Annex 5
Discretization of the FdV variational principle for XNEM

Content

A5.1. Discretization of the FdV variational principle

$$\Pi = \int_A W(\varepsilon_{ij})dA + \int_A \Sigma_{ij}\left[\frac{1}{2}(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}) - \varepsilon_{ij}\right]dA - \int_A F_i u_i\, dA - \int_{S_t} T_i u_i dS + \int_{S_u} r_i(\tilde{u}_i - u_i)dS$$

$$\delta\Pi = \delta\Pi_1 + \delta\Pi_2 + \delta\Pi_3 + \delta\Pi_4 + \delta\Pi_5 + \delta\Pi_6 = 0 \quad \text{(A5.1)}$$

$$\delta\Pi_1 = \sum_{I=1}^{N}\int_{A_I}(\sigma_{ij} - \Sigma_{ij})\,\delta\varepsilon_{ij}\,dA_I \quad \text{(A5.2)}$$

$$\delta\Pi_2 = \sum_{I=1}^{N}\int_{A_I}\Sigma_{ij}\left[\frac{1}{2}(\frac{\partial\delta u_i}{\partial X_j} + \frac{\partial\delta u_j}{\partial X_i})\right]dA_I \quad \text{(A5.3)}$$

$$\delta\Pi_3 = \sum_{I=1}^{N}\int_{A_I}\delta\Sigma_{ij}\left[\frac{1}{2}(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}) - \varepsilon_{ij}\right]dA_I \quad \text{(A5.4)}$$

$$\delta\Pi_4 = -\sum_{I=1}^{N}\int_{A_I}F_i\,\delta u_i\,dA_I \quad \text{(A5.5)}$$

$$\delta\Pi_5 = -\sum_{K=1}^{M_t}\int_{S_K}T_i\,\delta u_i\,dS_K \quad \text{(A5.6)}$$

$$\delta\Pi_6 = \sum_{K=1}^{M_u}\left[\int_{S_K}\delta r_i(\tilde{u}_i - u_i)dS_K - \int_{S_K}r_i\,\delta u_i\,dS_K\right] \quad \text{(A5.7)}$$

Development of $\delta\Pi_1$

$$\delta\Pi_1 = \sum_{I=1}^{N_O}\int_{A_I}(\sigma_{ij} - \Sigma_{ij})\delta\varepsilon_{ij}\,dA_I + \sum_{J\in\lambda_H}\int_{A_J}(\sigma_{ij} - \Sigma_{ij})\delta\varepsilon_{ij}\,dA_J + \sum_{L\in\lambda_C}\int_{A_L}(\sigma_{ij} - \Sigma_{ij})\delta\varepsilon_{ij}\,dA_L \quad \text{(A5.8)}$$

Let :

$$[HDH]^L = \begin{bmatrix} \int_{A_L}\langle H_1\rangle^L [D]^L \{H_1\}^L dA_L & \int_{A_L}\langle H_1\rangle^L [D]^L \{H_2\}^L dA_L \\ \int_{A_L}\langle H_2\rangle^L [D]^L \{H_1\}^L dA_L & \int_{A_L}\langle H_2\rangle^L [D]^L \{H_2\}^L dA_L \end{bmatrix} \quad \text{(A5.9)}$$

$$[IH]^L = \begin{bmatrix} \int_{A_L}\{H_1\}^L dA_L & \int_{A_L}\{H_2\}^L dA_L \end{bmatrix} \; ; \; \{H_\tau\}^L = \begin{bmatrix} \int_{A_L}\langle H_1\rangle^L [D]^L \{\tau\}^L dA_L \\ \int_{A_L}\langle H_2\rangle^L [D]^L \{\tau\}^L dA_L \end{bmatrix} \quad \text{(A5.10)}$$

$$\{K_\Sigma\}^L = \begin{Bmatrix} K_{\Sigma 1}^L \\ K_{\Sigma 2}^L \end{Bmatrix} \; ; \; \{K_\varepsilon\}^L = \begin{Bmatrix} K_{\varepsilon 1}^L \\ K_{\varepsilon 2}^L \end{Bmatrix} \; ; \; \{K_U\}^L = \begin{Bmatrix} K_{U1}^L \\ K_{U2}^L \end{Bmatrix} \quad \text{(A5.11)}$$

The rotation into the global frame of the last term of (A5.8) gives:

$$\sum_{L\in\lambda_C}\int_{A_L}(\sigma_{ij}-\Sigma_{ij})\delta\varepsilon_{ij}\,dA_I=\sum_{L\in\lambda_C}\int_{A_L}(\langle\tau\rangle^L-\langle P\rangle^L)\delta\{\gamma\}^L dA_L$$

$$=\sum_{L\in\lambda_C}\int_{A_L}(\langle\tau\rangle^L-\langle P^0\rangle^L-K_{\Sigma 1}^L\langle H_1\rangle^L-K_{\Sigma 2}^L\langle H_2\rangle^L)(\delta\{\gamma^0\}^L+\delta K_{\varepsilon 1}^L[D]^L\{H_1\}^L+\delta K_{\varepsilon 2}^L[D]^L\{H_2\})dA_L$$

$$=\sum_{L\in\lambda_C}\left[\int_{A_L}\langle\tau\rangle^L dA_L-A_L\langle P^0\rangle^L-K_{\Sigma 1}^L\int_{A_L}\langle H_1\rangle^L dA_L-K_{\Sigma 2}^L\int_{A_L}\langle H_2\rangle^L dA_L\right]\delta\{\gamma^0\}^L$$

$$+\sum_{L\in\lambda_C}\left[\begin{array}{l}\int_{A_L}\langle\tau\rangle^L[D]^L\{H_1\}^L dA_L-\langle P^0\rangle^L[D]^L\int_{A_L}\{H_1\}^L dA_L\\ -K_{\Sigma 1}^L\int_{A_L}\langle H_1\rangle^L[D]^L\{H_1\}^L dA_L-K_{\Sigma 2}^L\int_{A_L}\langle H_2\rangle^L[D]^L\{H_1\}^L dA_L\end{array}\right]\delta K_{\varepsilon 1}^L$$

$$+\sum_{L\in\lambda_C}\left[\begin{array}{l}\int_{A_L}\langle\tau\rangle^L[D]^L\{H_2\}^L dA_L-\langle P^0\rangle^L[D]^L\int_{A_L}\{H_2\}^L dA_L\\ -K_{\Sigma 1}^L\int_{A_L}\langle H_2\rangle^L[D]^L\{H_1\}^L dA_L-K_{\Sigma 2}^L\int_{A_L}\langle H_2\rangle^L[D]^L\{H_1\}^L dA_L\end{array}\right]\delta K_{\varepsilon 2}^L$$

or, using the notations (A5.9,A5.10,A5.11):

$$\begin{aligned}&\sum_{L\in\lambda_C}\int_{A_L}(\sigma_{ij}-\Sigma_{ij})\delta\varepsilon_{ij}\,dA_I=\\&\sum_{L\in\lambda_C}\langle\delta\gamma^0\rangle^L\left[\int_{A_L}\{\tau\}^L dA_L-A_L\{P^0\}^L-[IH]^L\{K_\Sigma\}^L\right]\\&+\sum_{L\in\lambda_C}\langle\delta K_\varepsilon\rangle^L\left[\{H_\tau\}^L-[IH]^{L,T}[D]^L\{P^0\}^L\right]\\&-\sum_{L\in\lambda_C}\langle\delta K_\varepsilon\rangle^L[HDH]^L\{K_\Sigma\}^L\end{aligned}\qquad\text{(A5.12)}$$

Recalling (II.34):

$$\sigma_{ij}=\frac{\partial W(\varepsilon_{ij})}{\partial\varepsilon_{ij}}\qquad\text{(II.34)}$$

On the other hand, in cells of type *O*, as a consequence of (II.34) and (VI.7), the constitutive stresses are constant.

Similarly, in cells of type *H*, as a consequence of (II.34) and (VI.10), the constitutive stresses are constant in parts *A* and *B*.

Consequently, the discretized form of $\delta\Pi_1$ is:

$$\begin{aligned}\delta\Pi_1 &= \sum_{I=1}^{N_O}\langle\delta\varepsilon\rangle^I A_I\left[\{\sigma\}^I-\{\Sigma\}^I\right]\\ &+\sum_{J\in\lambda_H}\left\{\langle\delta\varepsilon\rangle^{A,J}A_{A,J}\left[\{\sigma\}^{A,J}-\{\Sigma\}^{A,J}\right]+\langle\delta\varepsilon\rangle^{B,J}A_{B,I}\left[\{\sigma\}^{B,J}-\{\Sigma\}^{B,J}\right]\right\}\\ &+\sum_{L\in\lambda_C}\langle\delta\gamma^0\rangle^L\left[\int_{A_L}\{\tau\}^L dA_L-A_L\{P^0\}^L-[IH]^L\{K_\Sigma\}^L\right]\\ &+\sum_{L\in\lambda_C}\langle\delta K_\varepsilon\rangle^L\left[\{H_\tau\}^L-[IH]^{L,T}[D]^L\{P^0\}^L-[HDH]^L\{K_\Sigma\}^L\right]\end{aligned} \quad (A5.13)$$

with the notations:

$$\{\Sigma\}^I=\begin{Bmatrix}\Sigma_{11}^I\\ \Sigma_{22}^I\\ \Sigma_{12}^I\end{Bmatrix};\ \{\sigma\}^I=\begin{Bmatrix}\sigma_{11}^I\\ \sigma_{22}^I\\ \sigma_{12}^I\end{Bmatrix};\ \{\varepsilon\}^I=\begin{Bmatrix}\varepsilon_{11}^I\\ \varepsilon_{22}^I\\ 2\varepsilon_{12}^I\end{Bmatrix} \quad \text{(in global frame)} \quad (A5.14)$$

$$\{\gamma\}^L=\begin{Bmatrix}\gamma_{11}^L\\ \gamma_{22}^L\\ \gamma\varepsilon_{12}^L\end{Bmatrix};\{\gamma^0\}^L=\begin{Bmatrix}\gamma_{11}^{0L}\\ \gamma_{22}^{0L}\\ 2\gamma_{12}^{0L}\end{Bmatrix};\ \{P^0\}^L=\begin{Bmatrix}P_{11}^{0L}\\ P_{22}^{0L}\\ P_{12}^{0L}\end{Bmatrix} \quad \text{(in local frame)} \quad (A5.15)$$

Development of $\delta\Pi_2$

$$\begin{aligned}\delta\Pi_2 &= \sum_{I=1}^{N_O}\Sigma_{ij}^I\int_{A_I}\left[\frac{1}{2}\left(\frac{\partial\delta u_i}{\partial X_j}+\frac{\partial\delta u_j}{\partial X_i}\right)\right]dA_I\\ &+\sum_{J\in\lambda_H}\Sigma_{ij}^{A,J}\int_{A_J}\left[\frac{1}{2}\left(\frac{\partial\delta u_i}{\partial X_j}+\frac{\partial\delta u_j}{\partial X_i}\right)\right]dA_J+\sum_{J\in\lambda_H}\Sigma_{ij}^{B,J}\int_{A_J}\left[\frac{1}{2}\left(\frac{\partial\delta u_i}{\partial X_j}+\frac{\partial\delta u_j}{\partial X_i}\right)\right]dA_J\\ &+\sum_{L\in\lambda_C}\int_{A_L}\Sigma_{ij}^L\left[\frac{1}{2}\left(\frac{\partial\delta u_i}{\partial X_j}+\frac{\partial\delta u_j}{\partial X_i}\right)\right]dA_L\end{aligned} \quad (A5.16)$$

where the Σ_{ij}^L of the last term are given by (VI.15). Since the stresses in (VI.15) are given in the local frame $(Y_1,Y_2)^L$ attached to the crack tip L, they must be rotated to be expressed in the global frame (X_1,X_2) because the displacements are expressed in this global frame.

Another approach consists in using the local frame $(Y_1,Y_2)^L$ to express the displacements in the last term of (A5.13).

Let v_i^L be the components of the displacements in the local frame.

Integrating by part, the last term of (A5.16) becomes:

$$\int_{A_L}\Sigma_{ij}^L\left[\frac{1}{2}(\frac{\partial\delta u_i}{\partial X_j}+\frac{\partial\delta u_j}{\partial X_i})\right]dA_L=\int_{A_L}P_{ij}^L\left[\frac{1}{2}(\frac{\partial\delta v_i^L}{\partial Y_j^L}+\frac{\partial\delta v_j^L}{\partial Y_i^L})\right]dA_L$$
$$=\int_{C_L}M_j^L\;P_{ij}^L\,\delta v_i^L\;dC_L+\int_{A_L}\frac{\partial P_{ij}^L}{\partial Y_j^L}\,\delta v_i^L\;dA_L \qquad \text{(A5.17)}$$

where C_L is the contour of the cell and M_j^L are the components of the outward normal to C_L expressed in the local frame $(Y_1,Y_2)^L$

However, since $\{P\}^L=\{P^O\}^L+\{P^\Sigma\}^L$ where $\{P^O\}^L$ is constant and $\{P^\Sigma\}^L$ is an exact solution of the Theory of Elasticity, which satisfy the equilibrium equations, we have, in absence of body force,

$$\frac{\partial P_{ij}^L}{\partial Y_j^L}=0$$

so that the last term of (A5.17) vanishes.

The other terms of (A5.16) are also integrated by parts. This gives:

$$\delta\Pi_2=\delta\Pi_2^\Phi+\delta\Pi_2^H+\delta\Pi_2^V \qquad \text{(A5.18)}$$

with

$$\delta\Pi_2^\Phi=\sum_{I=1}^{N_O}\Sigma_{ij}^I\int_{C_I}N_j^I\;\delta u_i\;dC_I \qquad \text{(A5.19)}$$

$$\delta\Pi_2^H=\sum_{J\in\lambda_H}\Sigma_{ij}^{A,J}\int_{C_{A,J}}N_j^{A,J}\;\delta u_i\;dC_{A,J}+\sum_{J\in\lambda_H}\Sigma_{ij}^{B,J}\int_{C_{B,I}}N_j^{B,J}\;\delta u_i\;dC_{B,J} \qquad \text{(A5.20)}$$

$$\delta\Pi_2^V=\sum_{L\in\lambda_C}\int_{C_L}M_j^L\;P_{ij}^L\,\delta v_i\;\;dC_L \qquad \text{(A5.21)}$$

The assumed displacements are given by (VI.22):

$$u_i=\sum_{J=1}^{N}\Phi_J\,u_i^J+\sum_{J\in\Lambda}C(\underline{X})\;\Phi_J\;a_i^J$$

Their variations are given in the global frame (X_1,X_2) by:

$$\delta u_i=\delta u_i^\Phi+\delta u_i^C \qquad \text{(A5.22)}$$

with

$$\delta u_i^\Phi=\sum_{J=1}^{N}\Phi_J\,\delta u_i^J \qquad \text{(A5.23-a)}$$

$$\delta u_i^C=\sum_{J\in\Lambda}C(\underline{X})\;\Phi_J\;\delta a_i^J \qquad \text{(A5.23-b)}$$

They can also be expressed in a local frame (Y_1,Y_2) by:

$$\delta v_i=\delta v_i^\Phi+\delta v_i^C \qquad \text{(A5.24)}$$

with

$$\delta v_i^{\Phi} = \sum_{J=1}^{N} \Phi_J \, \delta v_i^J \tag{A5.25-a}$$

$$\delta v_i^{C} = \sum_{J \in \Lambda} C(\underline{X}) \, \Phi_J \, \delta b_i^J \tag{A5.25-b}$$

For the calculation of $\delta\Pi_2^{\Phi}$, we have:

$$\sum_{I=1}^{N_O} \Sigma_{ij}^I \int_{C_I} N_j^I \, \delta u_i^{\Phi} \, dC_I = \sum_{I=1}^{N_O} \sum_{J=1}^{N} \langle \delta u \rangle^J [A]^{IJ} \{\Sigma\}^I \tag{A5.26-a}$$

$$\sum_{I=1}^{N_O} \Sigma_{ij}^I \int_{C_I} N_j^I \, \delta u_i^{C} \, dC_I = \sum_{I=1}^{N_O} \sum_{J \in \Lambda} \langle \delta a \rangle^J \, [A_C]^{IJ} \{\Sigma\}^I \tag{A5.26-b}$$

where

$$\{\Sigma\}^I = \begin{Bmatrix} \Sigma_{11}^I \\ \Sigma_{22}^I \\ \Sigma_{12}^I \end{Bmatrix} \; ; \; \{u\}^I = \begin{Bmatrix} u_1^I \\ u_2^I \end{Bmatrix} \; ; \; \{a\}^I = \begin{Bmatrix} a_1^I \\ a_2^I \end{Bmatrix} ; \tag{A5.27}$$

$$[A]^{IJ} = \begin{bmatrix} A_1^{IJ} & 0 & A_2^{IJ} \\ 0 & A_2^{IJ} & A_1^{IJ} \end{bmatrix} \; ; \quad [A_C]^{IJ} = \begin{bmatrix} A_{C1}^{IJ} & 0 & A_{C2}^{IJ} \\ 0 & A_{C2}^{IJ} & A_{C1}^{IJ} \end{bmatrix} \tag{A5.28}$$

$$A_j^{IJ} = \oint_{C_I} N_j^I \, \Phi_J \, dC_I \quad ; \quad A_{Cj}^{IJ} = \oint_{C_I} N_j^I \, C(X) \, \Phi_J \, dC_I \tag{A5.29}$$

Finally, $\delta\Pi_2^{\Phi}$ becomes:

$$\delta\Pi_2^{\Phi} = \sum_{I=1}^{N_O} \sum_{J=1}^{N} \langle \delta u \rangle^J [A]^{IJ} \{\Sigma\}^I + \sum_{I=1}^{N_O} \sum_{J \in \Lambda} \langle \delta a \rangle^J \, [A_C]^{IJ} \{\Sigma\}^I \tag{A5.30}$$

$\delta\Pi_2^{H}$ is calculated in the same way:

$$\begin{aligned} \delta\Pi_2^{H} = & \sum_{I \in \lambda_H} \sum_{J=1}^{N} \langle \delta u \rangle^J [A_A]^{IJ} \{\Sigma\}^{A,I} + \sum_{I \in \lambda_H} \sum_{J=1}^{N} \langle \delta u \rangle^J [A_B]^{IJ} \{\Sigma\}^{B,I} \\ & + \sum_{I \in \lambda_H} \sum_{J \in \Lambda} \langle \delta a \rangle^J [A_{C,A}]^{IJ} \{\Sigma\}^{A,I} + \sum_{I \in \lambda_H} \sum_{J \in \Lambda} \langle \delta a \rangle^J [A_{C,B}]^{IJ} \{\Sigma\}^{B,I} \end{aligned} \tag{A5.31}$$

where the subcripts *A* and *B* refer to the 2 parts of the cell of type *H*.
In the calculation of contour integrals of the type $\int_{C_{A,I}} \bullet \, dC_{A,I}$ or $\int_{C_{B,I}} \bullet \, dC_{B,I}$, the contours include the crack lips (figure A5.1).

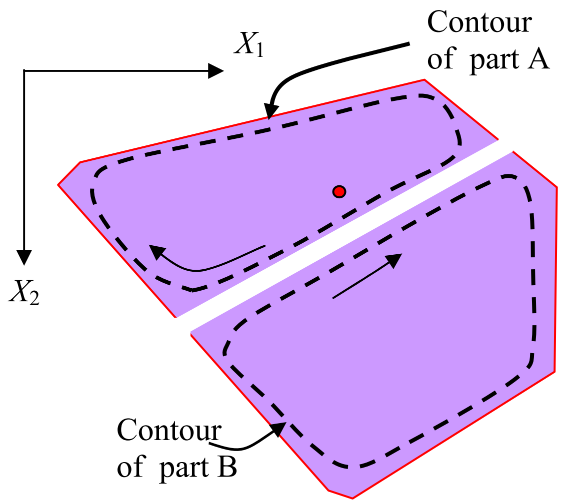

Figure A5.1. Cell of type H

For $\delta\Pi_2^V$, we have:

$$\delta\Pi_2^V = \sum_{L\in\lambda_C} \int_{C_L} \delta v_i \; P_{ij}^L M_j^L \, dC_L = \sum_{L\in\lambda_C} \int_{C_L} (\delta v_i^\Phi + \delta v_i^C) \, P_{ij}^L M_j^L \, dC_L \qquad \text{(A5.32)}$$

with P_{ij}^L given by:

$$\{P\}^L = \{P^O\}^L + \{P^\Sigma\}^L = \{P^O\}^L + K_{\Sigma 1}^L \{H_1\}^L + K_{\Sigma 2}^L \{H_2\}^L \;,\; L \in \lambda_C \qquad \text{(A5.33)}$$

The 1st term of $\delta\Pi_2^V$ writes:

$$\sum_{L\in\lambda_C} \int_{C_L} \delta v_i^\Phi \; P_{ij}^L M_j^L \, dC_L = \sum_{L\in\lambda_C} \sum_{J=1}^{N} \; \delta v_i^J \int_{C_L} \Phi_J \; P_{ij}^L M_j^L \, dC_L \qquad \text{(A5.34)}$$

Let

$$[M]^L = \begin{bmatrix} M_1^L & 0 & M_2^L \\ 0 & M_2^L & M_1^L \end{bmatrix} \qquad \text{(A5.35)}$$

$$[\eta]^{JL} = \Phi_J \begin{bmatrix} M_1^L & 0 & M_2^L \\ 0 & M_2^L & M_1^L \end{bmatrix} = \Phi_J \, [M]^L \quad ; \quad [W]^{JL} = \int_{C_L} [\eta]^{JL} dC_L \qquad \text{(A5.36)}$$

$$\{\eta\}_1^{JL} = \Phi_J \begin{bmatrix} M_1^L & 0 & M_2^L \\ 0 & M_2^L & M_1^L \end{bmatrix} \begin{Bmatrix} H_1^1 \\ H_1^2 \\ H_1^3 \end{Bmatrix}^L = \Phi_J \, [M]^L \{H_1\}^L \quad ; \quad \{V\}_1^{JL} = \int_{C_L} \{\eta\}_1^{JL} \, dC_L \qquad \text{(A5.37)}$$

$$\{\eta\}_2^{JL} = \Phi_J \begin{bmatrix} M_1^L & 0 & M_2^L \\ 0 & M_2^L & M_1^L \end{bmatrix} \begin{Bmatrix} H_2^1 \\ H_2^2 \\ H_2^3 \end{Bmatrix} = \Phi_J \, [M]^L \{H_2\}^L \quad ; \quad \{V\}_2^{JL} = \int_{C_L} \{\eta\}_2^{JL} \, dC_L \qquad \text{(A5.38)}$$

$$\left[V\right]^{JL} = \left[\{V\}_1^{JL} \quad \{V\}_2^{JL}\right] \; ; \quad \{v\}^{J,L} = \begin{Bmatrix} v_1^{J,L} \\ v_2^{J,L} \end{Bmatrix} \tag{A5.39}$$

Then, the term $\delta v_i^{J,L}\,\Phi_J\,P_{ij}^L M_j^L$ can be written in matrix form

$$\begin{aligned}
\delta v_i^{J,L}\,\Phi_J\,P_{ij}^L M_j^L &= \left\langle \delta v_1^{J,L} \quad \delta v_2^{J,L} \right\rangle \Phi_J \begin{bmatrix} M_1 & 0 & M_2 \\ 0 & M_2 & M_1 \end{bmatrix}^L \begin{Bmatrix} P_{11} \\ P_{22} \\ P_{12} \end{Bmatrix}^L = \left\langle \delta v \right\rangle^{J,L} \Phi_J\,\left[M\right]^L \{P\}^L \\
&= \left\langle \delta v \right\rangle^{J,L} \Phi_J\,\left[M\right]^L \left[\{P^0\}^L + K_{\Sigma 1}^L \{H_1\}^L + K_{\Sigma 2}^L \{H_2\}^L\right] \\
&= \left\langle \delta v \right\rangle^{J,L} \left[\Phi_J\,\left[M\right]^L \{P^0\}^L + K_{\Sigma 1}^L \Phi_J\,\left[M\right]^L \{H_1\}^L + K_{\Sigma 2}^L \Phi_J\,\left[M\right]^L \{H_2\}^L\right] \\
&= \left\langle \delta v \right\rangle^{J,L} \left[\left[\eta\right]^{JL} \{P^0\}^L + K_{\Sigma 1}^L \{\eta\}_1^{JL} + K_{\Sigma 2}^L \{\eta\}_1^{JL}\right]
\end{aligned} \tag{A5.40}$$

so that:

$$\begin{aligned}
\sum_{L\in\lambda_C} \int_{C_L} \delta v_i^{\Phi,L}\, P_{ij}^L M_j^L\, dC_L &= \sum_{L\in\lambda_C} \sum_{J=1}^{N} \left\langle \delta v \right\rangle^{J,L} \left[\left[W\right]^{JL} \{P^0\}^L + K_{\Sigma 1}^L\, \{V\}_1^{JL} + K_{\Sigma 2}^L\, \{V\}_2^{JL}\right] \\
&= \sum_{L\in\lambda_C} \sum_{J=1}^{N} \left\langle \delta v \right\rangle^{J,L} \left[\left[W\right]^{JL} \{P^0\}^L + \left[V\right]^{JL} \{K_\Sigma\}^L\right]
\end{aligned} \tag{A5.41}$$

In a similar way, we have, for the 2nd term of $\delta\Pi_2^V$:

$$\sum_{L\in\lambda_C} \int_{C_L} \delta v_i^{C,L}\, P_{ij}^L M_j^L\, dC_L = \sum_{L\in\lambda_C} \sum_{J\in\Lambda} \delta b_i^{J,L} \int_{C_L} C(\underline{X})\,\Phi_J\, P_{ij}^L M_j^L\, dC_L \tag{A5.42}$$

which can be written in the following matrix form:

$$\begin{aligned}
\sum_{L\in\lambda_C} \int_{C_L} \delta v_i^{\Phi,L}\, P_{ij}^L M_j^L\, dC_L &= \sum_{L\in\lambda_C} \sum_{J\in\Lambda} \left\langle \delta b \right\rangle^{J,L} \left[\left[W_C\right]^{JL} \{P^0\}^L + K_{\Sigma 1}^L\, \{V_C\}_1^{JL} + K_{\Sigma 2}^L\, \{V_C\}_2^{JL}\right] \\
&= \sum_{L\in\lambda_C} \sum_{J\in\Lambda} \left\langle \delta b \right\rangle^{J,L} \left[\left[W_C\right]^{JL} \{P^0\}^L + \left[V_C\right]^{JL} \{K_\Sigma\}^L\right]
\end{aligned} \tag{A5.43}$$

with

$$\left[\eta_C\right]^{JL} = C(\underline{X})\,\Phi_J \begin{bmatrix} M_1 & 0 & M_2 \\ 0 & M_2 & M_1 \end{bmatrix}^L = C(\underline{X})\,\Phi_J\,\left[M\right]^L \quad ; \quad \left[W_C\right]^{JL} = \int_{C_L} \left[\eta\right]^{JL} dC_L \tag{A5.44}$$

$$\{\eta_C\}_1^{JL} = C(\underline{X})\,\Phi_J \begin{bmatrix} M_1 & 0 & M_2 \\ 0 & M_2 & M_1 \end{bmatrix}^L \begin{Bmatrix} H_1^1 \\ H_1^2 \\ H_1^3 \end{Bmatrix}^L = C(\underline{X})\,\Phi_J\,\left[M\right]^L \{H_1\}^L \tag{A5.45}$$

$$\{\eta_C\}_2^{JL} = C(\underline{X})\,\Phi_J \begin{bmatrix} M_1 & 0 & M_2 \\ 0 & M_2 & M_1 \end{bmatrix}^L \begin{Bmatrix} H_2^1 \\ H_2^2 \\ H_2^3 \end{Bmatrix}^L = C(\underline{X})\,\Phi_J \,[M]^L \{H_2\}^L \qquad \text{(A5.46)}$$

$$\{V_C\}_1^{JL} = \int_{C_L} \{\eta_C\}_1^{JL}\, dC_L \;;\; \{V_C\}_2^{JL} = \int_{C_L} \{\eta_C\}_2^{JL}\, dC_L \;;\; [V_C]^{JL} = \left[\{V_C\}_1^{JL} \quad \{V_C\}_2^{JL}\right] \qquad \text{(A5.47)}$$

$$\{b\}^{J,L} = \begin{Bmatrix} b_1^{J,L} \\ b_2^{J,L} \end{Bmatrix} \qquad \text{(A5.48)}$$

Finally,

$$\begin{aligned} \delta\Pi_2^V = & \sum_{L\in\lambda_C}\sum_{J=1}^{N} \langle \delta v\rangle^{J,L}\left[[W]^{JL}\{P^0\}^L + [V]^{JL}\{K_\Sigma\}^L\right] \\ & + \sum_{L\in\lambda_C}\sum_{J\in\Lambda} \langle \delta b\rangle^{J,L}\left[[W_C]^{JL}\{P^0\}^L + [V_C]^{JL}\{K_\Sigma\}^L\right] \end{aligned} \qquad \text{(A5.49)}$$

Summarizing, we get:

$$\begin{aligned} \delta\Pi_2 = & \sum_{I=1}^{N_O}\sum_{J=1}^{N} \langle \delta u\rangle^J [A]^{IJ}\{\Sigma\}^I + \sum_{I=1}^{N_O}\sum_{J\in\Lambda} \langle \delta a\rangle^J\,[A_C]^{IJ}\{\Sigma\}^I \\ & + \sum_{I\in\lambda_H}\sum_{J=1}^{N} \langle \delta u\rangle^J [A_A]^{IJ}\{\Sigma\}^{A,I} + \sum_{I\in\Lambda}\sum_{J=1}^{N} \langle \delta u\rangle^J [A_B]^{IJ}\{\Sigma\}^{B,I} \\ & + \sum_{I\in\lambda_H}\sum_{J\in\Lambda} \langle \delta a\rangle^J [A_{C,A}]^{IJ}\{\Sigma\}^{A,I} + \sum_{I\in\Lambda}\sum_{J\in\Lambda} \langle \delta a\rangle^J [A_{C,B}]^{IJ}\{\Sigma\}^{B,I} \\ & + \sum_{L\in\lambda_C}\sum_{J=1}^{N} \langle \delta v\rangle^{J,L}\left[[W]^{JL}\{P^0\}^L + [V]^{JL}\{K_\Sigma\}^L\right] \\ & + \sum_{L\in\lambda_C}\sum_{J\in\Lambda} \langle \delta b\rangle^{J,L}\left[[W_C]^{JL}\{P^0\}^L + [V_C]^{JL}\{K_\Sigma\}^L\right] \end{aligned} \qquad \text{(A5.50)}$$

Development of $\delta\Pi_3$

$$\delta\Pi_3 = \sum_{I=1}^{N}\int_{A_I} \delta\Sigma_{ij}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)\right] dA_I - \sum_{I=1}^{N}\int_{A_I} \delta\Sigma_{ij}\,\varepsilon_{ij}\, dA_I$$

The first term is obtained by a calculation that is perfectly similar to that of $\delta\Pi_2$.

For the second term, we have:

$$\sum_{I=1}^{N}\int_{A_I} \delta\Sigma_{ij}\,\varepsilon_{ij}\, dA_I = \sum_{I=1}^{N_O}\int_{A_I} \delta\Sigma_{ij}\,\varepsilon_{ij}\, dA_I + \sum_{J\in\lambda_H}\int_{A_J} \delta\Sigma_{ij}\,\varepsilon_{ij}\, dA_J + \sum_{L\in\lambda_C}\int_{A_L} \delta\Sigma_{ij}\,\varepsilon_{ij}\, dA_L \qquad \text{(A5.51)}$$

$$\sum_{I=1}^{N_O}\int_{A_I} \delta\Sigma_{ij}\,\varepsilon_{ij}\, dA_I = \sum_{I=1}^{N_O} A_I \langle \varepsilon\rangle^I \delta\{\Sigma\}^I \qquad \text{(A5.52)}$$

$$\sum_{J\in\lambda_H}\int_{A_J}\delta\Sigma_{ij}\,\varepsilon_{ij}\,dA_J=\sum_{J\in\lambda_H}\left\{A_{A,J}\,\langle\varepsilon\rangle^{A,J}\,\delta\{\Sigma\}^{A,J}+A_{B,J}\,\langle\varepsilon\rangle^{B,J}\,\delta\{\Sigma\}^{B,J}\right\} \tag{A5.53}$$

$$\begin{aligned}\sum_{L\in\lambda_C}\int_{A_L}\delta\Sigma_{ij}\,\varepsilon_{ij}\,dA_L&=\sum_{L\in\lambda_C}\int_{A_L}\delta P_{ij}\,\gamma_{ij}\,dA_L=\sum_{L\in\lambda_C}\int_{A_L}\langle\gamma\rangle^L\{\delta P\}^L\,dA_L\\&=\sum_{L\in\lambda_C}\int_{A_L}\left\langle\left\langle\gamma^0\right\rangle^L+\left[K^L_{\varepsilon1}\langle H_1\rangle^L+K^L_{\varepsilon2}\langle H_2\rangle^L\right][D]^L\right\rangle\left\{\left\{\delta P^0\right\}^L+\delta K^L_{\Sigma1}\{H_1\}^L+\delta K^L_{\Sigma2}\{H_2\}\right\}dA_L\end{aligned} \tag{A5.54}$$

Using the notations $[HDH]^L$ and $[IH]^L$ introduced in (A5.9) and (A5.10), we get

$$\begin{aligned}\sum_{L\in\lambda_C}\int_{A_L}\delta\Sigma_{ij}\,\varepsilon_{ij}\,dA_L&=\sum_{L\in\lambda_C}A_L\left\langle\gamma^0\right\rangle^L\left\{\delta P^0\right\}^L\\&+\sum_{L\in\lambda_C}\left\langle\gamma^0\right\rangle^L[IH]^L\{\delta K_\Sigma\}^L+\sum_{L\in\lambda_C}\left\langle\delta P^0\right\rangle^L[D]^L[IH]^L\{K_\varepsilon\}^L\\&+\sum_{L\in\lambda_C}\langle\delta K_\Sigma\rangle^L[HDH]^L\{K_\varepsilon\}^L\end{aligned} \tag{A5.55}$$

Finally, we obtain:

$$\boxed{\begin{aligned}\delta\Pi_3&=\sum_{I=1}^{N_O}\sum_{J=1}^{N}\langle u\rangle^J[A]^{IJ}\{\delta\Sigma\}^I+\sum_{I=1}^{N_O}\sum_{J\in\Lambda}\langle a\rangle^J\,[A_C]^{IJ}\{\delta\Sigma\}^I\\&+\sum_{I\in\lambda_H}\sum_{J=1}^{N}\langle u\rangle^J[A_A]^{IJ}\{\delta\Sigma\}^{A,I}+\sum_{I\in\Lambda}\sum_{J=1}^{N}\langle u\rangle^J[A_B]^{IJ}\{\delta\Sigma\}^{B,I}\\&+\sum_{I\in\lambda_H}\sum_{J\in\Lambda}\langle a\rangle^J\left[A_{C,A}\right]^{IJ}\{\delta\Sigma\}^{A,I}+\sum_{I\in\Lambda}\sum_{J\in\Lambda}\langle a\rangle^J\left[A_{C,B}\right]^{IJ}\{\delta\Sigma\}^{B,I}\\&+\sum_{L\in\lambda_C}\sum_{J=1}^{N}\langle v\rangle^{J,L}\left[[W]^{JL}\left\{\delta P^0\right\}^L+[V]^{JL}\{\delta K_\Sigma\}^L\right]\\&+\sum_{L\in\lambda_C}\sum_{J\in\Lambda}\langle b\rangle^{J,L}\left[[W_C]^{JL}\left\{\delta P^0\right\}^L+[V_C]^{JL}\{\delta K_\Sigma\}^L\right]\\&-\sum_{I=1}^{N_O}A_I\langle\varepsilon\rangle^I\{\delta\Sigma\}^I-\sum_{J\in\lambda_H}\left\{A_{A,J}\,\langle\varepsilon\rangle^{A,J}\{\delta\Sigma\}^{A,J}+A_{B,J}\,\langle\varepsilon\rangle^{B,J}\{\delta\Sigma\}^{B,J}\right\}\\&-\sum_{L\in\lambda_C}A_L\left\langle\gamma^0\right\rangle^L\left\{\delta P^0\right\}^L\\&-\sum_{L\in\lambda_C}\left\langle\gamma^0\right\rangle^L[IH]^L\{\delta K_\Sigma\}^L-\sum_{L\in\lambda_C}\left\langle\delta P^0\right\rangle^L[D]^L[IH]^L\{K_\varepsilon\}^L\\&-\sum_{L\in\lambda_C}\langle\delta K_\Sigma\rangle^L[HDH]^L\{K_\varepsilon\}^L\end{aligned}} \tag{A5.56}$$

Development of $\delta\Pi_4$

$$\delta\Pi_4 = -\sum_{I=1}^{N} \int_{A_I} F_i \, \delta u_i \, dA_I = -\sum_{I=1}^{N} \int_{A_I} F_i \, \delta u_i^{\Phi} \, dA_I - \sum_{I=1}^{N} \int_{A_I} F_i \, \delta u_i^{C} \, dA_I \tag{A5.57}$$

$$\delta\Pi_4 = -\sum_{I=1}^{N} \int_{A_I} F_i \left[\sum_{J=1}^{N} \Phi_J \, \delta u_i^J) \right] dA_I - \sum_{I=1}^{N} \int_{A_I} F_i \left[\sum_{J\in\Lambda} C(\underline{X}) \, \Phi_J \;\; \delta a_i^J \right] dA_I \tag{A5.58}$$

Let

$$\widetilde{F}_i^{IJ} = \int_{A_I} F_i \Phi_J dA_I \;\; ; \;\; \{\widetilde{F}\}^{IJ} = \begin{Bmatrix} \widetilde{F}_1^{IJ} \\ \widetilde{F}_2^{IJ} \end{Bmatrix} \tag{A5.59}$$

$$\widetilde{F}_{C,i}^{IJ} = \int_{A_I} C(\underline{X}) \, F_i \;\; \Phi_J dA_I \;\; ; \;\; \{\widetilde{F}_C\}^{IJ} = \begin{Bmatrix} \widetilde{F}_{C1}^{IJ} \\ \widetilde{F}_{C2}^{IJ} \end{Bmatrix} \tag{A5.60}$$

We get:

$$\delta\Pi_4 = -\sum_{I=1}^{N} \sum_{J=1}^{N} \langle \delta u \rangle^J \{\widetilde{F}\}^{IJ} - \sum_{I=1}^{N} \sum_{J\in\Lambda} \langle \delta a \rangle^J \{\widetilde{F}_C\}^{IJ} \tag{A5.61}$$

Development of $\delta\Pi_5$

$$\delta\Pi_5 = -\sum_{K=1}^{M_t} \int_{S_K} T_i \, \delta u_i \, dS_K = -\sum_{K=1}^{M_t} \int_{S_K} T_i \, \delta u_i^{\Phi} \, dS_K - \sum_{K=1}^{M_t} \int_{S_K} T_i \, \delta u_i^{C} \, dS_K \tag{A5.62}$$

Let

$$\widetilde{T}_i^{KJ} = \int_{S_K} T_i \Phi_J dS_K \;\; ; \;\; \{\widetilde{T}\}^{KJ} = \begin{Bmatrix} \widetilde{T}_1^{KJ} \\ \widetilde{T}_2^{KJ} \end{Bmatrix} \tag{A5.63}$$

$$\widetilde{T}_{C,i}^{KJ} = \int_{S_K} C(\underline{X}) \, T_i \;\; \Phi_J dS_K \;\; ; \;\; \{\widetilde{T}_C\}^{KJ} = \begin{Bmatrix} \widetilde{T}_{C1}^{KJ} \\ \widetilde{T}_{C2}^{KJ} \end{Bmatrix} \tag{A5.64}$$

We get:

$$\delta\Pi_5 = -\sum_{K=1}^{M_t} \sum_{J=1}^{N} \langle \delta u \rangle^J \{\widetilde{T}\}^{KJ} - \sum_{K=1}^{M_t} \sum_{J\in\Lambda} \langle \delta a \rangle^J \{\widetilde{T}_C\}^{KJ} \tag{A5.65}$$

Development of $\delta\Pi_6$

$$\delta\Pi_6 = \sum_{K=1}^{M_u} \left[\int_{S_K} \delta r_i (\widetilde{u}_i - u_i) \, dS_K - \int_{S_K} r_i \, \delta u_i \, dS_K \right]$$

$$\delta\Pi_6 = \sum_{K=1}^{M_u^O} \delta r_i^K \int_{S_K} \tilde{u}_i dS_K + \sum_{K=1}^{M_u^A} \delta r_i^{A(K)} \int_{S_K} \tilde{u}_i^{A(K)} dS_K + \sum_{K=1}^{M_u^B} \delta r_i^{B(K)} \int_{S_K} \tilde{u}_i^{B(K)} dS_K$$

$$- \sum_{K=1}^{M_u^O} \delta r_i^K \left\{ \sum_{J=1}^{N} u_i^J \int_{s_K} \Phi_J dS_K + \sum_{J\in\Lambda} a_i^J \int_{s_K} C(\underline{X}) \Phi_J dS_K \right\}$$

$$- \sum_{K=1}^{M_u^A} \delta r_i^{A(K)} \left\{ \sum_{J=1}^{N} u_i^J \int_{s_K} \Phi_J dS_K + \sum_{J\in\Lambda} a_i^J \int_{s_K} C(\underline{X}) \Phi_J dS_K \right\}$$

$$- \sum_{K=1}^{M_u^B} \delta r_i^{B(K)} \left\{ \sum_{J=1}^{N} u_i^J \int_{s_K} \Phi_J dS_K + \sum_{J\in\Lambda} a_i^J \int_{s_K} C(\underline{X}) \Phi_J dS_K \right\}$$

$$- \sum_{K=1}^{M_u^O} r_i^K \left\{ \sum_{J=1}^{N} \delta u_i^J \int_{s_K} \Phi_J dS_K + \sum_{J\in\Lambda} \delta a_i^J \int_{s_K} C(\underline{X}) \Phi_J dS_K \right\}$$

$$- \sum_{K=1}^{M_u^A} r_i^{A(K)} \left\{ \sum_{J=1}^{N} \delta u_i^J \int_{s_K} \Phi_J dS_K + \sum_{J\in\Lambda} \delta a_i^J \int_{s_K} C(\underline{X}) \Phi_J dS_K \right\}$$

$$- \sum_{K=1}^{M_u^B} r_i^{B(K)} \left\{ \sum_{J=1}^{N} \delta u_i^J \int_{s_K} \Phi_J dS_K + \sum_{J\in\Lambda} \delta a_i^J \int_{s_K} C(\underline{X}) \Phi_J dS_K \right\} \quad \text{(A5.66)}$$

where $A(K)$ and $B(K)$ are the parts *A* and *B* of an edge *K* cut by the crack.

M_u^O is the number of edges not cut by the crack on which displacements are imposed.

M_u^A and M_u^B are the numbers of parts *A* and *B* of edges cut by the crack on which displacements are imposed.

Let

$$\tilde{U}_i^K = \int_{S_K} \tilde{u}_i dS_K \;;\quad \tilde{U}_i^{A(K)} = \int_{S_K} \tilde{u}_i^A dS_K \;;\quad \tilde{U}_i^{B(K)} = \int_{S_K} \tilde{u}_i^{B(K)} dS_K \quad \text{(A5.67)}$$

$$\{\tilde{U}\}^K = \begin{Bmatrix} \tilde{U}_1^K \\ \tilde{U}_2^K \end{Bmatrix} \;;\; \{\tilde{U}\}^{A(K)} = \begin{Bmatrix} \tilde{U}_1^{A(K)} \\ \tilde{U}_2^{A(K)} \end{Bmatrix} \;;\; \{\tilde{U}\}^{B(K)} = \begin{Bmatrix} \tilde{U}_1^{B(K)} \\ \tilde{U}_2^{B(K)} \end{Bmatrix} \quad \text{(A5.68)}$$

$$B^{KJ} = \int_{S_K} \Phi_J dS_K \;;\; B_C^{KJ} = \int_{S_K} C(\underline{X}) \Phi_J dS_K \quad \text{(A5.69)}$$

$$\{r\}^K = \begin{Bmatrix} r_1^K \\ r_2^K \end{Bmatrix} \quad \text{(A5.70)}$$

$$\delta\Pi_6 = \sum_{K=1}^{M_u^O} \delta r_i^K \widetilde{U}_i^K + \sum_{K=1}^{M_u^A} \delta r_i^{A(K)} \widetilde{U}_i^{A(K)} + \sum_{K=1}^{M_u^B} \delta r_i^{B(K)} \widetilde{U}_i^{B(K)}$$
$$- \sum_{K=1}^{M_u^O} \delta r_i^K \left\{ \sum_{J=1}^{N} u_i^J B^{KJ} + \sum_{J\in\Lambda} a_i^J B_C^{KJ} \right\} \quad - \sum_{K=1}^{M_u^A} \delta r_i^{A(K)} \left\{ \sum_{J=1}^{N} u_i^J B^{KJ} + \sum_{J\in\Lambda} a_i^J B_C^{KJ} \right\}$$
$$- \sum_{K=1}^{M_u^B} \delta r_i^{B(K)} \left\{ \sum_{J=1}^{N} u_i^J B^{KJ} + \sum_{J\in\Lambda} a_i^J B_C^{KJ} \right\}$$
$$- \sum_{K=1}^{M_u^O} r_i^K \left\{ \sum_{J=1}^{N} \delta u_i^J B^{KJ} + \sum_{J\in\Lambda} \delta a_i^J B_C^{KJ} \right\} \quad - \sum_{K=1}^{M_u^A} r_i^{A(K)} \left\{ \sum_{J=1}^{N} \delta u_i^J B^{KJ} + \sum_{J\in\Lambda} \delta a_i^J B_C^{KJ} \right\}$$
$$- \sum_{K=1}^{M_u^B} r_i^{B(K)} \left\{ \sum_{J=1}^{N} \delta u_i^J B^{KJ} + \sum_{J\in\Lambda} \delta a_i^J B_C^{KJ} \right\} \qquad \text{(A5.71)}$$

In matrix form, this equation becomes :

$$\delta\Pi_6 = \sum_{K=1}^{M_u^O} \langle \delta r \rangle^K \{\widetilde{U}\}^K + \sum_{K=1}^{M_u^A} \langle \delta r \rangle^{A(K)} \{\widetilde{U}\}^{A(K)} + \sum_{K=1}^{M_u^B} \langle \delta r \rangle^{B(K)} \{\widetilde{U}\}^{B(K)}$$
$$- \sum_{K=1}^{M_u^O} \langle \delta r \rangle^K \left\{ \sum_{J=1}^{N} B^{KJ} \{u\}^J + \sum_{J\in\Lambda} B_C^{KJ} \{a\}^J \right\}$$
$$- \sum_{K=1}^{M_u^O} \langle \delta r \rangle^{A(K)} \left\{ \sum_{J=1}^{N} B^{KJ} \{u\}^J + \sum_{J\in\Lambda} B_C^{KJ} \{a\}^J \right\}$$
$$- \sum_{K=1}^{M_u^O} \langle \delta r \rangle^{B(K)} \left\{ \sum_{J=1}^{N} B^{KJ} \{u\}^J + \sum_{J\in\Lambda} B_C^{KJ} \{a\}^J \right\}$$
$$- \sum_{K=1}^{M_u^O} \langle r \rangle^K \left\{ \sum_{J=1}^{N} B^{KJ} \{\delta u\}^J + \sum_{J\in\Lambda} B_C^{KJ} \{\delta a\}^J \right\}$$
$$- \sum_{K=1}^{M_u^O} \langle r \rangle^{A(K)} \left\{ \sum_{J=1}^{N} B^{KJ} \{\delta u\}^J + \sum_{J\in\Lambda} B_C^{KJ} \{\delta a\}^J \right\}$$
$$- \sum_{K=1}^{M_u^O} \langle r \rangle^{B(K)} \left\{ \sum_{J=1}^{N} B^{KJ} \{\delta u\}^J + \sum_{J\in\Lambda} B_C^{KJ} \{\delta a\}^J \right\} \qquad \text{(A5.72)}$$

Finally, we obtain

$$
\begin{aligned}
\delta\Pi_6 = &\sum_{K=1}^{M_u^O}\langle\delta r\rangle^K \left\{\{\tilde{U}\}^K - \sum_{J=1}^{N} B^{KJ}\{u\}^J - \sum_{J\in\Lambda} B_C^{KJ}\{a\}^J\right\} \\
&+\sum_{K=1}^{M_u^A}\langle\delta r\rangle^{A(K)} \left\{\{\tilde{U}\}^{A(K)} - \sum_{J=1}^{N} B^{KJ}\{u\}^J - \sum_{J\in\Lambda} B_C^{KJ}\{a\}^J\right\} \\
&+\sum_{K=1}^{M_u^B}\langle\delta r\rangle^{B(K)} \left\{\{\tilde{U}\}^{B(K)} - \sum_{J=1}^{N} B^{KJ}\{u\}^J - \sum_{J\in\Lambda} B_C^{KJ}\{a\}^J\right\} \\
&-\sum_{K=1}^{M_u^O}\langle r\rangle^K \left\{\sum_{J=1}^{N} B^{KJ}\{\delta u\}^J + \sum_{J\in\Lambda} B_C^{KJ}\{\delta a\}^J\right\} \\
&-\sum_{K=1}^{M_u^A}\langle r\rangle^{A(K)} \left\{\sum_{J=1}^{N} B^{KJ}\{\delta u\}^J + \sum_{J\in\Lambda} B_C^{KJ}\{\delta a\}^J\right\} \\
&-\sum_{K=1}^{M_u^B}\langle r\rangle^{B(K)} \left\{\sum_{J=1}^{N} B^{KJ}\{\delta u\}^J + \sum_{J\in\Lambda} B_C^{KJ}\{\delta a\}^J\right\}
\end{aligned}
\tag{A5.73}
$$

Final expression of the discretized FdV principle δΠ

Grouping the different terms, we get:

$$\delta\Pi = \delta\Pi_\varepsilon + \delta\Pi_{\gamma_0} + \delta\Pi_{K\varepsilon} + \delta\Pi_u + \delta\Pi_a + \delta\Pi_\Sigma + \delta\Pi_{P_0} + \delta\Pi_{K\Sigma} + \delta\Pi_t = 0 \tag{A5.74}$$

with

$$
\begin{aligned}
\delta\Pi_\varepsilon = &\sum_{I=1}^{N_O}\langle\delta\varepsilon\rangle^I A_I\left[\{\sigma\}^I - \{\Sigma\}^I\right] \\
&+\sum_{J\in\lambda_H}\left\{\langle\delta\varepsilon\rangle^{A,J} A_{A,J}\left[\{\sigma\}^{A,J} - \{\Sigma\}^{A,J}\right] + \langle\delta\varepsilon\rangle^{B,J} A_{B,I}\left[\{\sigma\}^{B,J} - \{\Sigma\}^{B,J}\right]\right\}
\end{aligned}
\tag{A5.75}
$$

$$\delta\Pi_{\gamma_0} = \sum_{L\in\lambda_C}\langle\delta\gamma^0\rangle^L\left[\int_{A_L}\{\tau\}^L dA_L - A_L\{P^0\}^L - [IH]^L\{K_\Sigma\}^L\right] \tag{A5.76}$$

$$\delta\Pi_{K\varepsilon} = \sum_{L\in\lambda_C}\langle\delta K_\varepsilon\rangle^L\left[\{H_\tau\}^L - [IH]^{L,T}[D]^L\{P^0\}^L - [HDH]^L\{K_\Sigma\}^L\right] \tag{A5.77}$$

$$
\begin{aligned}
\delta\Pi_u = &\sum_{I=1}^{N_O}\sum_{J=1}^{N}\langle\delta u\rangle^J[A]^{IJ}\{\Sigma\}^I + + \sum_{I\in\lambda_H}\sum_{J=1}^{N}\langle\delta u\rangle^J[A_A]^{IJ}\{\Sigma\}^{A,I} + \sum_{I\in\lambda_H}\sum_{J=1}^{N}\langle\delta u\rangle^J[A_B]^{IJ}\{\Sigma\}^{B,I} \\
&+\sum_{L\in\lambda_C}\sum_{J=1}^{N}\langle\delta v\rangle^{J,L}\left[[W]^{JL}\{P^0\}^L + [V]^{JL}\{K_\Sigma\}^L\right] - \sum_{I=1}^{N}\sum_{J=1}^{N}\langle\delta u\rangle^J\{\tilde{F}\}^{IJ} - \sum_{K=1}^{M_t}\sum_{J=1}^{N}\langle\delta u\rangle^J\{\tilde{T}\}^{IJ} \\
&-\sum_{K=1}^{M_u^O}\sum_{J=1}^{N}\langle\delta u\rangle^J B^{KJ}\{r\}^K - \sum_{K=1}^{M_u^A}\sum_{J=1}^{N}\langle\delta u\rangle^J B^{KJ}\{r\}^{A(K)} - \sum_{K=1}^{M_u^B}\sum_{J=1}^{N}\langle\delta u\rangle^J B^{KJ}\{r\}^{B(K)}
\end{aligned}
$$

(A5.78)

$$\begin{aligned}\delta\Pi_a &= \sum_{I=1}^{N_O}\sum_{J\in\Lambda}\langle\delta a\rangle^J\left[A_C\right]^{IJ}\{\Sigma\}^I + \sum_{I\in\lambda_H}\sum_{J\in\Lambda}\langle\delta a\rangle^J\left[A_{C,A}\right]^{IJ}\{\Sigma\}^{A,I} + \sum_{I\in\lambda_H}\sum_{J\in\Lambda}\langle\delta a\rangle^J\left[A_{C,B}\right]^{IJ}\{\Sigma\}^{B,I}\\ &+\sum_{L\in\lambda_C}\sum_{J\in\Lambda}\langle\delta b\rangle^{J,L}\left[\left[W_C\right]^{JL}\left\{P^0\right\}^L+\left[V_C\right]^{JL}\{K_\Sigma\}^L\right]-\sum_{I=1}^{N}\sum_{J\in\Lambda}\langle\delta a\rangle^J\left\{\widetilde{F}_C\right\}^{IJ}-\sum_{K=1}^{M_t}\sum_{J\in\Lambda}\langle\delta a\rangle^J\left\{\widetilde{T}_C\right\}^{KJ}\\ &-\sum_{K=1}^{M_u^O}\sum_{J\in\Lambda}\langle\delta a\rangle^J B_C^{KJ}\{r\}^K-\sum_{K=1}^{M_u^A}\sum_{J\in\Lambda}\langle\delta a\rangle^J B_C^{KJ}\{r\}^{A(K)}-\sum_{K=1}^{M_u^B}\sum_{J\in\Lambda}\langle\delta a\rangle^J B_C^{KJ}\{r\}^{B(K)}\end{aligned}$$

(A5.79)

$$\begin{aligned}\delta\Pi_\Sigma &= \sum_{I=1}^{N_O}\sum_{J=1}^{N}\langle u\rangle^J[A]^{IJ}\{\delta\Sigma\}^I+\sum_{I\in\lambda_H}\sum_{J=1}^{N}\langle u\rangle^J\left[A_A\right]^{IJ}\{\delta\Sigma\}^{A,I}+\sum_{I\in\lambda_H}\sum_{J=1}^{N}\langle u\rangle^J\left[A_B\right]^{IJ}\{\delta\Sigma\}^{B,I}\\ &+\sum_{I=1}^{N_O}\sum_{J\in\Lambda}\langle a\rangle^J\left[A_C\right]^{IJ}\{\delta\Sigma\}^I+\sum_{I\in\lambda_H}\sum_{J\in\Lambda}\langle a\rangle^J\left[A_{C,A}\right]^{IJ}\{\delta\Sigma\}^{A,I}+\sum_{I\in\lambda_H}\sum_{J\in\Lambda}\langle a\rangle^J\left[A_{C,B}\right]^{IJ}\{\delta\Sigma\}^{B,I}\\ &-\sum_{I=1}^{N_O}A_I\langle\varepsilon\rangle^I\{\delta\Sigma\}^I-\sum_{J\in\lambda_H}A_{A,J}\langle\varepsilon\rangle^{A,J}\{\delta\Sigma\}^{A,J}-\sum_{J\in\lambda_H}A_{B,J}\langle\varepsilon\rangle^{B,J}\{\delta\Sigma\}^{B,J}\end{aligned}$$

(A5.80)

$$\begin{aligned}\delta\Pi_{P_0} &= \sum_{L\in\lambda_C}\sum_{I=1}^{N}\langle v\rangle^{I,L}[W]^{IL}\left\{\delta P^0\right\}^L+\sum_{L\in\lambda_C}\sum_{J\in\Lambda}\langle b\rangle^{J,L}\left[W_C\right]^{JL}\left\{\delta P^0\right\}^L\\ &-\sum_{L\in\lambda_C}A_L\left\langle\gamma^0\right\rangle^L\left\{\delta P^0\right\}^L-\sum_{L\in\lambda_C}\left\langle\delta P^0\right\rangle^L[D]^L[IH]^L\{K_\varepsilon\}^L\end{aligned} \quad (A5.81)$$

$$\begin{aligned}\delta\Pi_{K\Sigma} &= \sum_{L\in\lambda_C}\sum_{I=1}^{N}\langle\delta K_\Sigma\rangle^L[V]^{IL,T}\{v\}^{I,L}+\sum_{L\in\lambda_C}\sum_{I\in\Lambda}\langle\delta K_\Sigma\rangle^L\left[V_C\right]^{IL,T}\{b\}^{I,L}\\ &-\sum_{L\in\lambda_C}\langle\delta K_\Sigma\rangle^L[IH]^{L,T}\left\{\gamma^0\right\}^L-\sum_{L\in\lambda_C}\langle\delta K_\Sigma\rangle^L[HDH]^L\{K_\varepsilon\}^L\end{aligned} \quad (A5.82)$$

$$\begin{aligned}\delta\Pi_r &= \sum_{K=1}^{M_u^O}\langle\delta r\rangle^K\left\{\left\{\widetilde{U}\right\}^K-\sum_{J=1}^{N}B^{KJ}\{u\}^J-\sum_{J\in\Lambda}B_C^{KJ}\{a\}^J\right\}\\ &+\sum_{K=1}^{M_u^A}\langle\delta t\rangle r^{A(K)}\left\{\left\{\widetilde{U}\right\}^{A(K)}-\sum_{J=1}^{N}B^{KJ}\{u\}^J-\sum_{J\in\Lambda}B_C^{KJ}\{a\}^J\right\}\\ &+\sum_{K=1}^{M_u^B}\langle\delta r\rangle^{B(K)}\left\{\left\{\widetilde{U}\right\}^{B(K)}-\sum_{J=1}^{N}B^{KJ}\{u\}^J-\sum_{J\in\Lambda}B_C^{KJ}\{a\}^J\right\}\end{aligned} \quad (A5.83)$$

with

$$\begin{Bmatrix}v_1\\v_2\end{Bmatrix}^L=\begin{bmatrix}\cos\alpha_L & \sin\alpha_L\\-\sin\alpha_L & \cos\alpha_L\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix} \Leftrightarrow \{v\}^L=[R]^L\{u\} \quad (A5.84)$$

A5.2. Euler equations deduced from the FdV variational principle

Euler equations deduced from $\delta\Pi_{\varepsilon}=0$

$$\boxed{\{\Sigma\}^{I}=\{\sigma\}^{I}=[C]^{I}\{\varepsilon\}^{I},\quad I=1,N_O} \tag{A5.85-a}$$

$$\boxed{\{\Sigma\}^{A,J}=\{\sigma\}^{A,J}=[C]^{J}\{\varepsilon\}^{A,J},\quad J\in\lambda_H} \tag{A5.85-b}$$

$$\boxed{\{\Sigma\}^{B,J}=\{\sigma\}^{B,J}=[C]^{J}\{\varepsilon\}^{B,J},\quad J\in\lambda_H} \tag{A5.85-c}$$

The assumed stresses in cells of types *O* and *H* are equal to the constitutive stresses deduced from the assumed strains in those cells.

Euler equations deduced from $\delta\Pi_{\gamma_0}=0$

$$\int_{A_L}\{\tau\}^{L}dA_L-A_L\{P^0\}^{L}-[IH]^{L}\{K_\Sigma\}^{L}=0\quad L\in\lambda_C \tag{A5.86}$$

But, from the elastic constitutive equation, we have

$$\{\tau\}^{L}=[C]^{L}\{\gamma\}^{L}=[C]^{L}(\{\gamma^{O}\}^{L}+[D]^{L}\{K_{\varepsilon1}^{L}\{H_1\}^{L}+K_{\varepsilon2}^{L}\{H_2\}\})=[C]^{L}\{\gamma^{0}\}^{L}+K_{\varepsilon1}^{L}\{H_1\}^{L}+K_{\varepsilon2}^{L}\{H_2\}$$

$$\int_{A_L}\{\tau\}^{L}dA_L=A_L[C]^{L}\{\gamma^{0}\}^{L}+K_{\varepsilon1}^{L}\int_{A_L}\{H_1\}^{L}dA_L+K_{\varepsilon2}^{L}\int_{A_L}\{H_2\}dA_L=A_L[C]^{L}\{\gamma^{0}\}^{L}+[IH]^{L}\{K_\varepsilon\}^{L}$$

So that (A5.86) becomes

$$\boxed{A_L\left\{[C]^{L}\{\gamma^{0}\}^{L}-\{P^{0}\}^{L}\right\}+[IH]^{L}\left\{\{K_\varepsilon\}^{L}-\{K_\Sigma\}^{L}\right\}=0,\quad L\in\lambda_C} \tag{A5.87}$$

Euler equations deduced from $\delta\Pi_{K\varepsilon}=0$

$$\{H_\tau\}^{L}-[IH]^{L,T}[D]^{L}\{P^{0}\}^{L}-[HDH]^{L}\{K_\Sigma\}^{L}=0,\quad L\in\lambda_C \tag{A5.88}$$

If we introduce the expression of $\{\tau\}^{L}$ in $\{H_\tau\}^{L}$ given by (A5.10), we get:

$$\{H_\tau\}^{L}=\begin{bmatrix}\int_{A_L}\langle H_1\rangle^{L}[D]^{L}\{\tau\}^{L}dA_L\\ \int_{A_L}\langle H_2\rangle^{L}[D]^{L}\{\tau\}^{L}dA_L\end{bmatrix}=\begin{bmatrix}\int_{A_L}\langle H_1\rangle^{L}\{\gamma\}^{L}dA_L\\ \int_{A_L}\langle H_2\rangle^{L}\{\gamma\}^{L}dA_L\end{bmatrix}=[IH]^{L,T}\{\gamma^{0}\}^{L}+[HDH]^{L}\{K_\varepsilon\}^{L}$$

so that (A5.88) becomes :

$$[IH]^{L,T}\left\{\{\gamma^{0}\}^{L}-[D]^{L}\{P^{0}\}^{L}\right\}+[HDH]^{L}\left\{\{K_\varepsilon\}^{L}-\{K_\Sigma\}^{L}\right\}=0,\quad L\in\lambda_C \tag{A5.89}$$

The solution of (A5.88) and (A5.89) is:

$$\boxed{\{P^{0}\}^{L}=[C]^{L}\{\gamma^{0}\}^{L},\quad L\in\lambda_C} \tag{A5.90}$$

$$\boxed{\{K_\varepsilon\}^{L}=\{K_\Sigma\}^{L},\quad L\in\lambda_C} \tag{A5.91}$$

The assumed stresses in cells of types *C* are equal to the constitutive stresses deduced from the assumed strains in those cells.

Euler equations deduced from $\delta\Pi_u = 0$

$$\sum_{I=1}^{N_O} [A]^{IJ}\{\Sigma\}^I + \sum_{I\in\lambda_H} [A_A]^{IJ}\{\Sigma\}^{A,I} + \sum_{I\in\lambda_H} [A_B]^{IJ}\{\Sigma\}^{B,I} + \sum_{L\in\lambda_C} [R]^{L,T}\left[[W]^{JL}\left\{P^0\right\}^L + [V]^{JL}\{K_\Sigma\}^L\right]$$
$$= \sum_{I=1}^{N}\left\{\tilde{F}\right\}^{IJ} + \sum_{K=1}^{M_t}\left\{\tilde{T}\right\}^{IJ} + \sum_{K=1}^{M_u^O} B^{KJ}\{r\}^K + \sum_{K=1}^{M_u^A} B^{KJ}\{r\}^{A(K)} + \sum_{K=1}^{M_u^B} B^{KJ}\{r\}^{B(K)} \qquad J = 1, N$$

(A5.92)

These are equilibrium equations associated with the degrees of freedom u_i^J of the cells of type *O*.

Euler equations deduced from $\delta\Pi_a = 0$

$$\sum_{I=1}^{N_O} [A_C]^{IJ}\{\Sigma\}^I + \sum_{I\in\lambda_H} [A_{C,A}]^{IJ}\{\Sigma\}^{A,I} + \sum_{I\in\lambda_H} [A_{C,B}]^{IJ}\{\Sigma\}^{B,I} + \sum_{L\in\lambda_C} [R]^{L,T}\left[[W_C]^{JL}\left\{P^0\right\}^L + [V_C]^{JL}\{K_\Sigma\}^L\right]$$
$$= \sum_{I=1}^{N}\left\{\tilde{F}_C\right\}^{IJ} + \sum_{K=1}^{M_t}\left\{\tilde{T}_C\right\}^{KJ} + \sum_{K=1}^{M_u^O} B_C^{KJ}\{r\}^K + \sum_{K=1}^{M_u^A} B_C^{KJ}\{r\}^{A(K)} + \sum_{K=1}^{M_u^B} B_C^{KJ}\{r\}^{B(K)} \quad , \quad J \in \Lambda$$

(A5.93)

These are equilibrium equations associated with the degrees of freedom a_i^J of the cells of type *H* and *C*.

Euler equations deduced from $\delta\Pi_\Sigma = 0$

$$A_I\{\varepsilon\}^I = \sum_{J=1}^{N} [A]^{IJ,T}\{u\}^J + \sum_{J\in\Lambda}[A_C]^{IJ,T}\{a\}^J, \quad I = 1, N_O \tag{A5.94-a}$$

$$A_{A,I}\{\varepsilon\}^{A,I} = \sum_{J=1}^{N} [A_A]^{IJ,T}\{u\}^J + \sum_{J\in\Lambda}[A_{C,A}]^{IJ,T}\{a\}^J, \quad I \in \lambda_H \tag{A5.94-b}$$

$$A_{B,I}\{\varepsilon\}^{B,I} = \sum_{J=1}^{N} [A_B]^{IJ,T}\{u\}^J + \sum_{J\in\Lambda}[A_{C,B}]^{IJ,T}\{a\}^J, \quad I \in \lambda_H \tag{A5.94-c}$$

These are compatibility equations between the assumed strains and the assumed displacements in the cells of types *O* and *H*.

Euler equations deduced from $\delta\Pi_{P_0} = 0$

$$A_L\left\{\gamma^0\right\}^L + [D]^L[IH]^L\{K_\varepsilon\}^L = \sum_{I=1}^{N}[W]^{IL,T}\{v\}^{I,L} + \sum_{I\in\Lambda}[W_C]^{IL,T}\{b\}^{I,L}, \quad L \in \lambda_C \tag{A5.95}$$

Euler equations deduced from $\delta\Pi_{K\Sigma} = 0$

$$[IH]^{L,T}\{\gamma^0\}^L + [HDH]^L\{K_\varepsilon\}^L = \sum_{I=1}^{N}[V]^{IL,T}\{v\}^{I,L} + \sum_{I\in\Lambda}[V_C]^{IL,T}\{b\}^{I,L}, L \in \lambda_C$$

(A5.96)

Equations (A5.95) and (A5.96) are compatibility equations between the assumed strains and the assumed displacements in the cells of type *C*.

Euler equations deduced from $\delta\Pi_t = 0$

$$\sum_{J=1}^{N} B^{KJ}\{u\}^J + \sum_{J\in\Lambda}^{N} B_C^{KJ}\{a\}^J = \{\tilde{U}\}^K, \quad K = 1, M_u^O \qquad \text{(A5.97-a)}$$

$$\sum_{J=1}^{N} B^{KJ}\{u\}^J + \sum_{J\in\Lambda}^{N} B_C^{KJ}\{a\}^J = \{\tilde{U}\}^{A(K)}, \quad K = 1, M_u^A \qquad \text{(A5.97-b)}$$

$$\sum_{J=1}^{N} B^{KJ}\{u\}^J + \sum_{J\in\Lambda}^{N} B_C^{KJ}\{a\}^J = \{\tilde{U}\}^{B(K)}, \quad K = 1, M_u^B \qquad \text{(A5.97-c)}$$

These are compatibility equations between the imposed displacements and the discretization parameters of the displacement field. They apply to every cell edge *K* or part of edge *A(K), B(K)* on which displacements are imposed.

A5.3. Calculation of $\{K_\varepsilon\}^L$

From (A5.95) we get successively

$$\{\gamma^0\}^L = \frac{1}{A_L}\left\{\sum_{I=1}^{N}[W]^{IL,T}\{v\}^{I,L} + \sum_{J\in\Lambda}[W_C]^{JL,T}\{b\}^{J,L} - [D]^L[IH]^L\{K_\varepsilon\}^L\right\},\ L\in\lambda_C$$

$$[IH]^{L,T}\{\gamma^0\}^L = \frac{1}{A_L}\left\{\sum_{I=1}^{N}[IH]^{L,T}[W]^{IL,T}\{v\}^{I,L} + \sum_{J\in\Lambda}[IH]^{L,T}[W_C]^{JL,T}\{b\}^{J,L} - [IH]^{L,T}[D]^L[IH]^L\{K_\varepsilon\}^L\right\},\ L\in\lambda_C$$

$$[IH]^{L,T}\{\gamma^0\}^L + [HDH]^L\{K_\varepsilon\}^L =$$
$$\frac{1}{A_L}\left\{\sum_{I=1}^{N}[IH]^{L,T}[W]^{IL,T}\{v\}^{I,L} + \sum_{J\in\Lambda}[IH]^{L,T}[W_C]^{JL,T}\{b\}^{J,L}\right\} + \left[[HDH]^L - \frac{1}{A_L}[IH]^{L,T}[D]^L[IH]^L\right]\{K_\varepsilon\}^L$$

Introducing this result in (A5.96) yields :

$$\left[[HDH]^L - \frac{1}{A_L}[IH]^{L,T}[D]^L[IH]^L\right]\{K_\varepsilon\}^L = \sum_{I=1}^{N}\left[[V]^{IL,T} - \frac{1}{A_L}[IH]^{L,T}[W]^{IL,T}\right]\{v\}^{I,L}$$
$$+\sum_{I\in\Lambda}\left[[V_C]^{IL,T} - \frac{1}{A_L}[IH]^{L,T}[W_C]^{JL,T}\right]\{b\}^{I,L},\quad L\in\lambda_C$$

From this equation, we can deduce $\{K_\varepsilon\}^L$:

$$\boxed{\{K_\varepsilon\}^L = \sum_{I=1}^{N}\left[M_{K\varepsilon v}\right]^{IL}\{v\}^{I,L} + \sum_{I\in\Lambda}\left[M_{K\varepsilon b}\right]^{IL}\{b\}^{I,L} \qquad L\in\lambda_C} \tag{A5.98}$$

with

$$[M_{K\varepsilon}]^L = [HDH]^L - \frac{1}{A_L}[IH]^{L,T}[D]^L[IH]^L \tag{A5.99-a}$$

$$\left[M_{K\varepsilon v}\right]^{IL} = [M_{K\varepsilon}]^{L,-1}\left[[V]^{IL,T} - \frac{1}{A_L}[IH]^{L,T}[W]^{IL,T}\right] \tag{A5.99-b}$$

$$\left[M_{K\varepsilon b}\right]^{IL} = [M_{K\varepsilon}]^{L,-1}\left[[V_C]^{IL,T} - \frac{1}{A_L}[IH]^{L,T}[W_C]^{JL,T}\right] \tag{A5.99-c}$$

A5.4. Calculation of $\{\gamma^0\}^L$

From (A5.96) we get successively

$$[HDH]^L\{K_\varepsilon\}^L = \sum_{I=1}^{N}[V]^{IL,T}\{v\}^{I,L} + \sum_{I\in\Lambda}[V_C]^{IL,T}\{b\}^{I,L} + -[IH]^{L,T}\{\gamma^0\}^L \quad , \quad L\in\lambda_C$$

$$\{K_\varepsilon\}^L = [HDH]^{L,-1}\left\{\sum_{I=1}^{N}[V]^{IL,T}\{v\}^{I,L} + \sum_{I\in\Lambda}[V_C]^{IL,T}\{b\}^{I,L}\right\} - [HDH]^{L,-1}[IH]^{L,T}\{\gamma^0\}^L, \quad L\in\lambda_C$$

$$[D]^L[IH]^L\{K_\varepsilon\}^L = [D]^L[IH]^L[HDH]^{L,-1}\left\{\sum_{I=1}^{N}[V]^{IL,T}\{v\}^{I,L} + \sum_{I\in\Lambda}[V_C]^{IL,T}\{b\}^{I,L}\right\}$$
$$-[D]^L[IH]^L[HDH]^{L,-1}[IH]^{L,T}\{\gamma^0\}^L, \quad L\in\lambda_C$$

Introducing this result in (A5.95) yields:

$$\left[A_L[1]-[D]^L[IH]^L[HDH]^{L,-1}[IH]^{L,T}\right]\{\gamma^0\}^L$$
$$=\sum_{I=1}^{N}\left[[W]^{IL,T}-[D]^L[IH]^L[HDH]^{L,-1}[V]^{IL,T}\right]\{v\}^{I,L}$$
$$+\sum_{J\in\Lambda}\left[[W_C]^{IL,T}-[D]^L[IH]^L[HDH]^{L,-1}[V_C]^{IL,T}\right]\{b\}^{I,L}, \quad L\in\lambda_C$$

From this equation, we can deduce $\{\gamma^0\}^L$:

$$\boxed{\{\gamma^0\}^L = \sum_{I=1}^{N}\left[M_{\gamma 0 v}\right]^{IL}\{v\}^{I,L} + \sum_{I\in\Lambda}\left[M_{\gamma 0 b}\right]^{IL}\{b\}^{I,L}, \quad L\in\lambda_C} \tag{A5.100}$$

with

$$\left[M_{\gamma 0}\right]^L = A_L[1]-[D]^L[IH]^L[HDH]^{L,-1}[IH]^{L,T} \tag{A5.101-a}$$

$$\left[M_{\gamma 0 v}\right]^{IL} = \left[M_{\gamma 0}\right]^{L,-1}\left[[W]^{IL,T}-[D]^L[IH]^L[HDH]^{L,-1}[V]^{IL,T}\right] \tag{A5.101-b}$$

$$\left[M_{\gamma 0 b}\right]^{IL} = \left[M_{\gamma 0}\right]^{L,-1}\left[[W_C]^{IL,T}-[D]^L[IH]^L[HDH]^{L,-1}[V_C]^{IL,T}\right] \tag{A5.101-c}$$