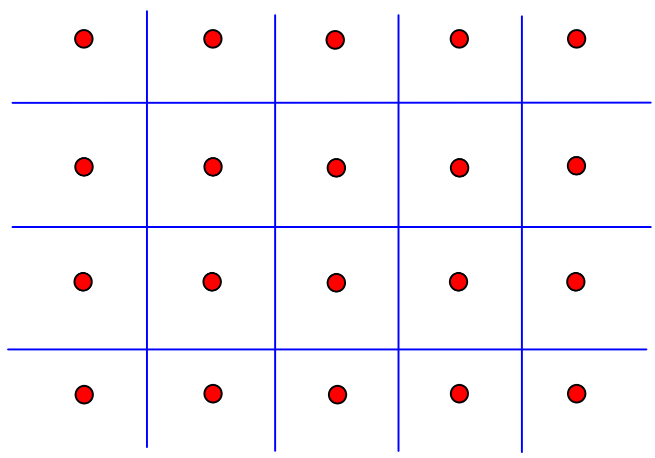


Figure A1.2. Voronoi polygons of a regular grid of nodes.

Let us calculate the value of the Laplace functions at point X located on an edge AD of a Voronoi polygon (figure A1.3).

The position of X is given by the distance: $-\frac{a}{2} \le u \le +\frac{a}{2}$.

Let $\xi = 2\frac{u}{a}$ $\quad -1 \le \xi \le +1$

We get the positions of points A,B,C,D: $\xi^A = -1$; $\xi^B = 1-\sqrt{2}$; $\xi^C = -1+\sqrt{2}$; $\xi^D = +1$

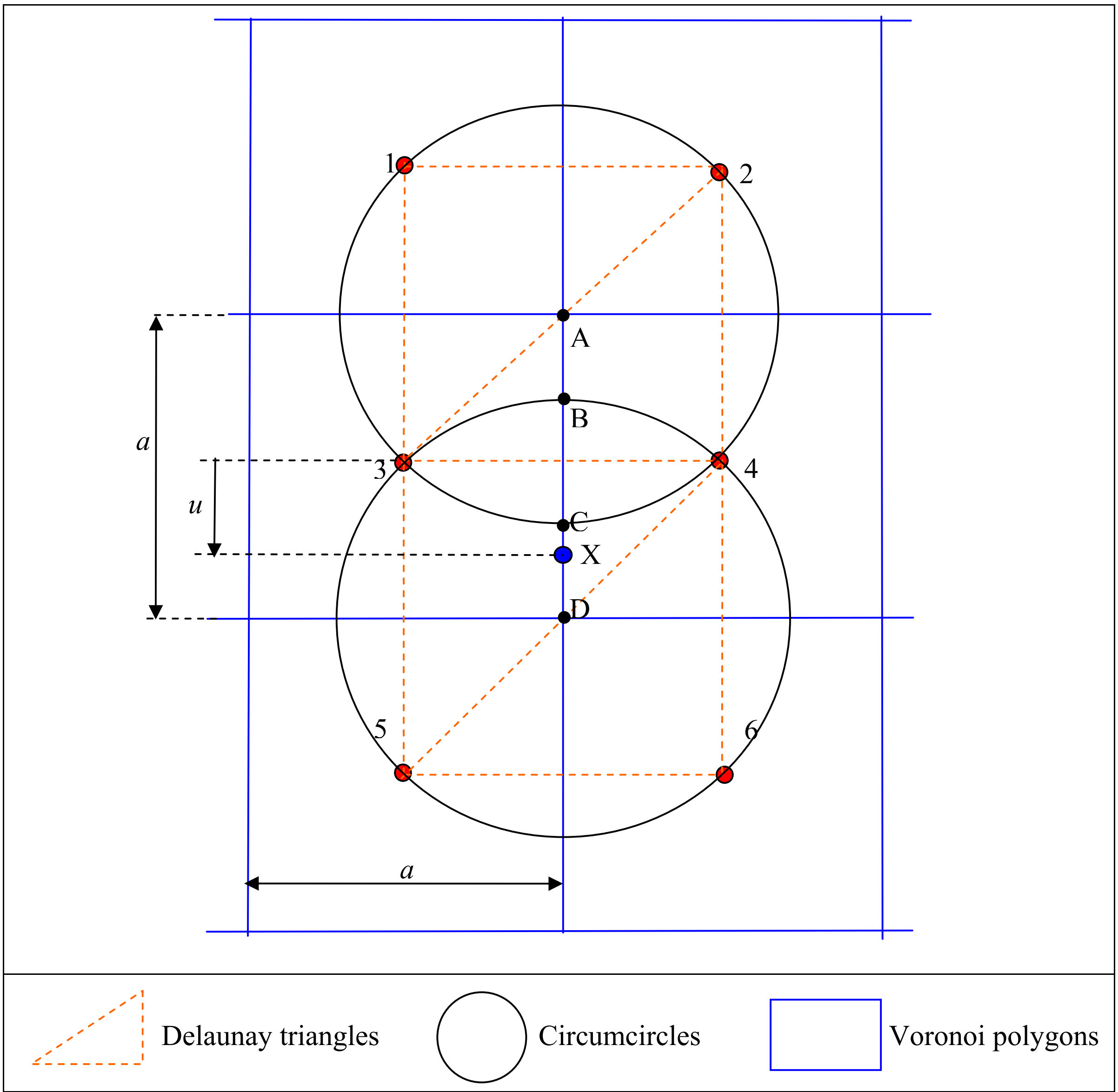


Figure A1.3. Calculation of Laplace functions on edge AD.

case 1: *X* between *A* and *B*

For $\xi^A \le \xi \le \xi^B$, we get from figure A1.4:

$$h_1 = h_2 = \frac{1}{2}a\sqrt{\xi^2 + 4\xi + 5} \qquad s_1 = s_2 = -\frac{a(\xi^2 + 2\xi - 1)}{4(\xi + 2)}\sqrt{\xi^2 + 4\xi + 5}$$

$$h_3 = h_4 = \frac{1}{2}a\sqrt{1 + \xi^2} \qquad s_3 = s_4 = \frac{a(\xi^2 + 2\xi - 1)\sqrt{1 + \xi^2}}{4\xi}$$

$$\alpha_1 = \alpha_2 = \frac{\xi^2 + 2\xi - 1}{2(\xi + 2)} \qquad \alpha_3 = \alpha_4 = \frac{\xi^2 + 2\xi - 1}{2\xi} \qquad \sum_{I=1,4}\alpha_I = \frac{2\xi^2 + 4\xi - 2}{\xi^2 + 2\xi}$$

$$\Phi_1 = \Phi_2 = -\frac{\xi}{4} \qquad \Phi_3 = \Phi_4 = \frac{2 + \xi}{4} \qquad \Phi_5 = \Phi_6 = 0$$

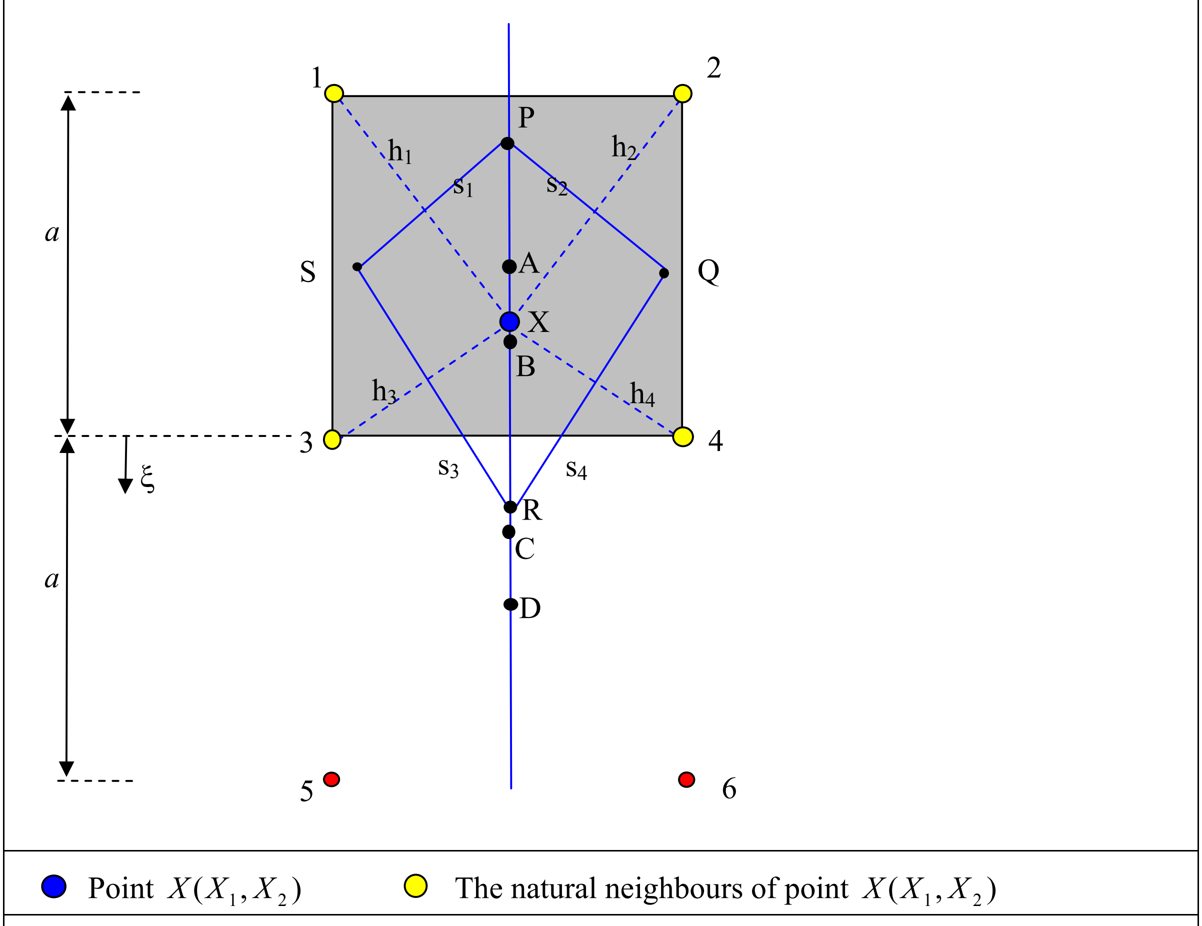


Figure A1.4. Calculation of Laplace functions on edge AD, zone AB.

case 2: *X* between *B* and *C*

For $\xi^B \le \xi \le \xi^C$, we get from figure A1.5:

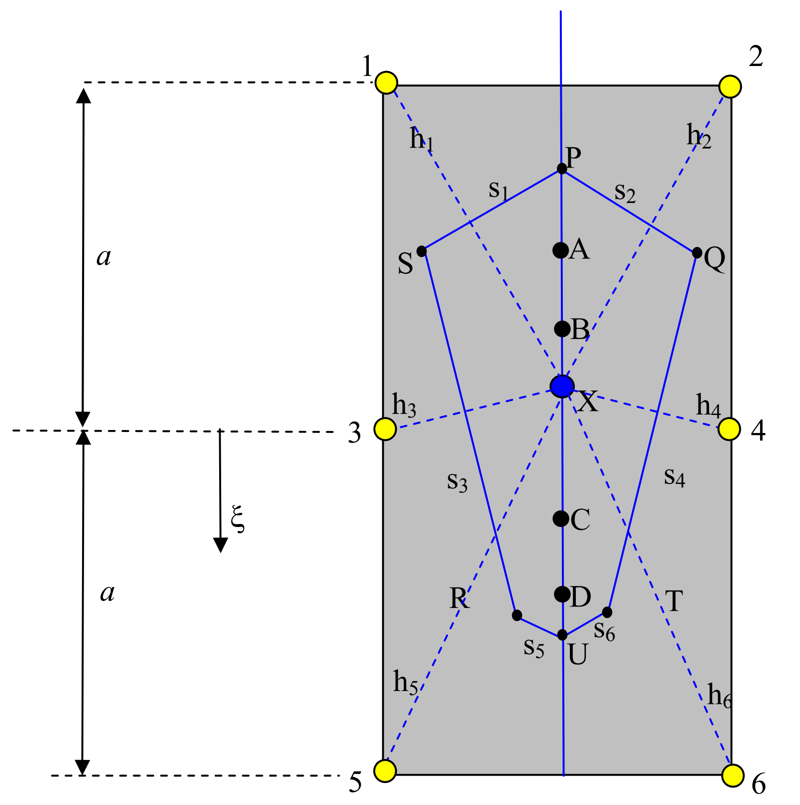
$$h_1 = h_2 = \frac{1}{2}a\sqrt{\xi^2+4\xi+5} \qquad s_1 = s_2 = -\frac{a(\xi^2+2\xi-1)}{4(\xi+2)}\sqrt{\xi^2+4\xi+5}$$

$$h_3 = h_4 = \frac{1}{2}a\sqrt{1+\xi^2} \qquad s_3 = s_4 = a\sqrt{1+\xi^2}$$

$$h_5 = h_6 = \frac{1}{2}a\sqrt{\xi^2-4\xi+5} \qquad s_5 = s_6 = -\frac{a(\xi^2-2\xi-1)}{4(\xi-2)}\sqrt{\xi^2-4\xi+5}$$

$$\alpha_1 = \alpha_2 = \frac{\xi^2+2\xi-1}{2(\xi+2)} \qquad \alpha_3 = \alpha_4 = 2 \qquad \alpha_5 = \alpha_6 = \frac{\xi^2-2\xi-1}{2(\xi-2)} \qquad \sum_{I=1,6}\alpha_I = \frac{4\xi^2-20}{\xi^2-4}$$

$$\Phi_1 = \Phi_2 = \frac{1}{8}\left(\frac{2}{5-\xi^2}-\xi\right) \qquad \Phi_3 = \Phi_4 = \frac{1}{2}\left(1-\frac{1}{5-\xi^2}\right) \qquad \Phi_5 = \Phi_6 = \frac{1}{8}\left(\frac{2}{5-\xi^2}+\xi\right)$$



Point $X(X_1, X_2)$ — The natural neighbours of point $X(X_1, X_2)$

Figure A1.5. Calculation of Laplace functions on edge AD, zone BC.

case 3: X between C and D

For $\xi^C \le \xi \le \xi^D$, we get from figure A1.6:

$$h_3 = h_4 = \frac{1}{2} a \sqrt{1+\xi^2} \qquad s_3 = s_4 = \frac{a(\xi^2 - 2\xi - 1)\sqrt{1+\xi^2}}{4\xi}$$

$$h_5 = h_6 = \frac{1}{2} a \sqrt{\xi^2 - 4\xi + 5} \qquad s_5 = s_6 = -\frac{a(\xi^2 - 2\xi - 1)}{4(\xi - 2)}\sqrt{\xi^2 - 4\xi + 5}$$

$$\alpha_3 = \alpha_4 = -\frac{\xi^2 - 2\xi - 1}{2\xi} \qquad \alpha_5 = \alpha_6 = \frac{\xi^2 - 2\xi - 1}{2(\xi - 2)} \qquad \sum_{I=3,6} \alpha_I = \frac{2\xi^2 - 4\xi - 2}{\xi^2 - 2\xi}$$

$$\Phi_1 = \Phi_2 = 0 \qquad \Phi_3 = \Phi_4 = \frac{2-\xi}{4} \qquad \Phi_5 = \Phi_6 = \frac{\xi}{4}$$

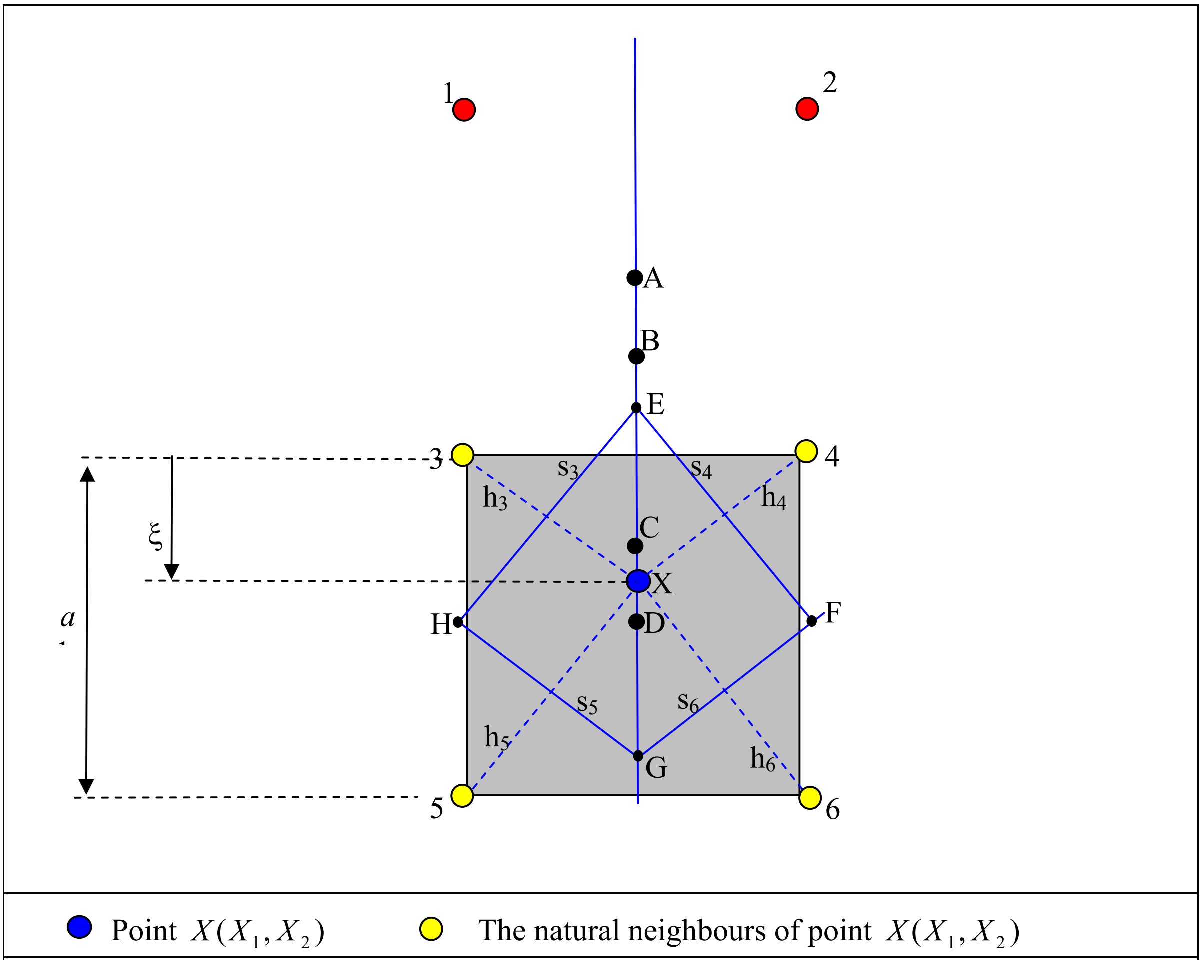


Figure A1.6. Calculation of Laplace functions on edge AD, zone CD.

Figure A1.7 illustrates the evolution of the Laplace functions with ξ.

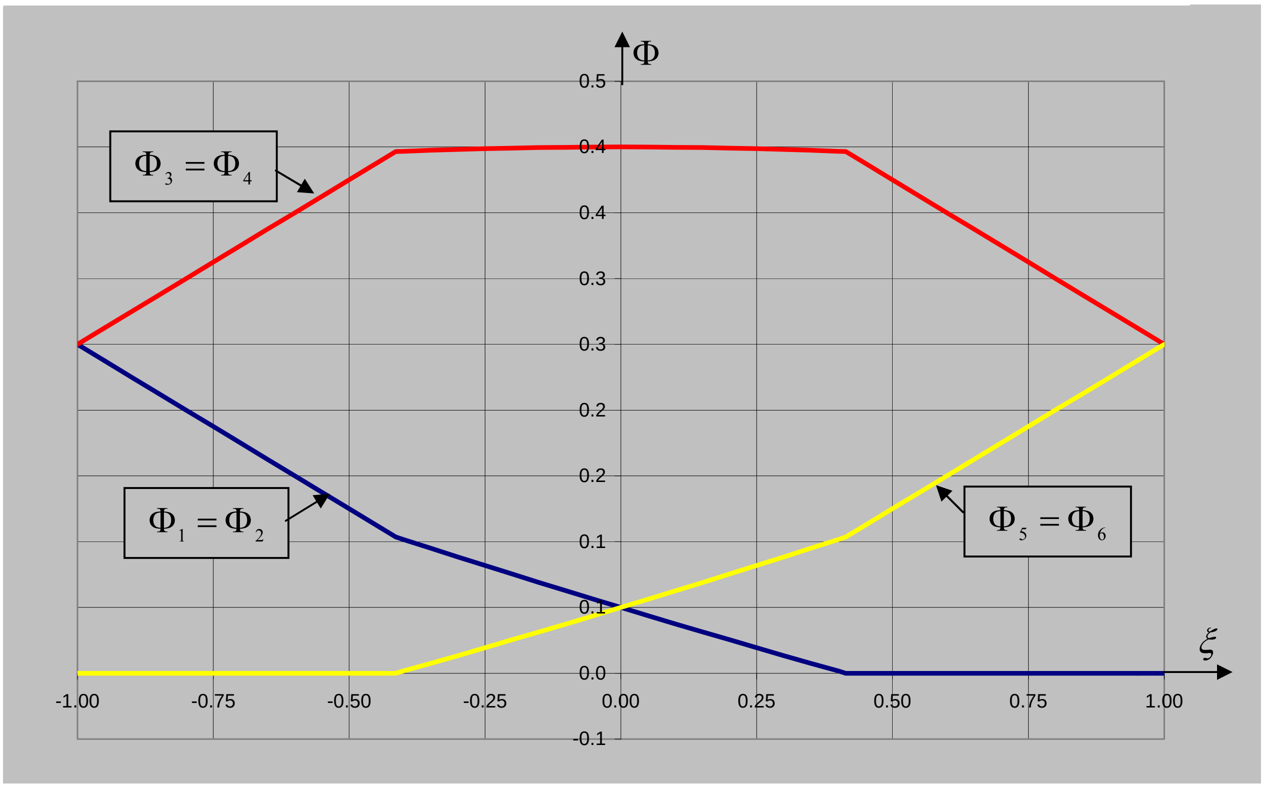


Figure A1.7. Laplace functions for a regular grid.

A1.2. Integration on an edge

The integrals of these functions on the interval $-1 \le \xi \le +1$ are easily computed.

Let

$$C_c = \frac{1}{20}\left\{5(\sqrt{2}-1)+\sqrt{5}\ln\left(\frac{\sqrt{5}+\sqrt{2}-1}{\sqrt{5}-\sqrt{2}+1}\right)\right\} = 0.1454585325461073 \qquad \text{(A1.1)}$$

$$C_m = \frac{1}{10}\left\{5(3-\sqrt{2})+\sqrt{5}\ln\left(\frac{\sqrt{5}-\sqrt{2}+1}{\sqrt{5}+\sqrt{2}-1}\right)\right\} = 0.7090829349077854 \qquad \text{(A1.2)}$$

Then, we find:

$$\int_{-1}^{+1}\Phi_1 d\xi \equiv \int_{-1}^{+1}\Phi_2 d\xi = C_c \qquad \text{(A1.3)}$$

$$\int_{-1}^{+1}\Phi_3 d\xi \equiv \int_{-1}^{+1}\Phi_4 d\xi = C_m \qquad \text{(A1.4)}$$

$$\int_{-1}^{+1}\Phi_5 d\xi \equiv \int_{-1}^{+1}\Phi_6 d\xi = C_c \qquad \text{(A1.5)}$$

A1.3. Calculation of A_j^{IJ}

Consider a Voronoi cell I and the 8 surrounding nodes (figure A1.8)

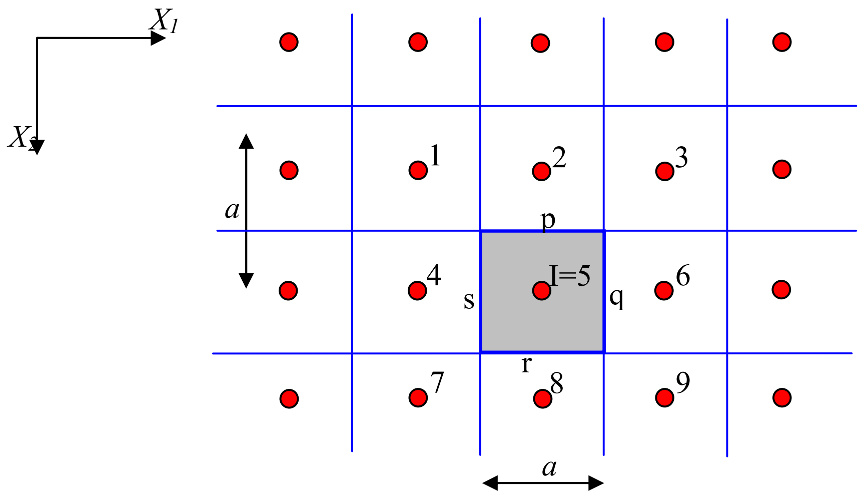


Figure A1.8. Cell I and surrounding nodes (here I=5).

By definition,

$$A_j^{IJ} = \oint_{c_I} n_j^I \Phi_J ds_I = \sum_{t=p,q,r,s} n_j^t \int_{s_t} \Phi_J ds_t = \frac{a}{2}\sum_{t=p,q,r,s} n_j^t \int_{-1}^{+1} \Phi_J d\xi_t$$

with the following values for n_j^t

Edge t	n_1^t	n_2^t
p	0	-1
q	1	0
r	0	1
s	-1	0

From section A1.1 above, we see that only the terms A_j^{IJ} for $J = 1\ to\ 9$ will differ from 0.

The calculation of these non zero terms is summarized in the table below.

Calculation of A_j^{IJ} for $J=1$ *to* 9 with $I=5$							
J	$I=5$	Contribution of edge				Total	Formula n°
		p	q	r	s		
J=1	A_1^{I1}	0	0	0	$-\frac{a}{2}C_c$	$A_1^{I1}=-\frac{a}{2}C_c$	(A1.6.a)
	A_2^{I1}	$-\frac{a}{2}C_c$	0	0	0	$A_2^{I1}=-\frac{a}{2}C_c$	(A1.6.b)
J=2	A_1^{I2}	0	$\frac{a}{2}C_c$	0	$-\frac{a}{2}C_c$	$A_1^{I2}=0$	(A1.6.c)
	A_2^{I2}	$-\frac{a}{2}C_m$	0	0	0	$A_2^{I2}=-\frac{a}{2}C_m$	(A1.6.d)
J=3	A_1^{I3}	0	$\frac{a}{2}C_c$	0	0	$A_1^{I3}=\frac{a}{2}C_c$	(A1.6.e)
	A_2^{I3}	$-\frac{a}{2}C_c$	0	0	0	$A_2^{I3}=-\frac{a}{2}C_c$	(A1.6.f)
J=4	A_1^{I4}	0	0	0	$-\frac{a}{2}C_m$	$A_1^{I4}=-\frac{a}{2}C_m$	(A1.6.g)
	A_2^{I4}	$-\frac{a}{2}C_c$	0	$\frac{a}{2}C_c$	0	$A_2^{I4}=0$	(A1.6.h)
J=5	A_1^{I5}	0	$\frac{a}{2}C_m$	0	$-\frac{a}{2}C_m$	$A_1^{I5}=0$	(A1.6.i)
	A_2^{I5}	$-\frac{a}{2}C_m$	0	$\frac{a}{2}C_m$	0	$A_2^{I5}=0$	(A1.6.j)
J=6	A_1^{I6}	0	$\frac{a}{2}C_m$	0	0	$A_1^{I6}=\frac{a}{2}C_m$	(A1.6.k)
	A_2^{I6}	$-\frac{a}{2}C_c$	0	$\frac{a}{2}C_c$	0	$A_2^{I6}=0$	(A1.6.l)
J=7	A_1^{I7}	0	0	0	$-\frac{a}{2}C_c$	$A_1^{I7}=-\frac{a}{2}C_c$	(A1.6.m)
	A_2^{I7}	0	0	$\frac{a}{2}C_c$	0	$A_2^{I7}=C_c$	(A1.6.n)
J=8	A_1^{I8}	0	$\frac{a}{2}C_c$	0	$-\frac{a}{2}C_c$	$A_1^{I8}=0$	(A1.6.o)
	A_2^{I8}	0	0	$\frac{a}{2}C_m$	0	$A_2^{I8}=\frac{a}{2}C_m$	(A1.6.p)
J=9	A_1^{I9}	0	$\frac{a}{2}C_c$	0	0	$A_1^{I9}=\frac{a}{2}C_c$	(A1.6.q)
	A_2^{I9}	0	0	$\frac{a}{2}C_c$	0	$A_2^{I9}=\frac{a}{2}C_c$	(A1.6.r)