Weighted tensorized fractional Brownian textures

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***Abstract*—This paper presents a new model of textures, obtained as realizations of a new class of fractional Brownian fields. These fields, called *weighted tensorized fractional Brownian fields*, are obtained by a relaxation of the tensor-product structure that appears in the definition of fractional Brownian sheets. Statistical properties such as self-similarity, stationarity of rectangular increments and regularity properties are obtained. An operator scaling extension is defined and we provide simulations of the fields using their spectral representation.**

***Index Terms*—Brownian fields, texture synthesis, stationary rectangular increments, self-similarity, anisotropy.**

I. Introduction

Classical extensions of the fractional Brownian motion in higher dimension. The modelization of phenomena, in particular textures, by random objects has led to the introduction of numerous stochastic processes and fields. The most famous and historically the first one is the well-known Brownian motion, which has been extended to fractional Brownian motions by Kolmogorov in the famous paper [1] from 1940, to define "Gaussian spirals" in Hilbert spaces. The first systematic study of fractional Brownian motion goes back to the article [2] from Mandelbrot and Van Ness. Given a Hurst parameter $H \in (0,1)$, the fractional Brownian motion B^H is the unique Gaussian process with stationary increments satisfying the self-similarity relation $B^H_{at} \stackrel{(d)}{=} a^H B_t$ for any $a, t > 0$, where $\stackrel{(d)}{=}$ means that the equality holds in the sense of finite-dimensional distributions. The fractional Brownian motion B^H can be characterized as the unique centered Gaussian process with covariance given by

$$\forall s,t \in \mathbb{R}_+, \quad \mathbb{E}(B^H_s B^H_t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}).$$

The later process can also be defined via a moving average formula or via its harmonizable representation :

$$\forall t \in \mathbb{R}_+,\, B^H_t = \int_{\mathbb{R}} \frac{e^{it\xi}-1}{|\xi|^{H+1/2}} d\hat{\mathbf{W}}(\xi), \tag{1}$$

see e.g. the monograph [3] for a precise definition of this last stochastic integral.

Several extensions have been proposed in higher dimensions. In particular two natural generalizations are provided. The first one is the Levy fractional Brownian motion (LFBM) of Hurst index $H \in (0,1)$, also called fractional Brownian field (see e.g. [4]). It is the unique real-valued isotropic Gaussian field Y^H with stationary increments satisfying the similarity property $Y^H_{a\mathbf{x}} \stackrel{(d)}{=} a^H Y^H_{\mathbf{x}}$, where "isotropic" means that the field is invariant in law by rotation. Again, it can be defined using its harmonizable representation:

$$Y^H_{\mathbf{x}} = \int_{\mathbb{R}^N} \frac{e^{i\langle \mathbf{x}, \boldsymbol{\xi}\rangle}-1}{\|\boldsymbol{\xi}\|^{H+\frac{N}{2}}} d\hat{\mathbf{W}}(\boldsymbol{\xi}) \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^N$.

A second famous extension is given by the fractional Brownian sheet (fBs) studied in [5], [6]. For a given vector $\mathbf{H} = (H_1, ... H_N) \in (0,1)^N$, the fBs of Hurst index $\mathbf{H}$ is a real-valued centered Gaussian random field $S^{\mathbf{H}}$ with covariance function given by

$$\mathbb{E}(S^{\mathbf{H}}_{\mathbf{x}} S^{\mathbf{H}}_{\mathbf{y}}) = \prod_{m=1}^N \frac{1}{2} \left(|x_m|^{2H_m} + |y_m|^{2H_m} - |x_m - y_m|^{2H_m} \right).$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N_+$. The fBs has the following harmonizable representation

$$S^{\mathbf{H}}_{\mathbf{x}} = \int_{\mathbb{R}^N} \prod_{m=1}^N \frac{e^{ix_m \xi_m}-1}{|\xi_m|^{H_m+\frac{1}{2}}} d\hat{\mathbf{W}}(\boldsymbol{\xi}). \tag{3}$$

Setting $H_m = \frac{1}{2}$ for each $m \in \{1, \ldots, N\}$ yields to the standard Brownian sheet. While LFBMs are isotropic, fBs exhibit a strong "tensor-product" structure even when $H_m = H$ for all $m \in \{1, \ldots, N\}$ and no longer have stationary increments but only rectangular stationary increments (see Definition III.2 below). Nevertheless, this field has been widely studied for its interesting mathematical aspects, including asymptotic properties [7], fractal dimensions [7], [8], geometric properties [9], local times [10], [11], stochastic differential equations [12], and many more.

Further developments. Focusing on the class of Gaussian fields, the two previously mentioned extensions of the fractional Brownian motions can be seen as particular cases of more general models. Important properties have been introduced in these models, such as anisotropy with different properties along directions [13], [14].

The Operator Scaling Gaussian Random Fields (OSGRF, also defined for α-stable fields) introduced in [15], [16] satisfy a matricial self-similarity condition, which is given by

$$\forall a > 0, \quad Z_{a^E \mathbf{x}} \stackrel{(d)}{=} a^H Z_{\mathbf{x}}$$

for some $H > 0$, where E is a $N \times N$ matrix with eigenvalues having positive real parts, and where $a^E = \exp(E\ln(a)) = \sum_{k\geq 0} \frac{\ln^k(a)E^k}{k!}$. These fields have been shown to exhibit anisotropic regularity properties [17], [18], which offer strategies for numerical estimations of the parameters of the model [19], [20].

Other extensions provide models with local changes in the Hurst parameter [21], [22], or changes in the local direction of anisotropy [23], [24], for example. For an overview of the various models, we refer to [24].

Goals, contribution and outline. The contribution of the paper is to provide a new class of fields called *Weighted tensorized fractional Brownian fields* (WTFBFs). They are defined through their harmonizable representation in (4). The importance of the "tensor-product" effect emerges as the parameter α goes from 1 to 0, yielding the fractional Brownian sheet for $\alpha = 0$ and a field closer to an LFBM in terms of regularity for $\alpha = 1$. The introduction of the WTFBFs is detailed in Section 2, and the fundamental properties of self-similarity and rectangular stationary increments are established in Section 3. Section 4 is dedicated to estimating the variance of the rectangular increments, leading to deductions regarding regularity properties. An operator-scaling extension is presented in Section 5, along with simulations of the fields obtained via the spectral representation.

II. Definition of the fields

Let $\alpha \in [0,1]$ and $H \in (0,1)$, we set

$$H_\alpha^+ := (1+\alpha)H \text{ and } H_\alpha^- := (1-\alpha)H$$

and we define the Gaussian field $\{X_{(x_1,x_2)}^{\alpha,H}\}_{(x_1,x_2)\in\mathbb{R}^2}$ by

$$X_{(x_1,x_2)}^{\alpha,H} := \int_{\mathbb{R}^2} \frac{(e^{ix_1\xi_1}-1)(e^{ix_2\xi_2}-1)}{\phi_{\alpha,H}(\xi_1,\xi_2)} d\hat{\mathbf{W}}(\boldsymbol{\xi}) \qquad (4)$$

where the function

$$\phi_{\alpha,H}(\xi_1,\xi_2) = \min(|\xi_1|,|\xi_2|)^{H_\alpha^- + \frac{1}{2}} \max(|\xi_1|,|\xi_2|)^{H_\alpha^+ + \frac{1}{2}}$$

denotes the square root of the inverse of the spectral density of the field. In the sequel, we also use the notation

$$\mathcal{K}_{(x_1,x_2)}^{\alpha,H}(\xi_1,\xi_2) := \frac{(e^{ix_1\xi_1}-1)(e^{ix_2\xi_2}-1)}{\phi_{\alpha,H}(\xi_1,\xi_2)}$$

for the kernel in the stochastic integral (4). Note that the field (4) is well-defined since this last kernel belongs to $L^2(\mathbb{R}^2)$. Note furthermore that the Fourier transform of $\mathcal{K}_{(x_1,x_2)}^{\alpha,H}$ is real. Indeed, one has

$$\begin{aligned}&\Im\left(e^{-i(t_1\xi_1+t_2\xi_2)}(e^{ix_1\xi_1}-1)(e^{ix_2\xi_2}-1)\right)\\ =&\sin((x_1-t_1)\xi_1+(x_2-t_2)\xi_2)-\sin(-t_1\xi_1+(x_2-t_2)\xi_2)\\ &-\sin((x_1-t_1)\xi_1-t_2\xi_2)+\sin(-t_1\xi_1-t_2\xi_2)\end{aligned}$$

which is an odd function in (ξ_1,ξ_2). It follows that

$$\int_{\mathbb{R}^2} \frac{\Im\left(e^{-i(t_1\xi_1+t_2\xi_2)}(e^{ix_1\xi_1}-1)(e^{ix_2\xi_2}-1)\right)}{\phi_{\alpha,H}(\xi_1,\xi_2)} d\boldsymbol{\xi} = 0.$$

It implies that $\widehat{\mathcal{K}_{(x_1,x_2)}^{\alpha,H}}$ is real, hence so is the field

$$\begin{aligned}X_{(x_1,x_2)}^{\alpha,H} &= \int_{\mathbb{R}^2} \mathcal{K}_{(x_1,x_2)}^{\alpha,H}(\xi_1,\xi_2)d\hat{\mathbf{W}}(\boldsymbol{\xi})\\ &= \int_{\mathbb{R}^2} \widehat{\mathcal{K}_{(x_1,x_2)}^{\alpha,H}}(\xi_1,\xi_2)d\mathbf{W}(\boldsymbol{\xi}).\end{aligned}$$

III. Basic properties

Proposition III.1. *For all $\alpha \in [0,1]$ and $H \in (0,1)$, the process $X^{\alpha,H}$ is self-similar : for all $a > 0$, $\{X_{(ax_1,ax_2)}^{\alpha,H}\}_{(x_1,x_2)\in\mathbb{R}^2} \overset{(d)}{=} \{a^{2H}X_{(x_1,x_2)}^{\alpha,H}\}_{(x_1,x_2)\in\mathbb{R}^2}$*

Proof. For all $(x_1,x_2)\in\mathbb{R}^2$ and $a>0$, we have

$$\begin{aligned}X_{(ax_1,ax_2)}^{\alpha,H} &= \int_{\mathbb{R}^2} \frac{(e^{iax_1\xi_1}-1)(e^{iax_2\xi_2}-1)}{\phi_{\alpha,H}(\xi_1,\xi_2)} d\hat{\mathbf{W}}(\boldsymbol{\xi})\\ &\overset{(d)}{=} \int_{\mathbb{R}^2} \frac{(e^{ix_1\eta_1}-1)(e^{ix_2\eta_2}-1)}{\phi_{\alpha,H}(\frac{\eta_1}{a},\frac{\eta_2}{a})} a^{-1} d\hat{\mathbf{W}}(\boldsymbol{\eta})\\ &= a^{2H} \int_{\mathbb{R}^2} \frac{(e^{ix_1\eta_1}-1)(e^{ix_2\eta_2}-1)}{\phi_{\alpha,H}(\eta_1,\eta_2)} d\hat{\mathbf{W}}(\boldsymbol{\eta})\\ &= a^{2H} X_{(x_1,x_2)}^{\alpha,H},\end{aligned}$$

where we used the change of variables $(\eta_1,\eta_2) = (a\xi_1,a\xi_2)$ in the stochastic integral. ☐

Classically, the stationarity of increments is a too strong property for stochastic fields, and it is preferable to use the following property of stationarity for rectangular increments [22], [25]–[27].

Definition III.2. *If $\{X_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$ is a field and if $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$, we set*

$$\begin{aligned}&\Delta X_{(x_1,x_2);(y_1,y_2)}\\ &:= X_{(x_1+y_1,x_2+y_2)} - X_{(y_1,x_2+y_2)} - X_{(x_1+y_1,y_2)} + X_{(y_1,y_2)}.\end{aligned}$$

We say that $\{X_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$ has stationary rectangular increments if, for any $(y_1,y_2)\in\mathbb{R}^2$, we have

$$\{\Delta X_{(x_1,x_2);(y_1,y_2)}\}_{(x_1,x_2)\in\mathbb{R}^2} \overset{(d)}{=} \{X_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}.$$

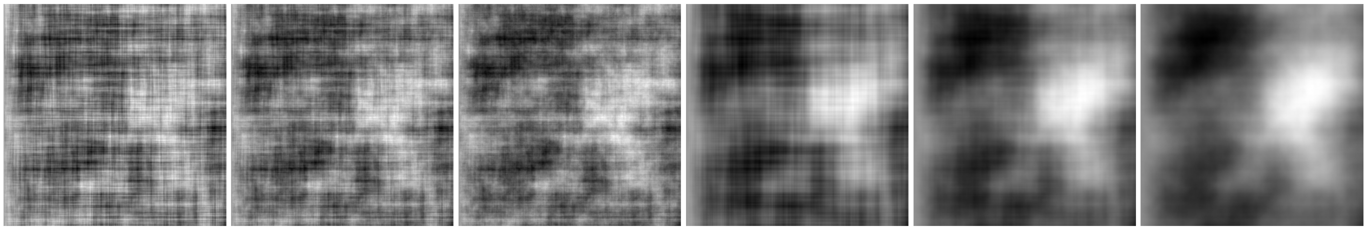
Proposition III.3. *For all $\alpha\in[0,1]$ and $H\in(0,1)$, the field $\{X_{(x_1,x_2)}^{\alpha,H}\}_{(x_1,x_2)\in\mathbb{R}^2}$ has stationary rectangular increments.*

Proof. First, we remark that for any $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$, we get from (4)

$$\Delta X_{(x_1,x_2);(y_1,y_2)}^{\alpha,H} = \int_{\mathbb{R}^2} e^{i(y_1\xi_1+y_2\xi_2)} \mathcal{K}_{(x_1,x_2)}^{\alpha,H}(\xi_1,\xi_2) d\hat{\mathbf{W}}(\boldsymbol{\xi}).$$

Thus, recalling [28, Corollary 6.3.2], we have

$$\begin{aligned}&\mathbb{E}\left(\exp\left(i\sum_{j=1}^n t^{(j)} \Delta X_{(x_1^{(j)},x_2^{(j)});(y_1,y_2)}^{\alpha,H}\right)\right)\\ &= \exp\left(-c_0 \int_{\mathbb{R}^2} \left|\sum_{j=1}^n t^{(j)} e^{i(y_1\xi_1+y_2\xi_2)} \mathcal{K}_{(x_1^{(j)},x_2^{(j)})}^{\alpha,H}(\xi_1,\xi_2)\right|^2 d\boldsymbol{\xi}\right)\\ &= \mathbb{E}\left(\exp\left(i\sum_{j=1}^n t^{(j)} X_{(x_1^{(j)},x_2^{(j)})}^{\alpha,H}\right)\right),\end{aligned}$$



(a) $\alpha = 0$ (b) $\alpha = 0.5$ (c) $\alpha = 1$ (d) $\alpha = 0$ (e) $\alpha = 0.5$ (f) $\alpha = 1$

Fig. 1. Weighted tensorized fractional Brownian fields simulated using a spectral representation approximation method, with parameters (a-c) $H = 0.3$ or (d-f) $H = 0.7$ and (a,d) $\alpha = 0$, (b,e) $\alpha = 0.5$ or (c,f) $\alpha = 1$.

for any $(x_1^{(1)}, x_2^{(1)}), \ldots, (x_1^{(n)}, x_2^{(n)}), (y_1, y_2) \in \mathbb{R}^2$ and any $t^{(1)}, \ldots, t^{(n)} \in \mathbb{R}$, with

$$c_0 := \frac{1}{2\pi} \int_0^{\pi} \cos(\theta)^2 \, d\theta.$$

The conclusion follows directly. $\square$

It is also noteworthy that, for all $\alpha \in [0,1]$ and $H \in (0,1)$, the field $\{X^{\alpha,H}_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$ does have some stationary increments, namely the horizontal and vertical ones.

Definition III.4. *A field $\{X_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$ has stationary horizontal increments if, for any $y_1 \in \mathbb{R}$,*

$$\begin{aligned}&\{X_{(x_1+y_1,x_2)} - X_{(y_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2} \\ &\quad \overset{(d)}{=} \{X_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}.\end{aligned}$$

Similarly, $\{X_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$ has stationary vertical increments if, for any $y_2 \in \mathbb{R}$,

$$\begin{aligned}&\{X_{(x_1,x_2+y_2)} - X_{(x_1,y_2)}\}_{(x_1,x_2)\in\mathbb{R}^2} \\ &\quad \overset{(d)}{=} \{X_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}.\end{aligned}$$

Proposition III.5. *For all $\alpha \in [0,1]$ and $H \in (0,1)$, the field $\{X^{\alpha,H}_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$ has stationary horizontal and vertical increments.*

Proof. For any $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have, on one hand,

$$X^{\alpha,H}_{(x_1+y_1,x_2)} - X^{\alpha,H}_{(y_1,x_2)} = \int_{\mathbb{R}^2} e^{iy_1\xi_1} \mathcal{K}^{\alpha,H}_{(x_1,x_2)}(\xi_1,\xi_2) d\hat{\mathbf{W}}(\boldsymbol{\xi})$$

and, on the other hand,

$$X^{\alpha,H}_{(x_1,x_2+y_2)} - X^{\alpha,H}_{(x_1,y_2)} = \int_{\mathbb{R}^2} e^{iy_2\xi_2} \mathcal{K}^{\alpha,H}_{(x_1,x_2)}(\xi_1,\xi_2) d\hat{\mathbf{W}}(\boldsymbol{\xi}).$$

Then, we conclude the proof in exactly the same way as in the proof of Proposition III.3. $\square$

IV. Variance of rectangular increments and Regularity properties

Proposition IV.1. *For all $\alpha \in [0,1]$ and $H \in (0,1)$, there is a constant $c_1 > 0$ such that the rectangular increments of $\{X^{\alpha,H}_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$ satisfy*

$$\begin{aligned}&\mathbb{E}(|\Delta X^{\alpha,H}_{(h_1,h_2);(x_1,x_2)}|^2) \\ &\quad \le c_1 \left(\max\{|h_1|,|h_2|\}^{1-\alpha} \min\{|h_1|,|h_2|\}^{1+\alpha}\right)^{2H}\end{aligned}$$

for all $(x_1,x_2), (h_1,h_2) \in \mathbb{R}^2$.

Proof. The isometry property of the stochastic integral gives

$$\begin{aligned}&\mathbb{E}\big(|\Delta X^{\alpha,H}_{(h_1,h_2);(x_1,x_2)}|^2\big) \\ &= \int_{\mathbb{R}^2} \frac{|e^{i(x_1+h_1)\xi_1} - e^{ix_1\xi_1}|^2 |e^{i(x_2+h_2)\xi_2} - e^{ix_2\xi_2}|^2}{(\phi_{\alpha,H}(\xi_1,\xi_2))^2} d\boldsymbol{\xi} \\ &= \frac{1}{|h_1|\,|h_2|} \int_{\mathbb{R}^2} \frac{|e^{i\eta_1}-1|^2 |e^{i\eta_2}-1|^2}{(\phi_{\alpha,H}(\frac{\eta_1}{h_1}, \frac{\eta_2}{h_2}))^2} d\boldsymbol{\eta}\end{aligned}$$

using the change of variables $(\eta_1,\eta_2) = (h_1\xi_1, h_2\xi_2)$. Notice now that if $|h_1| \ge |h_2|$, one has

$$(\phi_{\alpha,H}(\tfrac{\eta_1}{h_1}, \tfrac{\eta_2}{h_2}))^2 \ge \frac{\min(|\eta_1|,|\eta_2|)^{2H_\alpha^-+1} |\eta_2|^{2H_\alpha^+ + 1}}{|h_1|^{(2H_\alpha^-+1)} |h_2|^{(2H_\alpha^++1)}}.$$

This implies $\mathbb{E}\big(|\Delta X^{\alpha,H}_{(h_1,h_2);(x_1,x_2)}|^2\big) \le c_1 |h_1|^{2H_\alpha^-} |h_2|^{2H_\alpha^+}$ where

$$c_1 = \int_{\mathbb{R}^2} \frac{|e^{i\eta_1}-1|^2 |e^{i\eta_2}-1|^2}{\min(|\eta_1|,|\eta_2|)^{2H_\alpha^-+1} |\eta_2|^{2H_\alpha^++1}} d\boldsymbol{\eta} \tag{5}$$

if $|h_1| \ge |h_2|$. The same argument for $|h_1| < |h_2|$ leads to the conclusion. $\square$

Similarly, we obtain the following result regarding the horizontal and vertical increments.

Proposition IV.2. *For all $\alpha \in [0,1]$, $H \in (0,1)$ and every compact subset $K \subset \mathbb{R}$, there is a constant $c_K > 0$ such that the horizontal and vertical increments of the field $\{X^{\alpha,H}_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$ satisfy*

$$\mathbb{E}\big(|X^{\alpha,H}_{(x_1+h_1,x_2)} - X^{\alpha,H}_{(x_1,x_2)}|^2\big) \le c_K |h_1|^{2H_\alpha^+}$$

for all $x_1 \in \mathbb{R}$, $x_2 \in K$ and $h_1 \in \mathbb{R}$ such that $|h_1| \le |x_2|$, and

$$\mathbb{E}\big(|X^{\alpha,H}_{(x_1,x_2+h_2)} - X^{\alpha,H}_{(x_1,x_2)}|^2\big) \le c_K |h_2|^{2H_\alpha^+}$$

for all $x_1 \in K$, $x_2 \in \mathbb{R}$ and $h_2 \in \mathbb{R}$ such that $|h_2| \le |x_1|$.

Proof. It suffices to proceed as in the proof of Proposition IV.1 to write

$$\begin{aligned}&\mathbb{E}\big(|X^{\alpha,H}_{(x_1+h_1,x_2)} - X^{\alpha,H}_{(x_1,x_2)}|^2\big) \\ &= \int_{\mathbb{R}^2} \frac{|e^{i(x_1+h_1)\xi_1} - e^{ix_1\xi_1}|^2 |e^{ix_2\xi_2}-1|^2}{(\phi_{\alpha,H}(\xi_1,\xi_2))^2} d\boldsymbol{\xi} \\ &\le c_1 \max\{|h_1|,|x_2|\}^{2H_\alpha^-} \min\{|h_1|,|x_2|\}^{2H_\alpha^+} \\ &\le c_K |h_1|^{2H_\alpha^+}\end{aligned}$$

where c_1 is the constant given in Equation (5) and where we set $c_K = c_1 \sup_{y_2 \in K} |y_2|^{2H_\alpha^-}$. $\square$

The result indicates that the regularity along the axes is governed by H_α^+. Regarding the rectangular increments, a generalization of Kolmogorov's continuity theorem allows us to assert that there exists a modification of the field $\{X^{\alpha,H}_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$ which is *locally* (H_α^+, H_α^-)*-rectangular Hölder*. This means that for every bounded intervals I, J of $\mathbb{R}$, every $x_1 \in I$, $x_2 \in J$ and every $\varepsilon > 0$, there exists a positive finite random variable $C > 0$ such that almost surely

$$\begin{aligned} &|\Delta X^{\alpha,H}_{(h_1,h_2);(x_1,x_2)}| \\ &\leq C\left(\max\{|h_1|,|h_2|\}^{(1-\alpha)}\min\{|h_1|,|h_2|\}^{(1+\alpha)}\right)^{H-\varepsilon} \end{aligned}$$

for all $h_1, h_2 \in \mathbb{R}$ such that $x_1 + h_1 \in I$ and $x_2 + h_2 \in J$. This relation will be further investigated in an upcoming paper.

V. Anisotropic extension and Simulations

A. Anisotropic extension

The model introduced in the previous sections can be extended to provide anisotropic textures by imposing an operator scaling property. These fields then satisfy anisotropic properties of regularities that will be explored in a forthcoming work.

We consider $\beta_1, \beta_2 \in (1/2, 3/2)$ such that $\beta_1 + \beta_2 = 2$, and we set

$$X^{\alpha,H,\beta_1,\beta_2}_{(x_1,x_2)} := \int_{\mathbb{R}^2} \frac{(e^{ix_1\xi_1}-1)(e^{ix_2\xi_2}-1)}{\phi_{\alpha,H,\beta_1,\beta_2}(\xi_1,\xi_2)} d\hat{\mathbf{W}}(\boldsymbol{\xi}) \tag{6}$$

where

$$\phi_{\alpha,H,\beta_1,\beta_2}(\xi_1,\xi_2) = \phi_{\alpha,H}\left(|\xi_1|^{\frac{1}{\beta_1}}, |\xi_2|^{\frac{1}{\beta_2}}\right).$$

If $\max(\beta_1,\beta_2) - 1 < 2H < 3\min(\beta_1,\beta_2) - 1$, the corresponding field is well-defined and satisfies

$$X^{\alpha,H,\beta_1,\beta_2}_{a^D\mathbf{x}} \overset{(d)}{=} a^{2H} X^{\alpha,H,\beta_1,\beta_2}_{\mathbf{x}} \tag{7}$$

with $D = \mathrm{diag}(\beta_1,\beta_2)$ and $a^D\mathbf{x} = (a^{\beta_1}x_1, a^{\beta_2}x_2)$.

B. Simulation

Several strategies have been developed to simulate Gaussian random fields, as reviewed in [24]. Methods based on an explicit expression for the covariance of the field allow for exact simulations that preserve statistical properties such as stationarity. These methods have been used in [29] and [30]. When the covariance is not explicitly known but is known along radial directions, the turning-bands method introduced in [31], [32] can be employed, as done in [33] to simulate some anisotropic fields.

Using the spectral density of the field, approximations of AFBF have been obtained in [34], [35]. Since we only have an integral expression of the covariance, we employ the later strategy to generate a WTFBF. It relies on a discretization of the field in the Fourier domain. Its main drawbacks include the difficulty in obtaining results on the convergence of the approximation. There is also a possibility that the inverse Fourier transform "breaks" the statistical properties of the field. However, it is very fast and easy to perform as it involves fast Fourier transforms. Approximations based on wavelet methods could be used, but they are known to be quite slow in practice even if they provide the best approximation rate by a series in the case of FBF [36].

The results presented in Figures 1 and 2 are generated using a spectral representation approximation on a discrete grid of size $(M+1)\times(M+1)$, with $M = 512$. For a given WTFBF $\{X^{\alpha,H}_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$, the strategy involves generating W, a collection of independent standard complex Gaussian variables of size $(2M \times 2M)$. These variables are then multiplied by a function g. Next, in both directions successively, a 1D Fourier transform is applied, followed by subtracting the value of the field at the origin.

If we set $g(x,y) = (\phi_{\alpha,H}(x,y))^{-1}\mathbf{1}_{\{x\neq 0, y\neq 0\}}$ for $(x,y) \in \mathbb{R}^2$, the generated field $x^{\alpha,H}$ is given, for all $k_1, k_2 \in \{0,\ldots,M\}$, by

$$x^{\alpha,H}\left(\tfrac{k_1}{M},\tfrac{k_2}{M}\right) = \mathcal{R}\left(y_2\left(\tfrac{k_1}{M},\tfrac{k_2}{M}\right) - y_2\left(0,\tfrac{k_2}{M}\right)\right),$$

where for any $n_1 \in \{-M+1,\ldots,M\}$

$$y_1\left(n_1,\tfrac{k_2}{M}\right) = \sum_{n_2=-M+1}^{M} W(n_1,n_2)g(\pi n_1,\pi n_2)\, e^{-\frac{2i\pi n_2 k_2}{2M}},$$

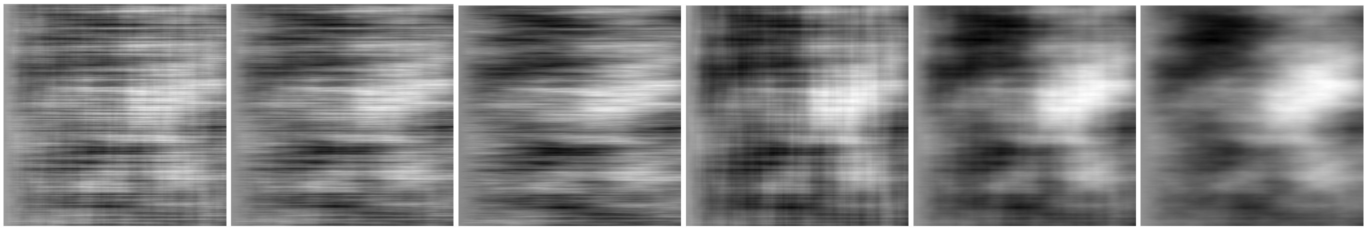
and

$$y_2\left(\tfrac{k_1}{M},\tfrac{k_2}{M}\right) = \pi\sum_{n_1=-M+1}^{M}\left(y_1\left(n_1,\tfrac{k_2}{M}\right) - y_1(n_1,0)\right)e^{-\frac{2i\pi n_1 k_1}{2M}}.$$

The same method is used to simulate the anisotropic extension. Figure 1 presents synthesized WTFBFs with various parameters H and α. Figure 2 shows anisotropic WTFBF. These fields $X^{\alpha,H,\beta_1,\beta_2}$ produce anisotropic textures, where the highest β determines the dominant direction. The images (a-f) illustrate the effects of the parameters α, β and H on the fields.

VI. Conclusion and perspectives

A new class of fractional Brownian fields has been defined using a tensor-product structure. These fields are indexed by two parameters: a Hurst index $H \in (0,1)$ and a parameter $\alpha \in [0,1]$ which measures the "deviation" of the corresponding field from the fractional Brownian sheet with the same Hurst index, obtained with $\alpha = 0$. These fields are demonstrated to be self-similar and to possess stationary rectangular, horizontal and vertical increments. The variance of these increments has been bounded to have a glimpse of the regularity of these fields. An anistropic extension has been proposed. Finally, simulations have been provided, for various values of the parameters H and α as well as for the anisotropic extension. In a forthcoming work, we aim at providing function spaces which are well-suited to consider the rectangular regularity of such fields. An analysis of these spaces, based on hyperbolic wavelet transform should be performed. Further development concerning the anisotropic extension could also be considered, as well as a N-dimensional definition of the *weighted tensorized fractional Brownian textures*.



(a) $\alpha = 0$ (b) $\alpha = 0.5$ (c) $\alpha = 1$ (d) $\alpha = 0$ (e) $\alpha = 0.5$ (f) $\alpha = 1$

Fig. 2. Anisotropic weighted tensorized fractional Brownian fields simulated using a spectral representation approximation method, with parameters (a-c) $H = 0.4$, $\beta_1 = 0.7$, $\beta_2 = 1.3$ and $\alpha = 0$, $\alpha = 0.5$ or $\alpha = 1$, and (d-f) $H = 0.6$, $\beta_1 = 0.85$, $\beta_2 = 1.15$ and $\alpha = 0$, $\alpha = 0.5$ or $\alpha = 1$.

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