Annex 3
Analytical calculation of $[V]$ and $[IH]$

Content

A3.1. Basic model

Consider a LFMVC as shown in the figure A3.1 where *X* and *Y* are the local axes associated with the crack tip.

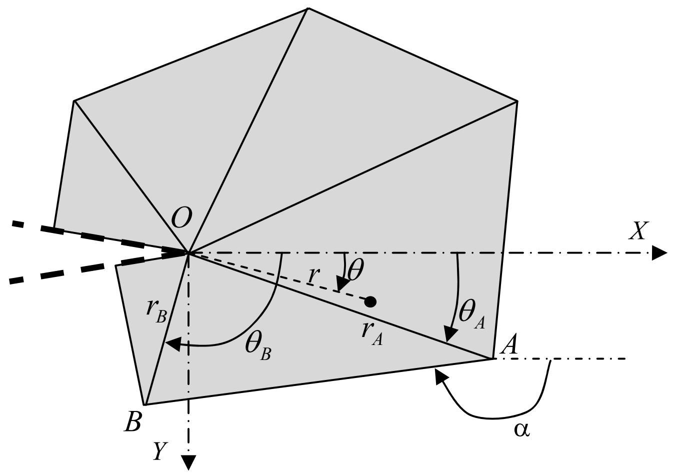

Figure A3.1. A LFMVC divided into triangles

We divide it into several triangles. The integration of $\left[V\right]$ and $\left[IH\right]$ over the domain thus transforms into the sum of integrations over these triangles.

Let us consider the integration over the triangle OAB as an example.

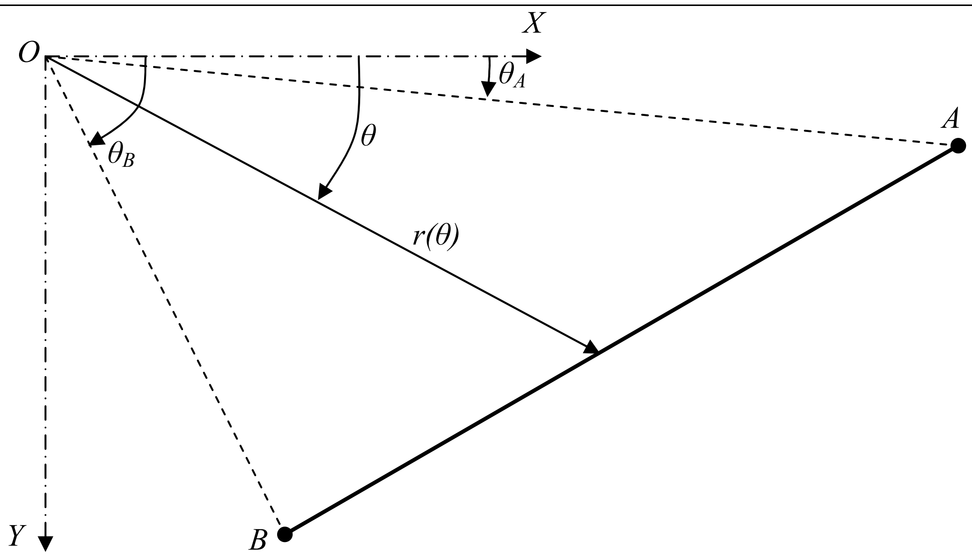

Figure A3.2. Equation of edge AB

$$\frac{X - X_A}{X_B - X_A} = \frac{Y - Y_A}{Y_B - Y_A}$$

$$X = r(\theta)\, cos\,\theta$$

$$Y = r(\theta)\, sin\,\theta$$

$$\frac{r(\theta)\, cos\,\theta - X_A}{X_B - X_A} = \frac{r(\theta)\, sin\,\theta - Y_A}{Y_B - Y_A}$$

Hence

$$r(\theta) = \frac{Y_A X_B - Y_B X_A}{(X_B - X_A)\, sin\,\theta - (Y_B - Y_A)\, cos\,\theta}$$

In this formula, the term $(X_B - X_A)\, sin\,\theta - (Y_B - Y_A)\, cos\,\theta$ can be equal to zero only if the edge *AB* is radial (i.e. if the line containing *A* and *B* also contains point *O*), which is impossible in the present context.

Let

$$a = Y_A X_B - Y_B X_A$$

$$s = X_B - X_A$$

$$c = Y_B - Y_A$$

Hence

$$r(\theta) = \frac{a}{s\, sin\,\theta - c\, cos\,\theta}$$

For any function $F(r,\theta)$ we have:

$$\int_{OAB} F\, dA = \int_{\theta_A}^{\theta_B} \int_0^{r(\theta)} F(r,\theta)\; r\, dr\, d\theta$$

A3.2. Analytical integration of $[V]$ over a triangle

$$[V_{OAB}]=\begin{bmatrix}\int\limits_{OAB}\langle H^{\Sigma 1}\rangle[D]\{H^{\Sigma 1}\}dA_I & \int\limits_{OAB}\langle H^{\Sigma 1}\rangle[D]\{H^{\Sigma 2}\}dA_I \\ \int\limits_{OAB}\langle H^{\Sigma 2}\rangle[D]\{H^{\Sigma 1}\}dA_I & \int\limits_{OAB}\langle H^{\Sigma 2}\rangle[D]\{H^{\Sigma 2}\}dA_I\end{bmatrix}=\int\limits_{OAB}\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}dA$$

Introducing the values of $\{H^{\Sigma 1}\},\{H^{\Sigma 2}\}$ given by the equations (V.20) and (V.21) and using the expression of $[D]$ appearing in (V.25), we get:

$$v_{11}=-\frac{cos^2\frac{\theta}{2}\left[\nu^*-3+(1+\nu^*)cos\,\theta\right]}{2\pi r E^*}$$

$$v_{12}=v_{21}=\frac{\left[\nu^*-1+(1+\nu^*)cos\,\theta\right]\,sin\,\theta}{2\pi r E^*}$$

$$v_{22}=\frac{9+\nu^*+4(\nu^*-1)cos\,\theta+3(1+\nu^*)cos\,2\theta}{8\pi r E^*}$$

Consequently,

$$\int\limits_{OAB}\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}dA=\int_{\theta_A}^{\theta_B}\int_0^{r(\theta)}\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}r\,dr\,d\theta=\int_{\theta_A}^{\theta_B}\int_0^{r(\theta)}\begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}dr\,d\theta$$

with:

$$w_{11}=-\frac{cos^2\frac{\theta}{2}\left[\nu^*-3+(1+\nu^*)cos\,\theta\right]}{2\pi E^*}$$

$$w_{12}=w_{21}=\frac{\left[\nu^*-1+(1+\nu^*)cos\,\theta\right]\,sin\,\theta}{2\pi E^*}$$

$$w_{22}=\frac{9+\nu^*+4(\nu^*-1)cos\,\theta+3(1+\nu^*)cos\,2\theta}{8\pi E^*}$$

Hence, $\begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}$ is not a function of r.

Therefore,

$$\int_0^{r(\theta)} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} r\,dr = \int_0^{r(\theta)} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} dr = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} r(\theta)$$

Finally,

$$[V_{OAB}] = \int_{OAB} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} dA = \int_{\theta_A}^{\theta_B} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} r(\theta)\, d\theta$$

Define the following constants and functions

$$q = c^2 + s^2$$

$$R_0 = \sqrt{q}$$

$$C_0 = \frac{a}{4\pi R_0^3\, E^*}$$

$$C_{11} = (\nu^* - 3)\, c^2 + 2(\nu^* - 1)\, s^2$$

$$C_{12} = (1 + \nu^*)\, c\, s$$

$$C_{22} = (\nu^* - 3)\, c^2 - 2(3 + \nu^*)\, s^2$$

$$f_1(\theta) = arctanh\left[\frac{s + c\, tan\,\frac{\theta}{2}}{R_0}\right]$$

$$f_2(\theta) = ln\,[c\, cos\,\theta - s\, sin\,\theta]$$

$$f_3(\theta) = 2(\nu^* - 1)\,[c\,\theta - s\, f_2(\theta)]$$

$$f_4(\theta) = (\nu^* - 1)\,[s\,\theta + c\, f_2(\theta)]$$

$$f_5(\theta) = c\, sin\,\theta - s\, cos\,\theta$$

$$f_6(\theta) = c\, cos\,\theta + s\, sin\,\theta$$

$$s_{11}(\theta) = C_0\ \left\{2C_{11}\, f_1(\theta) + R_0\left[(1 + \nu^*)\, f_5(\theta) + f_3(\theta)\right]\right\}$$

$$s_{12}(\theta) = 2C_0\left\{-C_{12}\, f_1(\theta) + R_0\left[(1 + \nu^*)\, f_6(\theta) + f_4(\theta)\right]\right\}$$

$$s_{22}(\theta) = C_0\ \left\{2C_{22}\, f_1(\theta) - R_0\left[3\,(1 + \nu^*)\, f_5(\theta) + f_3(\theta)\right]\right\}$$

Then

$$\int \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} r(\theta)\, d\theta = \begin{bmatrix} s_{11}(\theta) & s_{12}(\theta) \\ s_{21}(\theta) & s_{22}(\theta) \end{bmatrix}$$

We finally get :

$$\int_{OAB} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} dA = \begin{bmatrix} s_{11}(\theta_B) & s_{12}(\theta_B) \\ s_{12}(\theta_B) & s_{22}(\theta_B) \end{bmatrix} - \begin{bmatrix} s_{11}(\theta_A) & s_{12}(\theta_A) \\ s_{12}(\theta_A) & s_{22}(\theta_A) \end{bmatrix}$$

For the practical calculation, we use the following sequence to avoid the difficulties linked with the evaluation of the *arctanh* which is a complex number when its argument is not in the domain $[-1 , +1]$:

$$u_A = \frac{s + c \tan \dfrac{\theta_A}{2}}{R_0}$$

$$u_B = \frac{s + c \tan \dfrac{\theta_B}{2}}{R_0}$$

$$u_{AB} = \frac{(1+u_B)(1-u_A)}{(1+u_A)(1-u_B)}$$

$$u_{AB}^{lim} = \frac{\sin \dfrac{|\theta_A + \theta_B|}{2} + \sin \dfrac{\theta_B - \theta_A}{2}}{\sin \dfrac{|\theta_A + \theta_B|}{2} - \sin \dfrac{\theta_B - \theta_A}{2}}$$

$$f_{1AB} = \frac{1}{2} \ln [u_{AB}] \qquad if \quad c \neq 0$$

$$f_{1AB} = \frac{1}{2} \ln \left[u_{AB}^{lim}\right] \qquad if \quad c = 0$$

$$f_{2AB} = \ln [c \cos \theta_B - s \sin \theta_B] - \ln [c \cos \theta_A - s \sin \theta_A]$$

$$f_{3AB} = 2(\nu^* - 1)[c(\theta_B - \theta_A) - s\, f_{2AB}]$$

$$f_{4AB} = (\nu^* - 1)[s(\theta_B - \theta_A) + c\, f_{2AB}]$$

$$f_{5AB} = c(\sin \theta_B - \sin \theta_A) - s(\cos \theta_B - \cos \theta_A)$$

$$f_{6AB} = c(\cos \theta_B - \cos \theta_A) + s(\sin \theta_B - \sin \theta_A)$$

$$s_{11AB} = C_0 \left\{ 2 C_{11}\, f_{1AB} + R_0 \left[(1+\nu^*) f_{5AB} + f_{3AB} \right] \right\}$$

$$s_{12AB} = 2 C_0 \left\{ - C_{12}\, f_{1AB} + R_0 \left[(1+\nu^*) f_{6AB} + f_{4AB} \right] \right\}$$

$$s_{22AB} = C_0 \left\{ 2 C_{22}\, f_{1AB} - R_0 \left[3(1+\nu^*) f_{5AB} + f_{3AB} \right] \right\}$$

$$[V_{OAB}] = \begin{bmatrix} s_{11AB} & s_{12AB} \\ s_{12AB} & s_{22AB} \end{bmatrix}$$

A3.3. Analytical integration of $[IH]$ over a triangle

$$[IH_{OAB}] = \begin{Bmatrix} \int_{OAB} \langle H^{\Sigma 1} \rangle dA_I \\ \int_{OAB} \langle H^{\Sigma 2} \rangle dA_I \end{Bmatrix} = \begin{Bmatrix} \int_{\theta_A}^{\theta_B} \int_0^{r(\theta)} \langle H^{\Sigma 1} \rangle \, r \, dr \, d\theta \\ \int_{\theta_A}^{\theta_B} \int_0^{r(\theta)} \langle H^{\Sigma 2} \rangle \, r \, dr \, d\theta \end{Bmatrix}$$

Introducing the values of $\{H^{\Sigma 1}\}, \{H^{\Sigma 2}\}$ given by the equations (V.20) and (V.21), we get:

$$\begin{Bmatrix} \int_0^{r(\theta)} \langle H^{\Sigma 1} \rangle \, r \, dr \\ \int_0^{r(\theta)} \langle H^{\Sigma 2} \rangle \, r \, dr \end{Bmatrix} = \begin{bmatrix} IHr11(\theta) & IHr12(\theta) & IHr13(\theta) \\ IHr21(\theta) & IHr22(\theta) & IHr23(\theta) \end{bmatrix} = [IHr(\theta)]$$

with:

$$C_1(\theta) = \frac{[r(\theta)]^{\frac{3}{2}}}{3\sqrt{2\pi}}$$

$$IHr11(\theta) = C_1(\theta) \; cos\frac{\theta}{2} \; (2 - cos\,\theta + cos\,2\theta)$$

$$IHr12(\theta) = 2\,C_1(\theta) \; cos^3\frac{\theta}{2} \; (3 - 2\,cos\,\theta)$$

$$IHr13(\theta) = 2\,C_1(\theta) \; cos\frac{\theta}{2} \quad cos\frac{3\theta}{2} \; sin\frac{\theta}{2}$$

$$IHr21(\theta) = -\,C_1(\theta) \; sin\frac{\theta}{2} \; (4 + cos\,\theta + cos\,2\theta)$$

$$IHr22(\theta) = IHr13(\theta)$$

$$IHr23(\theta) = IHr11(\theta)$$

Then

$$\int [IHr(\theta)]\, d\theta = \begin{bmatrix} p11(\theta) & p12(\theta) & p13(\theta) \\ p21(\theta) & p22(\theta) & p23(\theta) \end{bmatrix}$$

with

$$C_2(\theta) = \frac{a\,\sqrt{r(\theta)}}{3\,q^2\,\sqrt{2\pi}}$$

$$g_1(\theta) = (3c^2 - s^2 - q\cos\theta)\sin\frac{\theta}{2}$$

$$g_2(\theta) = (c^2 + 5s^2 + q\cos\theta)\sin\frac{\theta}{2}$$

$$g_3(\theta) = (3c^2 s - 5s^3)\cos\frac{\theta}{2} + s\,q\cos\frac{3\theta}{2}$$

$$g_4(\theta) = (11c^2 s - 3s^3)\cos\frac{\theta}{2} + s\,q\cos\frac{3\theta}{2}$$

$$g_5(\theta) = (3c^3 - 5cs^2)\cos\frac{\theta}{2} + c\,q\cos\frac{3\theta}{2}$$

$$g_6(\theta) = (5c^3 + 13cs^2)\cos\frac{\theta}{2} - c\,q\cos\frac{3\theta}{2}$$

$$\begin{aligned}
p11(\theta) &= C_2(\theta)\ \left[g_3(\theta) - 2\,c\,g_2(\theta)\right]\\
p12(\theta) &= -C_2(\theta)\ \left[g_4(\theta) + 2\,c\,g_1(\theta)\right]\\
p13(\theta) &= C_2(\theta)\ \left[g_5(\theta) - 2\,s\,g_1(\theta)\right]\\
p21(\theta) &= C_2(\theta)\ \left[g_6(\theta) - 2\,s\,g_2(\theta)\right]\\
p22(\theta) &= p13(\theta)\\
p23(\theta) &= p11(\theta)
\end{aligned}$$

Finally

$$\left[IH_{OAB}\right] = \begin{bmatrix} p11(\theta_B) & p12(\theta_B) & p13(\theta_B) \\ p21(\theta_B) & p22(\theta_B) & p23(\theta_B) \end{bmatrix} - \begin{bmatrix} p11(\theta_A) & p12(\theta_A) & p13(\theta_A) \\ p21(\theta_A) & p22(\theta_A) & p23(\theta_A) \end{bmatrix}$$

A3.4. Special case: the edge *AB* crosses the axis of negative *X*

In a LFMVC, the segment *AB* will never cross the axis of negative *X* but, in chapter 6, the situation of figure A3.3 is encountered.

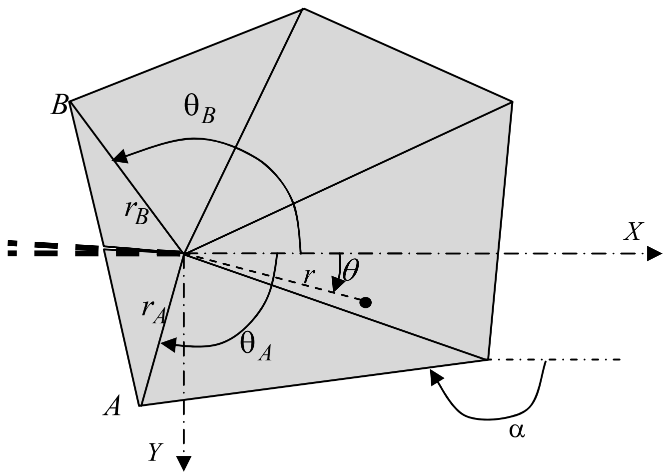

Figure A3.3. line AB crosses the negative *X* axis

In this case, the calculations of the preceding sections must be corrected to take account that θ jumps from $-\pi$ to $+\pi$ when the line *AB* crosses the crack.

Consequently, for any function $g(\theta)$, we have in this case:

$$\int_A^B g(\theta)\, d\theta = \int_{\theta_A}^{\pi} g(\theta)\, d\theta + \int_{-\pi}^{\theta_B} g(\theta)\, d\theta$$

Let

$$G(\theta) = \int g(\theta)\, d\theta$$

Then

$$\int_A^B g(\theta)\, d\theta = G(\pi) - G(\theta_A) + G(\theta_B) - G(-\pi) = \left[G(\theta_B) - G(\theta_A)\right] + \left[G(\pi) - G(-\pi)\right]$$

Hence, the previous results obtained for an edge *AB* not intersecting the axis of "negative *X*" must be corrected.

After some calculations, the correction on $\left[V_{OAB}\right]$ is found to be :

$$\text{correction on } \left[V_{OAB}\right] = \begin{bmatrix} s_{11}(\pi) & s_{12}(\pi) \\ s_{12}(\pi) & s_{22}(\pi) \end{bmatrix} - \begin{bmatrix} s_{11}(-\pi) & s_{12}(-\pi) \\ s_{12}(-\pi) & s_{22}(-\pi) \end{bmatrix} = \frac{a(-1+\nu^*)}{q\, E^*} \begin{bmatrix} c & s \\ s & -c \end{bmatrix}$$

In a similar way, the correction on $\left[IH_{OAB}\right]$ is

$$correction\ on\ \left[IH_{OAB}\right] = \begin{bmatrix} p11(\pi) & p12(\pi) & p13(\pi) \\ p21(\pi) & p22(\pi) & p23(\pi) \end{bmatrix} - \begin{bmatrix} p11(\pi) & p12(\pi) & p13(\pi) \\ p21(\pi) & p22(\pi) & p23(\pi) \end{bmatrix}$$

$$= \frac{8a\sqrt{2\frac{a}{c}}}{3\, q^2 \sqrt{\pi}} \begin{bmatrix} c\ s^2 & c^3 & c^2\ s \\ s^3 & c^2\ s & c\ s^2 \end{bmatrix}$$

Note that $c = Y_B - Y_A \neq 0$ because $c = Y_B - Y_A = 0$ would mean that the edge is parallel to *X*, that is parallel to the crack.

In such a case, the edge cannot intersect the crack.