Chapter I
Introduction

Content

I.1 Scope and objectives of study

There are many well known numerical methods to solve problems of Solid Mechanics, such as the finite elements method, the boundary elements method, the meshless methods, …

The natural neighbours method or natural elements method (NEM) has been developed since 1996 [SAMBRIDGE M. et al.(1996), SUKUMAR N. (1998), CUETO E. et al. (2003)] and can be considered as one of the many variants of the meshless methods.

Initially the natural neighbours method was developed to solve two-dimensional (2D) linear elastic problems. It has been subsequently extended to most domains of Solid Mechanics: non linear materials and large strains [YVONNET J. (2004), LORONG PH. et al. (2006), ILLOUL A. L. (2008)], 3D problems [ILLOUL A. L. (2008)], …

Classically, the development of this method is based on the virtual work principle.

A set of nodes are distributed over the domain to be studied and the displacement field is discretized with the help of interpolation functions that are not based on the finite element concept but only based on the nodes and the associated Voronoi polygons.

The goal of the present thesis is to propose an original approach of the natural neighbours method based on the Fraeijs de Veubeke variational principle.

Proposed in 1951 by FRAEIJS de VEUBEKE [FRAEIJS de VEUBEKE B. M. (1951)], this variational principle has 4 independent fields:

- u_i : the displacement field in the solid,
- Σ_{ij} : the stress field in the solid,
- ε_{ij} : the strain field in the solid,
- r_i : the support reactions on the part of the solid boundary submitted to imposed displacements.

These 4 fields can be discretized independently, which provides a large flexibility for the development of numerical methods.

So, the basic idea of this work was to explore the possibilities offered by the FRAEIJS de VEUBEKE (FdV) variational principle in the frame of the NEM.

In order to concentrate on this exploration, we decided to keep things as simple as possible.

This is why all the developments are made in the domain of 2D infinitesimal transformations to avoid the complexities involved by large displacements, large strains and three-dimensional problems.

In the same spirit, the software developed to test our developments is as simple as possible. It is written in Visual Fortran and we did not try to make it user friendly or to optimize its performances in terms of computation time or memory requirement. At the end of this thesis (annex 6) a short description of the developed software is given.

The new approach is developed in 3 domains:

- linear elastic problems
- materially non linear problems
- linear elastic fracture mechanics

The next section gives some details.

I.2 Outline of the thesis

The classical notions of Solid Mechanics, Plasticity and Finite elements are assumed to be known.

In chapter II, some of the less classical notions that will be used throughout this thesis are introduced:

- The notions of Delaunay tessellation, Voronoi cells and natural neighbours that constitute the basic tools of the NEM;
- The classical approach of the NEM based on the virtual work principle;
- The Fraeijs de Veubeke functional and variational principle.

On the other hand, although Linear Elastic Fracture Mechanics (LEFM) is a basic science that can be considered as a part of Solid Mechanics, some of the basic ingredients of this science are also briefly recalled because chapters V and VI of this thesis propose new approaches for the numerical solution of problems of LEFM.

In chapter III, linear elastic problems in 2D are treated with an approach of the NEM based on the FdV variational principle.

In the spirit of the NEM, the domain is decomposed into N Voronoi cells corresponding to N nodes distributed inside the domain and on its boundary.

The following discretization hypotheses are admitted:

1. The assumed displacements are interpolated between the nodes with the Laplace interpolation function.
2. The assumed support reactions are constant over each edge K of Voronoi cells on which displacements are imposed.
3. The assumed stresses are constant over each Voronoi cell.
4. The assumed strains are constant over each Voronoi cell.

Introducing these hypotheses in the FdV variational principle produces the set of equations governing the discretized solid.

These equations do not require the calculation of the derivatives of the Laplace interpolation functions and, in the absence of body forces, they only involve numerical integrations on the edges of the Voronoi cells.

These equations are recast in matrix form and it is shown that the discretization parameters associated with the assumptions on the stresses and on the strains can be eliminated at the Voronoi cell level so that the final system of equations only involves the nodal displacements and the assumed support reactions.

These support reactions can be further eliminated from the equation system if the imposed support conditions only involve displacements imposed as constant (in particular displacements imposed to zero) on a part of the solid contour.

Several applications are used to evaluate the method.

A set of patch tests are performed and show that this approach can pass the patch test up to machine precision and that there is no incompressibility locking.

Convergence studies are also made for the case of pure bending of a beam and the numerical solution is compared to the analytical solution of the Theory of Elasticity.

Finally, the case of a square membrane with a hole is also used for convergence evaluation and for comparison with the finite elements solution.

The main advantages of this approach are:

- In the absence of body forces, the calculation of integrals over the area of the domain is avoided: only integrations on the edges of the Voronoi cells are required, for which classical Gauss numerical integration with 2 integration points is sufficient to pass the patch test. In addition,
- The derivatives of the nodal shape functions are not required in the resulting formulation,
- Problems involving nearly incompressible materials are solved without locking.

In chapter IV, the NEM based on the FdV variational principle is extended to materially non linear solids in 2D.

Considering a solid of unit thickness in plane strain state, the material has a non linear constitutive equation but the displacements of the solid are assumed to be very small.

Hence, the problem considered in this chapter is geometrically linear and materially non linear.

The FdV variational principle for linear elasticity is extended to the elasto-plastic case in which the assumed velocity, stresses, strain rates and surface support reactions are discretized separately.

As in the linear elastic case, the domain is decomposed into N Voronoi cells corresponding to the N nodes distributed inside the domain and on its boundary. Since the displacements are assumed to be infinitesimal, it is not necessary to update this decomposition as the solid deforms.

The following discretization hypotheses are admitted:

1. The assumed velocities are interpolated between the nodes with the Laplace interpolation function
2. The assumed strain rates are constant over each Voronoi cell I
3. The assumed stresses are constant over each Voronoi cell I
4. The assumed support reactions are constant over each edge K of Voronoi cells on which displacements are imposed.

Introducing these hypotheses in the extension of the FdV variational principle produces the set of equations governing the discretized solid.

Although the material constitutive equation is non linear, the advantages of this method obtained in linear elasticity remain valid in the extension to the elasto-plasticity problems:

In chapter V, the NEM is extended to the domain of 2D Linear Elastic Fracture Mechanics (LEFM) using an approach based on the FdV variational principle.

The 2D domain contains N nodes (including nodes on the domain contour and nodes located at each crack tip) and the N Voronoi cells corresponding these nodes are built.

The cells corresponding to the crack tip nodes are called Linear Fracture Mechanics Voronoi Cells (LFMVC). The other ones are called Ordinary Cells (OVC).

In each LFMVC, the stress and the strain discretizations include not only a constant term but also additional terms corresponding to the solutions of LEFM for modes 1 and 2.

The following discretization hypotheses are admitted:

1. The assumed displacements are interpolated between the nodes with the Laplace interpolation function.
2. The assumed support reactions are constant over each edge *K* of Voronoi cells on which displacements are imposed
3. The assumed stresses are constant over each OVC
4. The assumed strains are constant over each OVC
5. The assumed stresses are assumed to have the distribution corresponding to modes 1 and 2 in each LFMVC
6. The assumed strains are assumed to have the distribution corresponding to modes 1 and 2 in each LFMVC

Introducing these hypotheses in the FdV variational principle produces the set of equations governing the discretized solid.

In this approach, the stress intensity coefficients are obtained as primary variables of the solution.

The present approach has also the following properties.

In the OVCs

- In the absence of body forces, the calculation of integrals over the area of the domain is avoided: only integrations on the edges of the Voronoi cells are required.
- The derivatives of the Laplace interpolation functions are not required.

In the LFMVCs,

- Some integrations on the area of the LFMVCs are required but they can be calculated analytically.
- The other integrals are integrals on the edges of the LFMVCs
- The derivatives of the Laplace interpolation functions are not required

Several applications are used to evaluate the method.

Patch test, translation test, mode 1 tests, mode 2 tests and single edge crack test confirm the validity of this approach.

Chapter VI develops an eXtended Natural nEighbours Method (XNEM) to solve 2D problems of LEFM. It is inspired by the eXtended Finite Elements Method (XFEM).

The crack is represented by a line that does not conform to the nodes or to the edges of the Voronoi cells.

The 2D domain contains N nodes (including nodes on the domain contour) but there is no node at the crack tips.

The N Voronoi cells corresponding these nodes are built.

Then the discretization of the displacement field is enriched with Heaviside functions allowing a displacement discontinuity at the level of the crack.

We have 3 types of cells:

- cells of type *O* that do not contain a crack;
- cells of type *H* that are divided into 2 parts by a crack;
- cells of type *C* that contain a crack tip.

We admit the following discretization hypotheses in the 3 different types of cells.

For of type O:

1. The assumed stresses are constant over an ordinary cell *I*.
2. The assumed strains are constant over an ordinary cell *I*.
3. The assumed support reactions are constant over each edge of ordinary Voronoi cells on which displacements are imposed.

For cells of type H that are cut into 2 parts by a crack:

1. The assumed stresses are constant over each part of the cell.
2. The assumed strains are constant over each part of the cell.
3. The assumed support reactions are piecewise constant over the edges on which displacements are imposed.

For cells of type C that contain a crack tip, the stress and strain fields are enriched in a similar way as the LFMVCs of chapter V, i.e. by introducing in the stress and strain discretizations the solutions of LEFM for mode 1 and mode 2.

Introducing all these assumptions and enrichments in the FdV variational principle produces the set of equations governing the discretized solid.

As in the method developed in chapter V, the stress intensity coefficients are obtained as primary variables of the solution in the present method.

The following properties, already obtained for the approach of chapter V, remain valid:

In the cells of types O and H:

- In the absence of body forces, the calculation of integrals over the area of the domain is avoided: only integrations on the edges of the Voronoi cells are required.
- The derivatives of the Laplace interpolation functions are not required.

In the cells of type C:

- Some integrations on the area of these cells are required but they can be calculated analytically.
- The other integrals are integrals on the edges of these cells.
- The derivatives of the Laplace interpolation functions are not required.

Applications similar to those of chapter V are presented to validate the approach.

Chapter VII contains the conclusions and some suggestions for future work.

I.3 Original contributions

The work presented in this thesis constitutes a contribution to the solution of some problems of Solid Mechanics.

The originality of the approach resides in the use of the FdV variational principle together with the NEM.

It has been developed in the domains of linear elasticity, materially non linear problems and linear elastic fracture mechanics and it has been shown that it leads to some interesting properties.

In particular, the facts that, even in the materially non linear case, it is not necessary to perform numerical integration on the domain area and to calculate the derivatives of the displacement interpolation functions constitute a serious advantage over the more classical NEM and FEM.

To the author's knowledge, this approach and the demonstration of its properties are entirely original.

Papers based on the developments of chapters III, IV and V have already been published in scientific journals and international conferences.

I.3.1.Papers published in peer reviewed journals

A natural neighbour method for linear elastic problems based on Fraeijs de Veubeke variational principle, Serge Cescotto and Xiang Li, *Int. J. Numer. Meth. Engng* 2007; **71**:1081-1101

A Natural Neighbor Method Based on Fraeijs de Veubeke Variational Principle for Elastoplasticity and Fracture Mechanics, Serge Cescotto, Li Xiang, Laurent Duchêne, *Journal of Engineering Materials and Technology*, April 2008, Vol. 130

A natural neighbour method for materially non linear problems based on Fraeijs de Veubeke variational principle, Li Xiang, Serge Cescotto, Barbara Rossi, *Acta Mechanica Sinica* (2009) 25:83-93

A natural neighbour method for fracture mechanics based on Fraeijs de Veubeke variational principle, Li Xiang, Serge Cescotto, submitted to *Int. J. Numer. Meth. Engng, under revision*.

I.3.2.Chapter of a book

A natural neighbour method based on Fraeijs de Veubeke variational principle, Li Xiang, Serge Cescotto, *Chapter 1 (99 pp) of the book "Computational Mechanics Research Trends", Hans P; Berger Editor, Nova Science Publisher, 2009, ISBN: 978-1-60876-057-2*

I.3.3.Papers in International Conference Proceedings

ESAFORM conference, April 18-20, 2007, Zaragoza, Spain

A natural neighbour method based on Fraeijs de Veubeke variational principle, Serge Cescotto, Li Xiang.

Fifth International Conference on Material processing Defects, July 18-20, 2007, Cornell University, Ithaca, New York, USA

A natural neighbour method based on Fraeijs de Veubeke variational principle, Serge Cescotto, Li Xiang, Laurent Duchêne

The 8th World Congress on Computational Mechanics (WCCM), 30 June- 4 July, 2008, Venice, Italy

Application of a three-fields natural neighbour method in linear fracture mechanics，X. LI, S.Cescotto, L. Duchene

The 8th World Congress on Computational Mechanics (WCCM), 30 June- 4 July, 2008, Venice, Italy

Application of a three-fields natural neighbour method in elastoplasticity, X. LI, S.Cescotto, B. Rossi

9th Int. Conf. on Technology of Plasticity (ICTP 2008) Geongyu, South Corea, September 8-12,2008

A natural neighbour method based on Fraeijs de Veubeke variational principle for elastoplasticity, X. Li , S. Cescotto , B. Rossi