Wavelet-Type Expansion of Generalized Hermite Processes with rate of convergence

A Ayache *, J. Hamonier †, L. Loosveldt‡

October 2, 2024

Abstract

Wavelet-type random series representations of the well-known Fractional Brownian Motion (FBM) and many other related stochastic processes and fields have started to be introduced since more than two decades. Such representations provide natural frameworks for approximating almost surely and uniformly rough sample paths at different scales and for study of various aspects of their complex erratic behavior.

Hermite process of an arbitrary integer order d, which extends FBM, is a paradigmatic example of a stochastic process belonging to the dth Wiener chaos. It was introduced very long time ago, yet many of its properties are still unknown when $d \geq 3$. In a paper published in 2004, Pipiras raised the problem to know whether wavelet-type random series representations with a well-localized smooth scaling function, reminiscent of those for FBM due to Meyer, Sellan and Taqqu, can be obtained for a Hermite process of any order d. He solved it in this same paper in the particular case $d = 2$ in which the Hermite process is called the Rosenblatt process. Yet, the problem remains unsolved in the general case $d \geq 3$. The main goal of our article is to solve it, not only for usual Hermite processes but also for generalizations of them. Another important goal of our article is to derive almost sure uniform estimates of the errors related with approximations of such processes by scaling functions parts of their wavelet-type random series representations.

Keywords: High order Wiener chaos, self-similar process, multiresolution analysis, FARIMA sequence, wavelet basis.
2020 MSC: Primary: 60G18, 42C40; secondary: 41A58.

1 Introduction and background

Fractional Brownian Motion (FBM) with Hurst parameter $h \in (0, 1)$, denoted $\{B_h(t)\}_{t\in\mathbb{R}}$, was introduced by Kolmogorov, in 1940, to generate Gaussian "spirals" in Hilbert spaces [15]. Its first systematic study was carried out in the

*Univ. Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France. antoine.ayache@univ-lille.fr

†Univ. Lille, CHU Lille, ULR 2694 - METRICS : Évaluation des technologies de santé et des pratiques médicales, F-59000 Lille, France. julien.hamonier@univ-lille.fr

‡**Corresponding author** Université de Liège, Département de Mathématique – zone Polytech 1, 12 allée de la Découverte, Bât. B37, B-4000 Liège. l.loosveldt@uliege.be

famous paper [18] by Mandelbrot and Van Ness, in 1968. It is the unique Gaussian process with $B_h(0) = 0$, mean zero and covariance function

$$\mathbb{E}[B_h(t)B_h(s)] = \frac{c_h}{2}\left(|t|^{2h} + |s|^{2h} - |t-s|^{2h}\right), \quad \text{for all } (t,s) \in \mathbb{R}^2,$$

where $c_h := \mathrm{Var}(B_h(1))$ is a positive constant only depending on the Hurst parameter h (when $c_h = 1$ then FBM is said to be standard). Among its most fundamental properties, FBM has stationary increments and is h-self-similar, meaning that, for all fixed $a > 0$, the processes $\{a^{-h}B_h(at)\}_{t\in\mathbb{R}}$ and $\{B_h(t)\}_{t\in\mathbb{R}}$ have the same finite-dimensional distributions. When $h = 1/2$, the process $\{B_{1/2}(t)\}_{t\in\mathbb{R}}$ is a usual Brownian motion. We refer for instance to the monograph [21] for a clear and concise presentation of various fundamental facts concerning FBM.

FBM appears naturally in many real-life applications in various domains, such as telecommunications, biology, finance, image processing, and so on. We refer for instance to [11] for a monograph with an overview of its different areas of applications. Thus, study of FBM and related processes has become a crucial issue since a long time. To this end, it is very useful to construct well appropriate representations for these processes. An important class of such representations consists of wavelet-type random series representations. More than two decades ago, they were introduced in the framework of FBM in several articles. We focus on the Meyer, Sellan and Taqqu seminal article [20] whose main goal was to obtain representations which clearly separate the low frequency part of FBM from its high frequency part, and, more importantly, to express the low frequency part in terms of a well-localized smooth scaling function. For a better understanding of our paper, we believe it useful to precisely present in our introduction the most classical one of these wavelet-type representations of FBM due to [20], since one of our principle aims is to extend it to Generalized Hermite process. The article [20] made use of the well-known class of the Meyer orthonormal wavelet bases of $L^2(\mathbb{R})$ as the main ingredient for constructing wavelet-type random series representations for FBM. Some fundamental properties of the two functions ϕ and ψ (scaling function and mother wavelet) generating such a basis are given the following remark.

Remark 1.1. The precise definitions of univariate scaling function and mother wavelet ϕ and ψ associated with a Meyer orthonormal wavelet basis of $L^2(\mathbb{R})$ can for instance be found in [8, p. 137-138]. These two functions ϕ and ψ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$ of infinitely differentiable functions whose derivatives of any order rapidly decay at infinity. Moreover, their Fourier transforms $\widehat{\phi}$ and $\widehat{\psi}$ are infinitely differentiable compactly supported functions satisfying

$$\operatorname{supp}\widehat{\phi} \subseteq \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right] \quad \text{and} \quad \operatorname{supp}\widehat{\psi} \subseteq \left[-\frac{8\pi}{3}, \frac{8\pi}{3}\right] \setminus \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right).$$

Notice that throughout our article, we use the rather common convention that $\mathcal{F}(f) = \widehat{f}$, the Fourier transform of an arbitrary function $f \in \mathcal{S}(\mathbb{R})$ is defined, for all $\xi \in \mathbb{R}$, as $\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\xi x} f(x)\,dx$, while $\mathcal{F}^{-1}(f)$, the inverse Fourier transform of f, is defined, for every $x \in \mathbb{R}$, as $\mathcal{F}^{-1}(f)(x) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{ix\xi} f(x)\,d\xi$.

The article [20] also made an extensive use of the notion of fractional primitive and derivative, which can be defined as follows:

Definition 1.2. Let f be an arbitrary function of the Schwartz class $\mathcal{S}(\mathbb{R})$. For all $h \in (1/2, 1)$ (resp. $h \in (0, 1/2]$), the fractional primitive of f of order $h - 1/2$ (resp. the fractional derivative of f of order $1/2 - h$) is the function denoted by f_h, which generally speaking belongs to $L^2(\mathbb{R})$, and which is defined through its Fourier transform $\widehat{f_h}$ by:

$$\widehat{f_h}(\xi) = (i\xi)^{1/2-h} \widehat{f}(\xi), \quad \text{for almost all } \xi \in \mathbb{R}. \tag{1.1}$$

One mentions that, using the common convention that, for all $(y, \alpha) \in \mathbb{R}^2$, when $y > 0$ one has $y_+^\alpha = y^\alpha$ and otherwise one has $y_+^\alpha = 0$, then, for any $h \in (1/2, 1)$, the fractional primitive f_h can be expressed as:

$$f_h(s) = \frac{1}{\Gamma(h - 1/2)} \int_{\mathbb{R}} (s - x)_+^{h-3/2} f(x) \, dx, \quad \text{for all } s \in \mathbb{R}, \tag{1.2}$$

where Γ is the usual "Gamma" Euler function defined, for all $z \in (0, +\infty)$, as $\Gamma(z) := \int_0^{+\infty} u^{z-1} e^{-u} \, du$. Also, one mentions that, when the Fourier transform $\widehat{f}$ of f vanishes on a neighbourhood of 0 (notice the univariate Meyer mother wavelet ψ satisfies this property), then one can drop the restriction $h \in (0, 1)$ and may allow h to be any real number. In the latter case, for all $h \in (1/2, +\infty)$ (resp. for all $h \in (-\infty, 1/2]$) the fractional primitive (resp. derivative) f_h can still be defined through its Fourier transform as in (1.1), and the equality (1.2) for fractional primitive remains valid. Moreover, for every $h \in \mathbb{R}$, one can easily check that f_h belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$.

Unfortunately, since for a univariate Meyer scaling function ϕ the Fourier transform $\widehat{\phi}$ does not vanish on a neighbourhood of 0, for all $h \in (0, 1)$, the fractional primitive or derivative ϕ_h, of ϕ, fails to be a smooth well-localized function. In order to overcome this serious difficulty, a clever idea of [20] was to "replace" ϕ_h by the so called fractional scaling function $\Phi_\Delta^{(\delta)}$, which belongs to $\mathcal{S}(\mathbb{R})$ and which was defined in [20] as follows:

Definition 1.3. The *fractional scaling function* of order $\delta \in \mathbb{R}$ of a univariate Meyer scaling function ϕ is the function $\Phi_\Delta^{(\delta)} \in \mathcal{S}(\mathbb{R})$ defined through its Fourier transform by:

$$\forall \, \xi \in \mathbb{R} \setminus \{0\}, \ \widehat{\Phi}_\Delta^{(\delta)}(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi} \right)^\delta \widehat{\phi}(\xi) \quad \text{and} \quad \widehat{\Phi}_\Delta^{(\delta)}(0) = 1.$$

Similarly to $\widehat{\phi}$, the function $\widehat{\Phi}_\Delta^{(\delta)}$ has a compact support satisfying

$$\operatorname{supp} \widehat{\Phi}_\Delta^{(\delta)} \subseteq \left[-\frac{4\pi}{3}, \frac{4\pi}{3} \right]. \tag{1.3}$$

Remark 1.4. Let δ and h be two arbitrary and fixed real numbers. One can check, from elementary properties of the Fourier transform (see e.g. the seminal book [26]), that the fractional scaling function $\Phi_\Delta^{(\delta)}$ and the fractional primitive or derivative ψ_h, of the univariate Meyer mother wavelet ψ, belong to $\mathcal{S}(\mathbb{R})$, which means that they are infinitely differentiable functions whose derivatives of any order rapidly decay at infinity, in other words one has, for all fixed $m \in \mathbb{N}_0$ and $L \in (0, +\infty)$,

$$\sup_{x \in \mathbb{R}} \left\{ (3 + |x|)^L \left(\left| \frac{d^m}{dx^m} \Phi_\Delta^{(\delta)}(x) \right| + \left| \frac{d^m}{dx^m} \psi_h(x) \right| \right) \right\} < +\infty. \tag{1.4}$$

Apart from the fact that $\Phi_{\Delta}^{(\delta)}$ is a very smooth and very well-localized function, another major advantage in expressing the low frequency part of FBM in terms of it is to draw connections between the latter process and FARIMA random walk time series (i.e. partial sums of FARIMA sequence (see Definition 1.6 below)), as shown by the following theorem of [20] which provides the most classical wavelet-type random series representation of FBM clearly separating its low and high frequency parts.

Theorem 1.5 (**Meyer, Sellan and Taqqu**). *For each fixed $J \in \mathbb{Z}$, the FBM $\{B_h(t)\}_{t\in\mathbb{R}}$ can be expressed as the following random series, which converges almost surely and uniformly in t on each compact interval of $\mathbb{R}$,*

$$\begin{aligned} B_h(t) &= \sum_{k\in\mathbb{Z}} 2^{-Jh} S_{J,k}^{(h)} \Big(\Phi_{\Delta}^{(h+1/2)}(2^J t - k) - \Phi_{\Delta}^{(h+1/2)}(-k) \Big) \\ &\quad + \sum_{j=J}^{+\infty} \sum_{k\in\mathbb{Z}} 2^{-jh} g_{j,k}^{\psi} \Big(\psi_{h+1}(2^j t - k) - \psi_{h+1}(-k) \Big), \end{aligned} \tag{1.5}$$

where:

- *$(g_{j,k}^{\psi})_{(j,k)\in\mathbb{Z}^2}$ is the sequence of the i.i.d. $\mathcal{N}(0,1)$ Gaussian random variables defined, for all $(j,k) \in \mathbb{Z}^2$, by the Wiener integral (with respect to a Brownian motion $\{B(x)\}_{x\in\mathbb{R}}$)*

$$g_{j,k}^{\psi} := 2^{j/2} \int_{\mathbb{R}} \psi(2^j x - k)\, dB(x); \tag{1.6}$$

- *given the sequence $(g_{J,k}^{\phi})_{k\in\mathbb{Z}}$ of the i.i.d. $\mathcal{N}(0,1)$ Gaussian random variables defined, for all $k \in \mathbb{Z}$, by*

$$g_{J,k}^{\phi} := 2^{J/2} \int_{\mathbb{R}} \phi(2^J x - k)\, dB(x), \tag{1.7}$$

- *$(S_{J,k}^{(h)})_{k\in\mathbb{Z}}$ is the Gaussian FARIMA random walk time series defined, for every $k \in \mathbb{Z}$, by*

$$S_{J,k}^{(h)} := \begin{cases} \sum_{\ell=1}^{k} Z_{J,\ell}^{(h-\frac{1}{2})} & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ -\sum_{\ell=k+1}^{0} Z_{J,\ell}^{(h-\frac{1}{2})} & \text{if } k < 0 \end{cases}$$

  with $(Z_{J,\ell}^{(h-\frac{1}{2})})_{\ell\in\mathbb{N}}$ the Gaussian FARIMA $(0, h-\frac{1}{2}, 0)$ sequence associated to $(g_{J,k}^{\phi})_{k\in\mathbb{Z}}$, see the next definition.

Definition 1.6. Let $(g_k)_{k\in\mathbb{Z}}$ be an arbitrary sequence of i.i.d. centred Gaussian random variables (for instance the sequence $(g_{J,k}^{\phi})_{k\in\mathbb{Z}}$ in the previous theorem). For each fixed $\delta \in (-1/2, 1/2)$, the Gaussian FARIMA $(0, \delta, 0)$ sequence associated to $(g_k)_{k\in\mathbb{Z}}$ is denoted by $(Z_l^{(\delta)})_{l\in\mathbb{Z}}$ and defined, for all $l \in \mathbb{Z}$, as:

$$Z_l^{(\delta)} := \gamma_0^{(\delta)} g_l + \sum_{p=1}^{+\infty} \gamma_p^{(\delta)} g_{l-p}\,, \quad \text{with } \gamma_0^{(\delta)} := 1 \text{ and } \gamma_p^{(\delta)} := \frac{\delta\, \Gamma(p+\delta)}{\Gamma(p+1)\Gamma(\delta+1)}. \tag{1.8}$$

Remark 1.7. Observe that, for the constant $a_\delta := \delta/\Gamma(\delta+1)$, it can be derived from the Stirling's formula that

$$\gamma_p^{(\delta)} \sim a_\delta\, p^{\delta-1}, \quad \text{when } p \text{ goes to } +\infty, \tag{1.9}$$

which implies that the random series in (1.8) is convergent in $L^2(\Omega)$, where Ω is the underlying probability space. Also notice that the latter series is almost surely convergent as well, thanks to the Kolmogorov's Three-Series theorem.

Remark 1.8. The FBM $\{B_h(t)\}_{t\in\mathbb{R}}$ can also be expressed as

$$B_h(t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{-jh} g_{j,k}^{\psi}\Big(\psi_{h+1}(2^j t-k) - \psi_{h+1}(-k)\Big), \tag{1.10}$$

where the series is convergent almost surely and uniformly in t on each compact interval of $\mathbb{R}$. Representations of the type (1.10) have turned out to be very useful in the study of local and global sample path behavior of various stochastic processes and fields extending FBM. Also it is worth mentioning that, even in the case of the FBM itself, whose sample path behavior was widely studied in the literature prior to wavelet theory, in the very recent article [12] the representation (1.10) has allowed to show that FBM sample paths have dense subsets of $\mathbb{R}$ of slow points and rapid points.

However, as explained in [20, 1, 25], the representation (1.5) is much more convenient than (1.10) for approximating the FBM $\{B_h(t)\}_{t\in\mathbb{R}}$. Indeed, according to (1.5), when J is large enough, $\{B_h(t)\}_{t\in\mathbb{R}}$ can be approximated by its low frequency part

$$B_{h,J}(t) = \sum_{k\in\mathbb{Z}} 2^{-Jh} S_{J,k}^{(h)}\left(\Phi_\Delta^{(h+1/2)}(2^J t-k) - \Phi_\Delta^{(h+1/2)}(-k)\right),$$

whose coefficients $S_{J,k}^{(h)}$, $k\in\mathbb{Z}$, can be rather easily obtained from the coefficients $S_{J-1,k}^{(h)}$, $k\in\mathbb{Z}$, of $\{B_{h,J-1}(t)\}_{t\in\mathbb{R}}$ by induction (pyramidal Mallat-type scheme); roughly speaking, this is due to the fact that the fractional scaling function $\Phi_\Delta^{(h+1/2)}$ generates a multiresolution analysis of $L^2(\mathbb{R})$ (see [20]).

In fact, FBM belongs to a much larger class of chaotic processes, the so-called Hermite processes. They are self-similar with stationary increments possessing a long-range dependence property. They first appeared in a natural way as limits of normalized partial sums of "strongly" correlated stationary Gaussian random sequences, in the so-called Non-Central Limit theorems established a long time ago by Taqqu, Dobrushin and Major [27, 28, 10]. Apart from the FBM, which is the Hermite process of order 1, any other Hermite process of arbitrary integer order $d \geq 2$ is non-Gaussian; in fact it belongs to the dth Wiener chaos, and it is even considered to be a paradigmatic example of a stochastic process in this chaos whose many properties are still unknown, though the second order chaos has turned out to be less difficult to study than the higher order chaoses. This fact have motivated many authors, interested in "conquering" non-Gaussian Wiener chaoses, to explore various issues related with them, we refer for instance to [5, 6, 7, 16, 24, 29, 30] to cite but a few works in this area.

By the end of the introduction of the paper [24] (see page 602 in it) published in 2004, Pipiras raised the problem to know whether wavelet-type random series representations with a well-localized smooth scaling function, reminiscent of the representation (1.5) of FBM, can be obtained for a Hermite process of any order d. He solved it in this same paper in the particular case $d = 2$ in which the Hermite process is called the Rosenblatt process. Moreover, some further advances have recently been made in this particular case $d = 2$ in the article [2] in which a rather sharp estimate of the almost sure uniform rate of convergence of the wavelet-type random series representing the Rosenblatt process has been obtained, and has even been shown to be valid in the extended framework of the generalized Rosenblatt process. For deriving this sharp estimate, the article [2] has introduced a new strategy which basically consists in expressing in a non-classical new way the approximation errors related with the approximation spaces of a multiresolution analysis of $L^2(\mathbb{R}^2)$, namely in terms of bivariate wavelet functions having two distinct dilation indices j_1 and j_2 (see Section 2 for more details).

So far, the challenging problem presented in the previous paragraph has remained completely open in the general case $d \geq 3$. In fact, for solving it, one has to face at least the following two major difficulties:

(a) To find in which way the low frequency part of an arbitrary Hermite process can be expressed in terms of FARIMA sequences and fractional scaling functions belonging to the Schwartz class.

(b) To show that a wavelet-type random series representation of any arbitrary Hermite process is almost surely uniformly convergent on compact intervals, and to estimate its almost sure uniform rate of convergent; the method introduced in [2] for reaching such a goal in the particular case of the generalized Rosenblatt process seems to be also useful in the general case of a Hermite process, yet some parts of it need to be significantly modified, in particular the crucial equality (2.33) in [2] fails to be true in the general case since, for $d \geq 3$, as far as we know, there is no generalized Plancherel formula which, loosely speaking, would be of the type: $\int_{\mathbb{R}^d} \prod_{l=1}^d f_l(x)\, dx = b_d \int_{\mathbb{R}^d} \prod_{l=1}^d \widehat{f_l}(\xi)\, d\xi$, where b_d is a universal constant only depending on d, and $\widehat{f_l}$ is the Fourier transform of the function f_l.

The main aim of our present article is to propose a solution for this open problem, not only for usual Hermite processes but also for the generalized Hermite processes, of any integer order $d \geq 3$, which were introduced by Bai and Taqqu in [4] and which extend the generalized Rosenblatt processes ($d = 2$) due to Maejima and Tudor [17]. Also, with this article, we hope to open the door to future development of simulation methods for such generalized chaotic processes for which no simulation method is available so far. We hope as well to open the door to that of new strategies allowing to study in depth their erratic local sample path behavior, as for instance to show the existence of slow points and rapid points, in the same spirit of what has been very recently done for FBM in [12] and for generalized Rosenblatt process in [9].

The generalized Hermite process of an arbitrary integer order $d \geq 2$ is denoted by $\{X_{\mathbf{h}}^{(d)}(t)\}_{t\in\mathbb{R}_+}$, because it depends on a vector-valued Hurst parameter $\mathbf{h} := (h_1, \ldots, h_d)$ whose coordinates h_l satisfy

$$h_1, \cdots, h_d \in (1/2, 1) \text{ and } \sum_{\ell=1}^{d} h_\ell > d - \frac{1}{2}. \tag{1.11}$$

This process belongs to the non-Gaussian dth Wiener chaos since it is defined, for each $t \in \mathbb{R}_+$, through the multiple Wiener integral:

$$X_{\mathbf{h}}^{(d)}(t) := \int_{\mathbb{R}^d}' K_{\mathbf{h}}^{(d)}(t, x_1, \ldots, x_d) dB(x_1) \cdots dB(x_d), \tag{1.12}$$

where $\{B(x)\}_{x \in \mathbb{R}}$ is a usual Brownian motion on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and where the deterministic kernel function $K_{\mathbf{h}}^{(d)}$ is given, for every $t \in \mathbb{R}_+$ and for Lebesgue almost all $(x_1, \ldots, x_d) \in \mathbb{R}^d$, by

$$K_{\mathbf{h}}^{(d)}(t, x_1, \cdots, x_d) := \frac{1}{\prod_{\ell=1}^{d} \Gamma(h_\ell - 1/2)} \int_0^t \prod_{j=1}^{d} (s - x_\ell)_+^{h_\ell - 3/2} \, ds. \tag{1.13}$$

Observe that the symbol $\int_{\mathbb{R}^d}'$ in (1.12) denotes integration over $\mathbb{R}^d$ with diagonals $\{x_\ell = x_{\ell'}\}$, $\ell \neq \ell'$, excluded. Also observe that when all the coordinates $h_1, \ldots, h_d$ of the vector-valued parameter $\mathbf{h}$ are equal, then the process $\{X_{\mathbf{h}}^{(d)}(t)\}_{t \in \mathbb{R}_+}$ reduces to usual Hermite process.

The remaining of our article is organized as follows. In Section 2, we present the main lines of our strategies as well as some major ingredients in them including some preliminary proofs, and we state our three main theorems. Sections 3, 4 and 5 are completely devoted to the proofs of our three main theorems. Some important results on multiple Wiener integrals, which are very useful for us, are given in Appendix A. At last the statements of some technical Lemmas, borrowed from the article [2] and used in many our proofs, are recalled in Appendix B.

2 Strategies, main results and some major ingredients

Let us start by briefly recalling some fundamental definitions and facts from wavelet analysis in $L^2(\mathbb{R}^d)$ which will be useful for justifying our strategies.

Definition 2.1. A *multiresolution analysis* of the Hilbert space $L^2(\mathbb{R}^d)$ is a sequence $(V_j^d)_{j \in \mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ satisfying the following four properties:

(a) for all $j \in \mathbb{Z}$, $V_j^d \subseteq V_{j+1}^d$;

(b) $\bigcap_{j \in \mathbb{Z}} V_j^d = \{0\}$ and $\bigcup_{j \in \mathbb{Z}} V_j^d$ is dense in $L^2(\mathbb{R}^d)$;

(c) for all $j \in \mathbb{Z}$, $V_j^d = \{f(2^j \cdot) : f \in V_0^d\}$;

(d) there exists a function $\Phi \in V_0^d$, called scaling function, such that the sequence $\big(\Phi(\cdot - \mathbf{k})\big)_{\mathbf{k} \in \mathbb{Z}^d}$ is an orthonormal basis of V_0^d. Notice that in the univariate case $d = 1$, this function Φ is denoted by ϕ as in the previous Section 1.

Remark 2.2. It clearly results from (c) and (d) in Definition 2.1, that, for all fixed $j \in \mathbb{Z}$, the sequence $\big(2^{jd/2}\Phi(2^j \cdot - \mathbf{k})\big)_{\mathbf{k} \in \mathbb{Z}^d}$ is an orthonormal basis of V_j^d.

Usually, one denotes by W_J^d the orthogonal complement of V_J^d in V_{J+1}^d. Then, it follows from (a) and (b) in Definition 2.1 that, for all fixed $J \in \mathbb{Z}$, the following very important equalities hold:

$$V_J^d = \overset{\perp}{\bigoplus_{-\infty<j<J}} W_j^d \text{ and } L^2(\mathbb{R}^d) = V_J^d \overset{\perp}{\oplus} \left(\overset{\perp}{\bigoplus_{J\le j<+\infty}} W_j^d \right) = \overset{\perp}{\bigoplus_{j\in\mathbb{Z}}} W_j^d\,. \tag{2.1}$$

Using (2.1), with $d = 1$ and an arbitrary J, one can derive from the following fundamental theorem (see e.g. [8, 19]) orthonormal bases for the subspace $V_J^1 \subset L^2(\mathbb{R})$ and for the whole space $L^2(\mathbb{R})$.

Theorem 2.3. *There is a function $\psi \in W_0^1$, called mother wavelet, such that, for all fixed $j \in \mathbb{Z}$, the sequence of functions $\big(2^{j/2}\psi(2^j \cdot - k)\big)_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j^1. Then, the important equalities (2.1), imply, for all fixed $J \in \mathbb{Z}$, that:*

(a) the sequence of functions $\big(2^{j/2}\psi(2^j \cdot - k)\big)_{j<J,\, k\in\mathbb{Z}}$ is an orthonormal basis for the space V_J^1;

(b) the sequences of functions $\big(2^{J/2}\phi(2^J \cdot - k)\big)_{k\in\mathbb{Z}} \cup \big(2^{j/2}\psi(2^j \cdot - k)\big)_{j\ge J,\, k\in\mathbb{Z}}$ and $(2^{j/2}\psi(2^j \cdot - k))_{(j,k)\in\mathbb{Z}^2}$ are two orthonormal bases for the space $L^2(\mathbb{R})$.

Such bases are called orthonormal wavelet bases.

Thanks to the tensor product method (see e.g. [8, 19]), for any integer $d > 1$ one can construct from a multiresolution analysis $(V_j^1)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ a multiresolution analysis $(V_j^d)_{j\in\mathbb{Z}}$ for $L^2(\mathbb{R}^d)$. Namely, for each $j \in \mathbb{Z}$, the space V_j^d is defined as $V_j^d := (V_j^1)^{\otimes_d}$ the tensor product of the space V_j^1, d times with itself. Then a scaling function Φ, which can be associated in a natural way to such a multiresolution analysis $(V_j^d)_{j\in\mathbb{Z}}$, is $\Phi := \phi^{\otimes_d}$, the tensor product of the univariate scaling function ϕ, d times with itself. In such a setting, it is well known that, for any fixed $J \in \mathbb{Z}$, an orthonormal wavelet basis of the space $(V_J^d)^\perp$ (the orthogonal complement of V_J^d in $L^2(\mathbb{R}^d)$) is:

$$\Big\{ 2^{jd/2} \prod_{l=1}^{d} \psi^{(\eta_l)}(2^j x_l - k_l) \;:\; j \in \mathbb{Z} \text{ and } j \ge J,$$
$$(\eta_1, \ldots, \eta_d) \in \{0,1\}^d \setminus \{0\}^d,\ (k_1, \ldots, k_d) \in \mathbb{Z}^d \Big\},$$

where $\psi^{(0)}$ and $\psi^{(1)}$ respectively denote the univariate scaling function and mother wavelet ϕ and ψ (see Theorem 2.3). Nevertheless, a major ingredient of strategies of our article consists in making use of another much less classical orthonormal wavelet basis of $(V_J^d)^\perp$. This idea comes from the article [2] in which $d = 2$ and whose main goal was to estimate almost sure rate of uniform convergence of wavelet-type random series representation of the generalized Rosenblatt process.

In order to precisely define the non classical orthonormal wavelet basis of $(V_J^d)^\perp$ we intend to use, we need to introduce some further notations. For all multi-indices $\mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d$ and $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, we denote by

$\psi_{\mathbf{j},\mathbf{k}}$ the multivariate wavelet function belonging to $L^2(\mathbb{R}^d)$ defined as the tensor product:

$$\psi_{\mathbf{j},\mathbf{k}} := \bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell},$$

where the univariate wavelet functions ψ_{j_l,k_l} are defined, for every $x \in \mathbb{R}$, as: $\psi_{j_l,k_l}(x) := 2^{j_l/2}\psi(2^{j_l}x - k_l)$. Observe that the previous definition of V_J^d through tensor product, and the point (a) in Theorem 2.3 imply that the collection of functions

$$\left\{\psi_{\mathbf{j},\mathbf{k}} \,:\, \mathbf{j},\mathbf{k} \in \mathbb{Z}^d \text{ and } \max_{\ell \in [\![1,d]\!]} j_\ell < J\right\},$$

where $[\![1,d]\!] := \{1,2,\ldots,d\}$, is an orthonormal basis of the subspace $V_J^d \subset L^2(\mathbb{R}^d)$; while the point (b) in this same theorem entails that the collection of functions $\{\psi_{\mathbf{j},\mathbf{k}} \,:\, \mathbf{j},\mathbf{k} \in \mathbb{Z}^d\}$ is an orthonormal basis of the whole space $L^2(\mathbb{R}^d)$, since $L^2(\mathbb{R}^d) = \big(L^2(\mathbb{R})\big)^{\otimes_d}$. Combining these two results, it turns out that the collection of functions

$$\left\{\psi_{\mathbf{j},\mathbf{k}} \,:\, \mathbf{j},\mathbf{k} \in \mathbb{Z}^d \text{ and } \max_{\ell \in [\![1,d]\!]} j_\ell \geq J\right\} \tag{2.2}$$

is an orthonormal basis of the subspace $(V_J^d)^\perp \in L^2(\mathbb{R}^d)$ which is the orthogonal complement of V_J^d in $L^2(\mathbb{R}^d)$.

Let us now precisely explain the connection between the latter basis and the error of approximation of a generalized Hermite process by the scaling function part of its wavelet-type random series representation. For each fixed $t \in \mathbb{R}_+$ and integer $J \geq 1$, the two functions of $L^2(\mathbb{R}^d)$ $(x_1,\ldots,x_d) \mapsto K_{\mathbf{h},J}^{(d)}(t,x_1,\cdots,x_d)$ and $(x_1,\ldots,x_d) \mapsto K_{\mathbf{h},J}^{(d,\perp)}(t,x_1,\cdots,x_d)$ respectively denote the two orthogonal projections of the function $(x_1,\ldots,x_d) \mapsto K_{\mathbf{h}}^{(d)}(t,x_1,\cdots,x_d)$ (see (1.12) and (1.13)) onto V_J^d and $(V_J^d)^\perp$. One clearly has that

$$K_{\mathbf{h}}^{(d)}(t,\bullet) - K_{\mathbf{h},J}^{(d)}(t,\bullet) = K_{\mathbf{h},J}^{(d,\perp)}(t,\bullet),$$

which leads us to define the approximation and details processes associated with the generalized Hermite process $\{X_{\mathbf{h}}^{(d)}(t)\}_{t\in\mathbb{R}_+}$ in the following way:

Definition 2.4. Let $d \in \mathbb{N}$ and $\mathbf{h}$ satisfying the conditions (1.11). For all $J \in \mathbb{N}$, the *approximation process at scale* J of the generalized Hermite process $\{X_{\mathbf{h}}^{(d)}(t)\}_{t\in\mathbb{R}_+}$ is the process defined, for all $t \in \mathbb{R}_+$, by the multiple Wiener integral:

$$X_{\mathbf{h},J}^{(d)}(t) := \int_{\mathbb{R}^d}' K_{\mathbf{h},J}^{(d)}(t,x_1,\ldots,x_d)dB(x_1)\ldots dB(x_d); \tag{2.3}$$

in fact $\{X_{\mathbf{h},J}^{(d)}(t)\}_{t\in\mathbb{R}_+}$ can be viewed as the scaling function part of the wavelet-type random series representation of $\{X_{\mathbf{h}}^{(d)}(t)\}_{t\in\mathbb{R}_+}$. The *details process at scale* J is defined, for all $t \in \mathbb{R}_+$, as:

$$X_{\mathbf{h},J}^{(d,\perp)}(t) := X_{\mathbf{h}}^{(d)}(t) - X_{\mathbf{h},J}^{(d)}(t) = \int_{\mathbb{R}^d}' K_{\mathbf{h},J}^{(d,\perp)}(t,x_1,\ldots,x_d)dB(x_1)\ldots dB(x_d); \tag{2.4}$$

in fact $\{X_{\mathbf{h},J}^{(d,\perp)}(t)\}_{t\in\mathbb{R}_+}$ can be viewed as the error stemming from the approximation of $\{X_{\mathbf{h}}^{(d)}(t)\}_{t\in\mathbb{R}_+}$ by $\{X_{\mathbf{h},J}^{(d)}(t)\}_{t\in\mathbb{R}_+}$.

Observe that combining (2.3) with the Wiener isometry and the fact that $\big(2^{J\frac{d}{2}}\Phi(2^J\cdot-\mathbf{k})\big)_{\mathbf{k}\in\mathbb{Z}^d}$ is an orthonormal basis of V_J^d, one gets, for each fixed $t\in\mathbb{R}_+$, that

$$X_{\mathbf{h},J}^{(d)}(t)=\sum_{\mathbf{k}\in\mathbb{Z}^d}\mu_{J,\mathbf{k}}\mathfrak{K}_{J,\mathbf{k}}^{(d,\mathbf{h})}(t), \tag{2.5}$$

where the sequence $(\mu_{J,\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ of random variables in the dth order Wiener chaos and the deterministic sequence $\big(\mathfrak{K}_{J,\mathbf{k}}^{(d,\mathbf{h})}(t)\big)\big)_{\mathbf{k}\in\mathbb{Z}^d}$ of $\ell^2(\mathbb{Z}^d)$ are given, for all $\mathbf{k}\in\mathbb{Z}^d$, by:

$$\mu_{J,\mathbf{k}}:=2^{J\frac{d}{2}}\int_{\mathbb{R}^d}'\phi(2^Jx_1-k_1)\cdots\phi(2^Jx_d-k_d)\,dB(x_1)\ldots dB(x_d) \tag{2.6}$$

and

$$\mathfrak{K}_{J,\mathbf{k}}^{(d,\mathbf{h})}(t):=2^{J\frac{d}{2}}\int_{\mathbb{R}^d}K_{\mathbf{h}}^{(d)}(t,x_1,\ldots,x_d)\phi(2^Jx_1-k_1)\cdots\phi(2^Jx_d-k_d)\,dx_1\cdots dx_d. \tag{2.7}$$

Also observe that combining (2.4) with the Wiener isometry and the fact that the collection of functions in (2.2) is an orthonormal basis of $(V_J^d)^{\perp}$, one obtains, for each fixed $t\in\mathbb{R}_+$, that

$$X_{\mathbf{h},J}^{(d,\perp)}(t)=\sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2\\ \max_{\ell\in[\![1,d]\!]}j_\ell\geq J}}\varepsilon_{\mathbf{j},\mathbf{k}}\,\mathcal{K}_{\mathbf{j},\mathbf{k}}^{(d,\mathbf{h})}(t), \tag{2.8}$$

where the dth order Wiener chaos random variables $\varepsilon_{\mathbf{j},\mathbf{k}}$ and the deterministic coefficients $\mathcal{K}_{\mathbf{j},\mathbf{k}}^{(d,\mathbf{h})}(t)$ are given by:

$$\varepsilon_{\mathbf{j},\mathbf{k}}:=\int_{\mathbb{R}^d}'\psi_{\mathbf{j},\mathbf{k}}(x_1,\ldots,x_d)\,dB(x_1)\ldots dB(x_d) \tag{2.9}$$

and

$$\mathcal{K}_{\mathbf{j},\mathbf{k}}^{(d,\mathbf{h})}(t):=\int_{\mathbb{R}^d}K_{\mathbf{h}}^{(d)}(t,x_1,\ldots,x_d)\psi_{\mathbf{j},\mathbf{k}}(x_1,\ldots,x_d)\,dx_1\cdots dx_d. \tag{2.10}$$

One mentions in passing that, so far, one only knows that the random series in (2.5) and (2.8) are unconditionally convergent in $L^2(\Omega)$, for each fixed $t\in\mathbb{R}_+$.

Remark 2.5. Similarly to the article [20], from now and till the end of our article we always assume that the univariate scaling function and mother wavelet ϕ and ψ are associated with an orthonormal Meyer wavelet basis of $L^2(\mathbb{R})$ (see Remark 1.1). Then, it results from (1.2), (1.13), (2.10), Fubini theorem and the changes of variable $y_\ell=2^{j_\ell}x_\ell-k_\ell$ (for all $\ell\in[\![1,d]\!]$), that

$$\mathcal{K}_{\mathbf{j},\mathbf{k}}^{(d,\mathbf{h})}(t)=2^{j_1(1-h_1)+\cdots+j_d(1-h_d)}\int_0^t\prod_{\ell=1}^d\psi_{h_\ell}(2^{j_\ell}s-k_\ell)\,ds, \tag{2.11}$$

where ψ_{h_l} is the fractional primitive of order $h_l-1/2$ of ψ. Also, one can derive from (2.7) and similar arguments that

$$\mathfrak{K}_{J,\mathbf{k}}^{(d,\mathbf{h})}(t)=2^{-J(h_1+\cdots+h_d-d)}\int_0^t\prod_{\ell=1}^d\phi_{h_\ell}(2^Js-k_\ell)\,ds, \tag{2.12}$$

where ϕ_{h_l} is the fractional primitive of order $h_l - 1/2$ of ϕ. Notice that one knows from (1.1) that the Fourier transform of ϕ_{h_l} satisfies

$$\widehat{\phi}_{h_l}(\xi) = (i\xi)^{1/2-h_l}\widehat{\phi}(\xi), \quad \text{for almost all } \xi \in \mathbb{R}. \tag{2.13}$$

In view of the fact that the functions ϕ_{h_l} fail to belong to the Schwartz class $\mathcal{S}(\mathbb{R})$ and are even badly localized functions, one of the main goal of our article will be to introduce, in the same spirit of what has been done for the approximation process of FBM in [20] and for that of Rosenblatt process in [24], a modified version of the random series representation (2.5) in which the deterministic coefficients are expressed in terms of "nice" fractional scaling functions (see Definition 1.3) belonging to the Schwartz class. In order to adapt ideas of [20, 24] to the framework of the generalized Hermite process $\{X_{\mathbf{h}}^{(d)}(t)\}_{t\in\mathbb{R}_+}$, which is much more complex than those of FBM and Rosenblatt process, we need to introduce, for each fixed $J \in \mathbb{Z}$, the sequence of random variables $(\sigma_{J,\mathbf{k}}^{(\mathbf{h})})_{\mathbf{k}\in\mathbb{Z}^d}$, defined, for all $\mathbf{k} \in \mathbb{Z}^d$, as:

$$\sigma_{J,\mathbf{k}}^{(\mathbf{h})} := \sum_{\mathbf{p}\in\mathbb{N}_0^d} \Big(\prod_{l=1}^{d} \gamma_{p_l}^{(h_l-1/2)}\Big)\mu_{J,\mathbf{k}-\mathbf{p}}, \tag{2.14}$$

where the deterministic coefficients $\gamma_{p_l}^{(h_l-1/2)}$ are given by the third and the second equalities in (1.8) with $p = p_l$ and $\delta = h_l - 1/2$. Notice that the following Proposition 2.7 shows, among other things, that the definition (2.14) makes sense. Roughly speaking, the sequence $(\sigma_{J,\mathbf{k}}^{(\mathbf{h})})_{\mathbf{k}\in\mathbb{Z}^d}$ can be viewed as a generalized FARIMA sequence. In fact, it can be expressed in terms of usual FARIMA sequences (see Proposition 2.7 below). In order to provide the latter expression of $\sigma_{J,\mathbf{k}}^{(\mathbf{h})}$, we need the following definition:

Definition 2.6. Let S be an arbitrary finite subset of $\mathbb{N}$ whose cardinality is denoted by $\#S$. Then, for any integer m such that $0 \le m \le \lfloor \#S/2 \rfloor$, one denotes by $\mathcal{P}_m^S$ the finite set of the partitions of S with m (non ordered) pairs and $\#S - 2m$ singletons. Moreover, for the sake of simplicity, when $S = [\![1, n]\!]$ with $n \in \mathbb{N}$ being arbitrary, one sets $\mathcal{P}_m^{(n)} = \mathcal{P}_m^{[\![1,n]\!]}$.

Proposition 2.7. *For all fixed $(J, \boldsymbol{k}) \in \mathbb{Z} \times \mathbb{Z}^d$, the random series in (2.14) is convergent almost surely and in $L^\gamma(\Omega)$, for any $\gamma \in (0, +\infty)$. Moreover, one has that*

$$\sigma_{J,\boldsymbol{k}}^{(\mathbf{h})} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P\in\mathcal{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}\big[Z_{J,k_{\ell_r}}^{(h_{\ell_r}-1/2)} Z_{J,k_{\ell'_r}}^{(h_{\ell'_r}-1/2)}\big] \prod_{s=m+1}^{d-m} Z_{J,k_{\ell''_s}}^{(h_{\ell''_s}-1/2)}, \tag{2.15}$$

where the indices ℓ_r, ℓ'_r and ℓ''_s are such that

$$P = \Big\{\{\ell_1, \ell'_1\}, \dots, \{\ell_m, \ell'_m\}, \{\ell''_{m+1}\}, \dots, \{\ell''_{d-m}\}\Big\},$$

and where, for all $\delta \in (0, 1/2)$, $(Z_{J,q}^{(\delta)})_{q\in\mathbb{Z}}$, is the FARIMA $(0, \delta, 0)$ sequence (see Definition 1.6) associated with the sequence $(g_{J,k}^{\phi})_{k\in\mathbb{Z}}$ of i.i.d. $\mathcal{N}(0, 1)$ Gaussian random variables introduced in (1.7).

We mention in passing that, in the particular case $d=2$, $\sigma_{J,\mathbf{k}}^{(\mathbf{h})}=\sigma_{J,k_1,k_2}^{(h_1,h_2)}$ reduces to

$$\sigma_{J,k_1,k_2}^{(h_1,h_2)}=Z_{J,k_1}^{(h_1-1/2)}Z_{J,k_2}^{(h_2-1/2)}-\mathbb{E}[Z_{J,k_1}^{(h_1-1/2)}Z_{J,k_2}^{(h_2-1/2)}].$$

Also, we mention that, in the particular case $d=3$, $\sigma_{J,\mathbf{k}}^{(\mathbf{h})}=\sigma_{J,k_1,k_2,k_3}^{(h_1,h_2,h_3)}$ reduces to

$$\begin{aligned}
&\sigma_{J,k_1,k_2,k_3}^{(h_1,h_2,h_3)}\\
&=Z_{J,k_1}^{(h_1-1/2)}Z_{J,k_2}^{(h_2-1/2)}Z_{J,k_3}^{(h_3-1/2)}-\mathbb{E}[Z_{J,k_1}^{(h_1-1/2)}Z_{J,k_2}^{(h_2-1/2)}]Z_{J,k_3}^{(h_3-1/2)}\\
&\quad-\mathbb{E}[Z_{J,k_1}^{(h_1-1/2)}Z_{J,k_3}^{(h_3-1/2)}]Z_{J,k_2}^{(h_2-1/2)}-\mathbb{E}[Z_{J,k_2}^{(h_2-1/2)}Z_{J,k_3}^{(h_3-1/2)}]Z_{J,k_1}^{(h_1-1/2)}.
\end{aligned}$$

Before proving Proposition 2.7, let us state the first main theorem of our article which provides a modified version of the random series representation (2.5) obtained through the generalized FARIMA sequence $(\sigma_{J,\mathbf{k}}^{(\mathbf{h})})_{\mathbf{k}\in\mathbb{Z}^d}$ (see (2.15) and (2.14)) as well as "nice" fractional scaling functions (see Definition 1.3) belonging to the Schwartz class.

Theorem 2.8. *The approximation process* $\{X_{\mathbf{h},J}^{(d)}(t)\}_{t\in\mathbb{R}_+}$*, defined in* (2.3)*, can be expressed, for all* $t\in\mathbb{R}_+$*, as:*

$$X_{\mathbf{h},J}^{(d)}(t)=2^{-J(h_1+\ldots+h_d-d)}\sum_{\mathbf{k}\in\mathbb{Z}^d}\left(\int_0^t\prod_{\ell=1}^d\Phi_\Delta^{(h_\ell-1/2)}(2^Js-k_\ell)\,ds\right)\sigma_{J,\boldsymbol{k}}^{(\mathbf{h})},\quad(2.16)$$

where the series is convergent in $L^2(\Omega)$*. Moreover this series is also almost surely uniformly convergent in* t *on each compact interval of* $\mathbb{R}_+$*.*

Remark 2.9. Let f be an arbitrary function in the Schwartz class $\mathcal{S}(\mathbb{R})$ and let $(a_p)_{p\in\mathbb{Z}}$ be an arbitrary slowly increasing sequence of real numbers, that is we have, for some constants $\kappa>0$ and $\mu>0$ and for every $p\in\mathbb{Z}$, $|a_p|\le\kappa(1+|p|)^\mu$. It is known (see for instance [20]) that, if we set $A_0=0$ and $A_q-A_{q-1}=a_q$ for all $q\in\mathbb{Z}\setminus\{0\}$ and $\widetilde{f}(y)=\int_{y-1}^y f(v)\,dv$ for all $y\in\mathbb{R}$, then the function $\widetilde{f}$ belongs to $\mathcal{S}(\mathbb{R})$ and the sequence $\{A_k\}_{k\in\mathbb{Z}}$ is slowly increasing. Moreover, using an Abel transform, for all $t\in\mathbb{R}$, we have

$$\sum_{k\in\mathbb{Z}}a_k\int_0^t f(v-k)\,dv=\sum_{q\in\mathbb{Z}}A_q(\widetilde{f}(t-q)-\widetilde{f}(-q)).\qquad(2.17)$$

In order to apply (2.17) in the framework of Theorem 2.8, we define, for each $(J,q,\mathbf{n})\in\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}^{d-1}$, the random variable $S_{J,q,\mathbf{n}}^{(\mathbf{h})}$ as:

$$S_{J,q,\mathbf{n}}^{(\mathbf{h})}=\begin{cases}\sum_{p=1}^q\sigma_{J,(p,\mathbf{n}+p)}^{(\mathbf{h})} & \text{if } q>0\\ 0 & \text{if } q=0\\ -\sum_{p=q+1}^0\sigma_{J,(p,\mathbf{n}+p)}^{(\mathbf{h})} & \text{if } q<0,\end{cases}\qquad(2.18)$$

with the convention that $\mathbf{n}+p:=(n_1+p,\ldots,n_{d-1}+p)$. Also, for every $\mathbf{n}=(n_1,\ldots,n_{d-1})\in\mathbb{Z}^{d-1}$, we define the function $\widetilde{\Phi}_{\Delta,\mathbf{n}}^{(\mathbf{h})}$, belonging to $\mathcal{S}(\mathbb{R})$, as:

$$\widetilde{\Phi}_{\Delta,\mathbf{n}}^{(\mathbf{h})}(y):=\int_{y-1}^y\Phi_\Delta^{(h_1-1/2)}(v)\prod_{\ell=1}^{d-1}\Phi_\Delta^{(h_{\ell+1}-1/2)}(v-n_\ell)\,dv,\quad\text{for all } y\in\mathbb{R}.$$

Then, using Theorem 2.8, Fubini theorem, the change of variable $v = 2^J u$, the change of indices $k_1 = p$ and $n_{l-1} = k_l - k_1$ (for all $l \in [\![2, d]\!]$), the slow increase property for the sequence $\big(\sigma^{(\mathbf{h})}_{J,(p,\mathbf{n}+p)}\big)_{p\in\mathbb{Z}}$ provided by (3.22), (2.17), a slow increase property (derived from (3.22) and (2.18)) for the sequence $\big(S^{(\mathbf{h})}_{J,q,\mathbf{n}}\big)_{q\in\mathbb{Z}}$ with a random constant[1] $\kappa(\mathbf{n}) = \mathcal{O}\big(\log^{d/2}(3+|\mathbf{n}|)\big)$, and the inequality

$$\sup_{y\in[0,Y]}\ \sup_{(q,\mathbf{n})\in\mathbb{Z}\times\mathbb{Z}^{d-1}} \Big\{\big(3+|p|+|\mathbf{n}|\big)^L \big|\widetilde{\Phi}^{(\mathbf{h})}_{\Delta,\mathbf{n}}(y)\big|\Big\} < \infty, \quad \text{for all fixed } Y, L > 0,$$

we obtain that

$$X^{(d)}_{\mathbf{h},J}(t) = 2^{-J(h_1+\ldots+h_d+1-d)} \sum_{\mathbf{n}\in\mathbb{Z}^{d-1}} \sum_{q\in\mathbb{Z}} S^{(\mathbf{h})}_{J,q,\mathbf{n}} \Big(\widetilde{\Phi}^{(\mathbf{h})}_{\Delta,\mathbf{n}}(2^J t - q) - \widetilde{\Phi}^{(\mathbf{h})}_{\Delta,\mathbf{n}}(-q)\Big), \tag{2.19}$$

where the convergence of the random series holds almost surely and uniformly in t on each compact interval of $\mathbb{R}_+$. Notice that the random series representation (2.19) for the approximation $\{X^{(d)}_{\mathbf{h},J}(t)\}_{t\in\mathbb{R}_+}$ of the generalized Hermite process is reminiscent of that of the low frequency part (that is the scaling function part) in the representation of FBM in (1.5).

The proof of Theorem 2.8 will be given in Section 3. Let us now focus on the proof of the fundamental Proposition 2.7. Its starting point consists in an expression of the dth order Wiener chaos random variable $\mu_{J,\mathbf{k}}$ (see (2.6)) in terms of the i.i.d Gaussian random variables $g^{\phi}_{J,k}$ and Hermite polynomials H_n. We mention in passing that a rather similar expression also holds for the dth order Wiener chaos random variable $\varepsilon_{\mathbf{j},\mathbf{k}}$ (see (2.9)); it will be useful for us later. For giving these expressions for $\mu_{J,\mathbf{k}}$ and $\varepsilon_{\mathbf{j},\mathbf{k}}$ it is convenient to make use of the very common notation for multiple Wiener integral: for any $n \in \mathbb{N}$ and $f \in L^2(\mathbb{R}^n)$,

$$I_n(f) = \int'_{\mathbb{R}^n} f(x_1,\ldots,x_n)\, dB(x_1)\ldots dB(x_n).$$

It is known (see e.g. equation (1) in [14]) that, for any univariate functions $\varphi_1,\ldots,\varphi_p$ of $L^2(\mathbb{R})$ which are orthonormal and for every $n_1,\ldots,n_p \in \mathbb{N}$, one has

$$I_{n_1+\cdots+n_p}\Big(\varphi_1^{\otimes n_1}\otimes\cdots\otimes\varphi_p^{\otimes n_p}\Big) = \prod_{\ell=1}^{p} H_{n_\ell}\Big(\int_{\mathbb{R}} \varphi_\ell(x)\, dB(x)\Big), \tag{2.20}$$

where we recall the following definition.

Definition 2.10. For all $n \in \mathbb{Z}_+$, the nth Hermite polynomial is the polynomial of degree n denoted by H_n and defined, for every $x \in \mathbb{R}$, as:

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

For instance, the first four Hermite polynomials are $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$ and $H_3(x) = x^3 - 3x$.

[1] All along this paper, if $\mathbf{n} \in \mathbb{Z}^d$, we use the notation $|\mathbf{n}| = \sum_{\ell=1}^{d} |n_\ell|$.

The equality (2.20) will play a crucial role in the sequel; for the sake of completeness its proof is given in Appendix A. In order to apply it to the multiple Wiener integrals in (2.6) and (2.9), we need to introduce some notations. In fact, any $(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^d)^2$ can be viewed as a finite sequence $\big((j_m,k_m)\big)_{1\le m\le d}$ whose d terms (j_m,k_m) belong to $\mathbb{Z}^2$ and some of them can be equal to each other. The positive integer $p(\mathbf{j},\mathbf{k}) \le d$ denotes the number of the distinct terms of the sequence $(\mathbf{j},\mathbf{k}) = \big((j_m,k_m)\big)_{1\le m\le d}$, and the latter terms are denoted by $(\widetilde{j_\ell},\widetilde{k_\ell})$, $1 \le l \le p(\mathbf{j},\mathbf{k})$; moreover the notation $(\widetilde{j_\ell},\widetilde{k_\ell})_{n_\ell}$, where $n_\ell \in \{1,\dots,d\}$, means that $(\widetilde{j_\ell},\widetilde{k_\ell})$ has the multiplicity n_ℓ, that is there are exactly n_ℓ terms of the sequence $\big((j_m,k_m)\big)_{1\le m\le d}$ which are equal to $(\widetilde{j_\ell},\widetilde{k_\ell})$. At last, it is clear that $\sum_{\ell=1}^{p(\mathbf{j},\mathbf{k})} n_\ell = d$. Using these notations and (1.6), we can derive from (2.9) and (2.20) that, for all $(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^2)^d$,

$$\varepsilon_{\mathbf{j},\mathbf{k}} = \prod_{\ell=1}^{p(\mathbf{j},\mathbf{k})} H_{n_\ell}\left(g^{\psi}_{\widetilde{j_\ell},\widetilde{k_\ell}}\right). \tag{2.21}$$

Similar arguments and (1.7) allow to shown that, for all $J \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$,

$$\mu_{J,\mathbf{k}} = \prod_{\ell=1}^{p(\{J\}^d,\mathbf{k})} H_{n_\ell}\left(g^{\phi}_{J,\widetilde{k_\ell}}\right); \tag{2.22}$$

observe that the positive integer n_ℓ in (2.22) is the multiplicity of $\widetilde{k_\ell}$ in $\mathbf{k}$. In order to connect the random variables $\mu_{J,\mathbf{k}}$ to FARIMA sequences (see Definition 1.6), we have to rewrite the expression (2.22) in a way that gives us an easier "access" to the i.i.d Gaussian random variables $g^{\phi}_{J,k}$ in it. To this end, we recall that, for any $n \in \mathbb{N}$, the nth Hermite polynomial H_n satisfies, for all $x \in \mathbb{R}$, the equality:

$$H_n(x) = \sum_{m=0}^{\lfloor n/2\rfloor} (-1)^m a_m^{(n)} x^{n-2m}, \tag{2.23}$$

where $a_m^{(n)}$ is the number of partitions of $[\![1,n]\!]$ with m (non ordered) pairs and $n-2m$ singletons.

Lemma 2.11. *Using notations already introduced in Definition 2.6, for all $J \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$, the random variable $\mu_{J,\mathbf{k}}$ in* (2.22) *can be rewritten as:*

$$\mu_{J,\mathbf{k}} = \sum_{m=0}^{\lfloor d/2\rfloor} (-1)^m \sum_{P\in\mathcal{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}[g^{\phi}_{J,k_{\ell_r}} g^{\phi}_{J,k_{\ell'_r}}] \prod_{s=m+1}^{d-m} g^{\phi}_{J,k_{\ell''_s}}, \tag{2.24}$$

where the indices ℓ_r, ℓ'_r and ℓ''_s are such that

$$P = \Big\{\{\ell_1,\ell'_1\},\dots,\{\ell_m,\ell'_m\},\{\ell''_{m+1}\},\dots,\{\ell''_{d-m}\}\Big\}.$$

Proof. Let us proceed by induction on the positive integer d. It easily follows from (2.22) and Definition 2.10 that the equality (2.24) is satisfied in the two particular cases $d=1$ and $d=2$. In the sequel, one assumes that $d>2$ and that (2.24) holds for any positive integer n such that $n<d$. Let us first show

that these assumptions allow to prove (2.24) when the d indices forming the multi-index $\mathbf{k}$ are all equal together, that is $\mathbf{k} = (k_1, \ldots, k_1)$. Indeed, the latter equality implies, for all $\ell_r, \ell'_r \in [\![1, d]\!]$, that $\mathbb{E}[g^{\phi}_{J,k_{\ell_r}} g^{\phi}_{J,k_{\ell'_r}}] = 1$, which in turn entails that

$$\sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P \in \mathcal{P}^{(d)}_m} \prod_{r=1}^{m} \mathbb{E}[g^{\phi}_{J,k_{\ell_r}} g^{\phi}_{J,k_{\ell'_r}}] \prod_{s=m+1}^{d-m} g^{\phi}_{J,k_{\ell''_s}} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m a^{(d)}_m (g^{\phi}_{J,k_1})^{d-2m}$$
$$= H_d(g^{\phi}_{J,k_1}) = \mu_{J,\mathbf{k}},$$

where the second and the third equalities respectively follow from (2.23) and (2.22). From now on, we focus on the case in which the d indices forming the multi-index $\mathbf{k}$ are not equal together. Thus, there exists a unique integer a satisfying $1 \le a < d$ such that one has $\mathbf{k} = (k_1, \ldots, k_1, k_{a+1}, \ldots, k_d)$ with $k_1 \ne k_\ell$, for all $a < \ell \le d$; in fact a is nothing else than the multiplicity of the first index of $\mathbf{k}$. Then, one can derive from (2.22), (2.23) and the induction hypothesis that

$$\mu_{J,\mathbf{k}} = H_a(g^{\phi}_{J,k_1}) \prod_{\ell=2}^{p} H_{n_\ell}(\mu_{J,\widetilde{k}_\ell}) = \left(\sum_{m=0}^{\lfloor a/2 \rfloor} (-1)^m a^{(d)}_m (g^{\phi}_{J,k_1})^{d-2m} \right) \prod_{\ell=2}^{p} H_{n_\ell}(\mu_{J,\widetilde{k}_\ell})$$
$$= \left(\sum_{m=0}^{\lfloor a/2 \rfloor} (-1)^m \sum_{P_1 \in \mathcal{P}^{(a)}_m} \prod_{r=1}^{m} \mathbb{E}[g^{\phi}_{J,k_{\ell_r}} g^{\phi}_{J,k_{\ell'_r}}] \prod_{s=m+1}^{a-m} g^{\phi}_{J,k_{\ell''_s}} \right) \times \ldots$$
$$\ldots \times \left(\sum_{n=0}^{\lfloor (d-a)/2 \rfloor} (-1)^n \sum_{P_2 \in \mathcal{P}^{[\![a+1,d]\!]}_n} \prod_{t=1}^{n} \mathbb{E}[g^{\phi}_{J,k_{\ell_t}} g^{\phi}_{J,k_{\ell'_t}}] \prod_{u=n+1}^{d-a-n} g^{\phi}_{J,k_{\ell''_u}} \right)$$
$$= \sum_{v=0}^{\lfloor d/2 \rfloor} (-1)^v \sum_{m,n\,:\,m+n=v} \left(\sum_{P \in \mathcal{P}^{(d,a)}_{v,[m,n]}} \prod_{r=1}^{v} \mathbb{E}[g^{\phi}_{J,k_{\ell_r}} g^{\phi}_{J,k_{\ell'_r}}] \prod_{s=v+1}^{d-v} g^{\phi}_{J,k_{\ell''_s}} \right),$$

where $\mathcal{P}^{(d,a)}_{v,[m,n]}$ is the subset of $\mathcal{P}^{(d)}_v$ of the partitions of $[\![1, d]\!]$ with m (non ordered) pairs of integers in $[\![1, a]\!]$ and n (non ordered) pairs of integers in $[\![a + 1, d]\!]$; notice that when $\lfloor a/2 \rfloor + \lfloor (d - a)/2 \rfloor < m + n \le \lfloor d/2 \rfloor$ then $\mathcal{P}^{(d,a)}_{v,[m,n]}$ becomes an empty set, therefore the sum over it reduces to zero.

Finally, notice that when $P' \in \mathcal{P}^{(d)}_v$ is a partition with at least a (non ordered) pair $\{\ell, \ell'\}$ such that $\ell \in [\![1, a]\!]$ and $\ell' \in [\![a+1, d]\!]$, then $\mathbb{E}[g^{\phi}_{J,k_\ell} g^{\phi}_{J,k_{\ell'}}] = 0$, thus, using the fact that $\mathcal{P}^{(d,a)}_{v,[m',n']} \cap \mathcal{P}^{(d,a)}_{v,[m'',n'']} = \emptyset$ when $(m', n') \ne (m'', n'')$, one gets that

$$\sum_{v=0}^{\lfloor d/2 \rfloor} (-1)^v \sum_{m,n\,:\,m+n=v} \left(\sum_{P \in \mathcal{P}^{(d,a)}_{v,[m,n]}} \prod_{r=1}^{v} \mathbb{E}[g^{\phi}_{J,k_{\ell_r}} g^{\phi}_{J,k_{\ell'_r}}] \prod_{s=v+1}^{d-v} g^{\phi}_{J,k_{\ell''_s}} \right)$$
$$= \sum_{v=0}^{\lfloor d/2 \rfloor} (-1)^v \sum_{P \in \mathcal{P}^{(d)}_v} \prod_{r=1}^{v} \mathbb{E}[g^{\phi}_{J,k_{\ell_r}} g^{\phi}_{J,k_{\ell'_r}}] \prod_{s=v+1}^{d-v} g^{\phi}_{J,k_{\ell''_s}},$$

which shows that (2.24) is valid. □

We are now in position to prove Proposition 2.7.

Proof of Proposition 2.7. Combining Lemma 2.11 with Remark 1.7, it can easily be shown that, when the integer n goes to $+\infty$, the partial sum of order n of the random series in (2.14), that is the dth order Wiener chaos random variable

$$\sum_{\mathbf{p}\in[\![0,n]\!]^d} \Big(\prod_{l=1}^{d} \gamma_{p_l}^{(h_l-1/2)}\Big)\mu_{J,\mathbf{k}-\mathbf{p}}\,,$$

converges almost surely to

$$\sum_{m=0}^{\lfloor d/2\rfloor}(-1)^m \sum_{P\in\mathcal{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}[Z_{J,k_{\ell_r}}^{(h_{\ell_r}-1/2)} Z_{J,k_{\ell'_r}}^{(h_{\ell'_r}-1/2)}] \prod_{s=m+1}^{d-m} Z_{J,k_{\ell''_s}}^{(h_{\ell''_s}-1/2)}.$$

The fact that the convergence also holds in $L^\gamma(\Omega)$, for any $\gamma\in(0,+\infty)$, can be derived from a general result in [13] according to which any sequence of random variables belonging to a finite order Wiener chaos converges in $L^\gamma(\Omega)$ as soon as it converges in probability. □

The following theorem, which provides, for $\|\cdot\|_{I,\infty}$ the uniform norm on any compact interval $I\subset\mathbb{R}_+$, an almost sure estimate of the error stemming from the approximation of $\{X_{\mathbf{h}}^{(d)}(t)\}_{t\in I}$ by $\{X_{\mathbf{h},J}^{(d)}(t)\}_{t\in I}$ is the second main result of our article. This theorem will be proved in Section 4.

Theorem 2.12. *For any compact interval $I\subset\mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J\in\mathbb{N}$,*

$$\|X_{\mathbf{h}}^{(d)} - X_{\mathbf{h},J}^{(d)}\|_{I,\infty} = \|X_{\mathbf{h},J}^{(d,\perp)}\|_{I,\infty} \le CJ^{\frac{d}{2}}2^{-J(h_1+\cdots+h_d-d+1/2)}. \tag{2.25}$$

Before stating the third and the last main result of our article, let us explain the motivation behind it. As the collection of functions $\{\psi_{\mathbf{j},\mathbf{k}} : \mathbf{j},\mathbf{k}\in\mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, one can also wish to give a random series representation for the generalized Hermite process $\{X_{\mathbf{h}}^{(d)}(t)\}_{t\in\mathbb{R}_+}$ using this basis. Indeed, similarly to (2.8), it can be shown that

$$X_{\mathbf{h}}^{(d)}(t) = \sum_{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2} \varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{K}_{\mathbf{j},\mathbf{k}}^{(d,\mathbf{h})}(t), \tag{2.26}$$

where the random series is unconditionally convergent in $L^2(\Omega)$, for each fixed $t\in\mathbb{R}_+$. Roughly speaking, our third main result shows that when the partial sums of the random series in (2.26) are well-chosen, then its convergence holds in a much stronger sense: almost surely for the uniform norm $\|\cdot\|_{[0,T],\infty}$, where the fixed real number $T>2$ is arbitrary. Also, our third main result provides an almost sure estimate of the rate of convergence of the series for the uniform norm $\|\cdot\|_{[0,T],\infty}$. In order to precisely explain how the partial sums have to be chosen, we need the following definition:

Definition 2.13. Let $T>2$, $b>0$, $b'>0$ and $g>0$ be four fixed arbitrary real numbers. For all $N\in\mathbb{N}$, we define the two disjoint finite subsets of $(\mathbb{Z}^d)^2$

$$\mathcal{S}_N^+ := \{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \,:\, -2^{Nb} \le \min_{\ell\in[\![1,d]\!]} j_\ell,\ 0\le \max_{\ell\in[\![1,d]\!]} j_\ell < N,\ \max_{\ell\in[\![1,d]\!]} |k_\ell| \le 2^{N+1}T\}$$

and

$$\mathcal{S}_N^- := \{(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^d)^2 \,:\, -2^{Nb'} \leq \min_{\ell\in[\![1,d]\!]} j_\ell \leq \max_{\ell\in[\![1,d]\!]} j_\ell < 0,\, \max_{\ell\in[\![1,d]\!]} |k_\ell| \leq 2^{Ng}\}.$$

We are now in position to state our third and last main result.

Theorem 2.14. *Let $T>2$, $b>0$, $b'>0$ and $g>0$ be four fixed arbitrary real numbers. For all $t\in\mathbb{R}^+$ and $N\in\mathbb{N}$, let $\widetilde{X}^{(d)}_{\mathbf{h},N}(t)$ be the dth order Wiener chaos random variable defined by*

$$\widetilde{X}^{(d)}_{\mathbf{h},N}(t) := \sum_{(\mathbf{j},\mathbf{k})\in\mathcal{S}_N^+\cup\mathcal{S}_N^-} \varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{K}^{(d,\mathbf{h})}_{\boldsymbol{j},\boldsymbol{k}}(t). \tag{2.27}$$

There exists an almost surely finite random variable C (depending on T,b,b',g) for which one has, almost surely, for all $N\in\mathbb{N}$,

$$\|X^{(d)}_{\mathbf{h}} - \widetilde{X}^{(d)}_{\mathbf{h},N}\|_{[0,T],\infty} \leq CN^{\frac{d}{2}}2^{-N(h_1+\cdots+h_d-d+1/2)}. \tag{2.28}$$

To prove Theorems 2.12 and 2.14, we will need a logarithmic bound for the the sequence of random variables $(\varepsilon_{\mathbf{j},\mathbf{k}})_{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2}$. We get it from the following lemma which is a straightforward consequence of Lemma 2 in [3] and of the fact that the $g^\psi_{j,k} := I_1(\psi_{j,k})$, $(j,k)\in\mathbb{Z}^2$, are $\mathcal{N}(0,1)$ Gaussian random variables.

Lemma 2.15. *There are Ω^* an event of probability 1 and C_1^* a positive random variable of finite moment of any order, such that, for all $\omega\in\Omega^*$ and for each $(j,k)\in\mathbb{Z}^2$, one has*

$$\big|g^\psi_{j,k}(\omega)\big| \leq C_1^*(\omega)\sqrt{\log\big(3+|j|+|k|\big)}. \tag{2.29}$$

Next, observe that, for any $n\in\mathbb{N}$, there exists a constant $\alpha_n>0$ such that, for all $x\in\mathbb{R}$

$$|H_n(x)| \leq \alpha_n\big(1+|x|^n\big)\,; \tag{2.30}$$

the latter inequality is a straightforward consequence of the fact that H_n is a polynomial function of degree n. Then, combining (2.21) with (2.29) and (2.30), one obtains the following lemma.

Lemma 2.16. *Let Ω^* be the same event of probability 1 as in Lemma 2.15. There is C_d^* a positive random variable of finite moment of any order, such that, on Ω^*, one has, for all $(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^2)^d$,*

$$|\varepsilon_{\mathbf{j},\mathbf{k}}| \leq C_d^* \prod_{\ell=1}^{p(\mathbf{j},\mathbf{k})} \left(\sqrt{\log(3+|\widetilde{j}_\ell|+|\widetilde{k}_\ell|)}\right)^{n_\ell} = C_d^*\prod_{m=1}^{d}\sqrt{\log(3+|j_m|+|k_m|)}. \tag{2.31}$$

To prove Theorem 2.12, we will also need to know precisely when two random variables $\varepsilon_{\mathbf{j},\mathbf{k}}$ and $\varepsilon_{\mathbf{r},\mathbf{s}}$ are correlated. For this purpose, it is useful to define the set $\mathscr{D}(\mathbf{j},\mathbf{k})$.

Definition 2.17. Using the same notations as in (2.21), for all $(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2$, the set $\mathscr{D}(\mathbf{j},\mathbf{k})$ is defined as:

$$\mathscr{D}(\mathbf{j},\mathbf{k}) := \big\{(\widetilde{j}_\ell,\widetilde{k}_\ell)_{n_l} : 1\leq l\leq p(\mathbf{j},\mathbf{k})\big\}\,.$$

Remark 2.18. For any arbitrary two elements $(\mathbf{j},\mathbf{k}) = \big((j_m,k_m)\big)_{1\le m\le d}$ and $(\mathbf{r},\mathbf{s}) = \big((r_m,s_m)\big)_{1\le m\le d}$ of $(\mathbb{Z}^d)^2$, a necessary and sufficient condition for having $\mathscr{D}(\mathbf{j},\mathbf{k}) = \mathscr{D}(\mathbf{r},\mathbf{s})$ is that there exists a permutation σ of the set $\{1,\dots,d\}$ for which one has $(j_m,k_m) = (r_{\sigma(m)}, s_{\sigma(m)})$, for all $m\in\{1,\dots,d\}$. Thus, being given an arbitrary element $(\mathbf{j},\mathbf{k})$ of $(\mathbb{Z}^d)^2$, there are at most $d\,!-1$ other elements $(\mathbf{r},\mathbf{s})$ of $(\mathbb{Z}^d)^2$ which satisfy $\mathscr{D}(\mathbf{j},\mathbf{k}) = \mathscr{D}(\mathbf{r},\mathbf{s})$. Notice that, in this case, as a consequence of equality (2.21), one has $\varepsilon_{\mathbf{j},\mathbf{k}} = \varepsilon_{\mathbf{r},\mathbf{s}}$.

Let us also recall that, if G is any arbitrary $\mathcal{N}(0,1)$ Gaussian random variable then, one has

$$\mathbb{E}\big[H_m(G)H_n(G)\big] = \delta_{m,n} m!\,, \quad \text{for any } m,n\in\mathbb{Z}_+, \tag{2.32}$$

where $\delta_{m,n} = 1$ when $m = n$ and $\delta_{m,n} = 0$ otherwise. A straightforward consequence of (2.32) is that

$$\mathbb{E}\big[H_n(G)\big] = 0\,, \quad \text{for all integer } n\ge 1. \tag{2.33}$$

Relation (2.32) is the keystone of the proof of the following proposition.

Proposition 2.19. *For every $(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^2)^d$ and $(\mathbf{r},\mathbf{s})\in(\mathbb{Z}^2)^d$, one has*

$$\mathbb{E}[\varepsilon_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{r},\mathbf{s}}] = \begin{cases} \mathbb{E}[\varepsilon^2_{\mathbf{j},\mathbf{k}}] = \displaystyle\prod_{\ell=1}^{p(\mathbf{j},\mathbf{k})} n_l! \le d\,! & \text{if } \mathscr{D}(\boldsymbol{j},\boldsymbol{k}) = \mathscr{D}(\boldsymbol{r},\boldsymbol{s}),\\ 0 & \text{otherwise.}\end{cases} \tag{2.34}$$

Proof. First notice that, in view of Remark 2.18, (2.21), the independence of the $\mathcal{N}(0,1)$ Gaussian random variables $g^{\psi}_{\widetilde{j}_\ell,\widetilde{k}_\ell}$ with $\ell\in\{1,\dots,p(\mathbf{j},\mathbf{k})\}$ and (2.32), the equality (2.34) is clearly satisfied when $\mathscr{D}(\mathbf{j},\mathbf{k}) = \mathscr{D}(\mathbf{r},\mathbf{s})$. So, from now on, one assumes that $\mathscr{D}(\mathbf{j},\mathbf{k}) = \{(\widetilde{j}_1,\widetilde{k}_1)_{n_1},\dots,(\widetilde{j}_p,\widetilde{k}_p)_{n_p}\}$ (where $p = p(\mathbf{j},\mathbf{k})$) is not equal to $\mathscr{D}(\mathbf{r},\mathbf{s}) = \{(\widetilde{r}_1,\widetilde{s}_1)_{m_1},\dots,(\widetilde{r}_q,\widetilde{s}_q)_{m_q}\}$ (where $q = p(\mathbf{r},\mathbf{s})$) which happens in two different cases.

The first case consists in the situation where one has $\{(\widetilde{j}_1,\widetilde{k}_1),\dots,(\widetilde{j}_p,\widetilde{k}_p)\} \ne \{(\widetilde{r}_1,\widetilde{s}_1),\dots,(\widetilde{r}_q,\widetilde{s}_q)\}$, which implies that there exists at least one element of one of these two sets which does not belong to the other set. For sake of simplicity, one assumes that $(\widetilde{j}_1,\widetilde{k}_1)\notin\{(\widetilde{r}_1,\widetilde{s}_1),\dots,(\widetilde{r}_q,\widetilde{s}_q)\}$. Then, using (2.21), the fact that the $\mathcal{N}(0,1)$ Gaussian random variable $g^{\psi}_{\widetilde{j}_1,\widetilde{k}_1}$ is independent of the Gaussian vector $\big(g^{\psi}_{\widetilde{j}_2,\widetilde{k}_2},\dots,g^{\psi}_{\widetilde{j}_p,\widetilde{k}_p}, g^{\psi}_{\widetilde{r}_1,\widetilde{s}_1},\dots,g^{\psi}_{\widetilde{r}_q,\widetilde{s}_q}\big)$, and (2.33), one gets that

$$\mathbb{E}[\varepsilon_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{r},\mathbf{s}}] = \underbrace{\mathbb{E}\big[H_{n_1}(g^{\psi}_{\widetilde{j}_1,\widetilde{k}_1})\big]}_{=0}\mathbb{E}\left[\prod_{\ell=2}^{p} H_{n_\ell}(g^{\psi}_{\widetilde{j}_\ell,\widetilde{k}_\ell})\prod_{\ell'=1}^{q} H_{m_{\ell'}}(g^{\psi}_{\widetilde{j}_{\ell'},\widetilde{k}_{\ell'}})\right] = 0.$$

The second case consists in the situation where one has $p = q$, $\{(\widetilde{j}_1,\widetilde{k}_1),\dots,(\widetilde{j}_p,\widetilde{k}_p)\} = \{(\widetilde{r}_1,\widetilde{s}_1),\dots,(\widetilde{r}_p,\widetilde{s}_p)\}$ and $n_{\ell_0}\ne m_{\ell_0}$ for some $\ell_0\in\{1,\dots,p\}$. For sake of simplicity, one assumes that $\ell_0 = 1$. Then, using (2.21), the fact that the $\mathcal{N}(0,1)$ Gaussian random variable $g^{\psi}_{\widetilde{j}_1,\widetilde{k}_1}$ is independent of the Gaussian vector $\big(g^{\psi}_{\widetilde{j}_2,\widetilde{k}_2},\ldots,g^{\psi}_{\widetilde{j}_p,\widetilde{k}_p}\big)$, and (2.32), one obtains that

$$\mathbb{E}[\varepsilon_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{r},\mathbf{s}}]=\underbrace{\mathbb{E}\big[H_{n_1}(g^{\psi}_{\widetilde{j}_1,\widetilde{k}_1})H_{m_1}(g^{\psi}_{\widetilde{j}_1,\widetilde{k}_1})\big]}_{=0}\mathbb{E}\left[\prod_{\ell=2}^{p}H_{n_\ell}(g^{\psi}_{\widetilde{j}_\ell,\widetilde{k}_\ell})H_{m_\ell}(g^{\psi}_{\widetilde{j}_\ell,\widetilde{k}_\ell})\right]=0.$$

□

3 Proof of Theorem 2.8

In this section, we aim at proving Theorem 2.8. The main four steps of the proof are the following.

In the first step, we show that, for each fixed $J\in\mathbb{N}$, the generalized FARIMA sequence $(\sigma^{(\mathbf{h})}_{J,\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$, defined through the random series in (2.14), can be represented through multiple Wiener integral. To this end, for any fixed $\delta\in(0,1/2)$, we introduce, via the sequence of coefficients $(\gamma^{(\delta)}_p)_{p\in\mathbb{N}_0}$ (see Definition 1.6) and the univariate Meyer scaling function ϕ (see Remark 1.1), the real-valued function $\Phi^{(-\delta)}$ defined as

$$\Phi^{(-\delta)}(x)=\sum_{p=0}^{+\infty}\gamma^{(\delta)}_p\phi(x+p),$$

where the convergence of the series holds in $L^2(\mathbb{R})$. Then, using the isometry property of multiple Wiener integral, it turns out that one has almost surely, for all $J\in\mathbb{N}$ and $\mathbf{k}\in\mathbb{Z}^d$,

$$\sigma^{(\mathbf{h})}_{J,\mathbf{k}}=\int'_{\mathbb{R}^d}2^{J\frac{d}{2}}\prod_{\ell=1}^{d}\Phi^{(1/2-h_\ell)}(2^Ju_\ell-k_\ell)\,dB(u_1)\cdots dB(u_d).$$

Also, in the first step, we prove that the Fourier transform of $\Phi^{(-\delta)}$ is given, for almost all $\xi\in\mathbb{R}$, by $\widehat{\Phi}^{(-\delta)}(\xi)=\big(1-e^{i\xi}\big)^{-\delta}\,\widehat{\phi}(\xi)$.

In the second step, we show that, for each fixed $t\in\mathbb{R}_+$ and $J\in\mathbb{N}$, the series of deterministic functions of the variable $\mathbf{u}=(u_1,\ldots,u_d)\in\mathbb{R}^d$

$$\mathcal{V}^{(d)}_{\mathbf{h},J}(t,\mathbf{u}):=\sum_{\mathbf{k}\in\mathbb{Z}^d}\left(\int_0^t\prod_{\ell=1}^{d}\Phi^{(h_\ell-1/2)}_{\Delta}(2^Js-k_\ell)\,ds\right)2^{J\frac{d}{2}}\prod_{\ell=1}^{d}\Phi^{(1/2-h_\ell)}(2^Ju_\ell-k_\ell)$$

is convergent, and even normally convergent, in $L^2_{\mathbf{u}}(\mathbb{R}^d)$. Then, using the isometry property of multiple Wiener integral, it turns out that the random series in the right-hand side of (2.16) is, for each fixed $t\in\mathbb{R}_+$ and $J\in\mathbb{N}$, convergent in $L^2(\Omega)$ and satisfies almost surely

$$\sum_{\mathbf{k}\in\mathbb{Z}^d}\left(\int_0^t\prod_{\ell=1}^{d}\Phi^{(h_\ell-1/2)}_{\Delta}(2^Js-k_\ell)\,ds\right)\sigma^{(\mathbf{h})}_{J,\mathbf{k}}=\int'_{\mathbb{R}^d}\mathcal{V}^{(d)}_{\mathbf{h},J}(t,\mathbf{u})\,dB(u_1)\cdots dB(u_d).$$

In the third step, we show that, for each fixed $t\in\mathbb{R}_+$ and $J\in\mathbb{N}$, the two functions $\mathbf{u}\mapsto K^{(d)}_{\mathbf{h},J}(t,\mathbf{u})$ (see (2.3)) and $\mathbf{u}\mapsto 2^{-J(h_1+\cdots+h_d-d)}\mathcal{V}^{(d)}_{\mathbf{h},J}(t,\mathbf{u})$ are equal for almost all $\mathbf{u} \in \mathbb{R}^d$; this result is obtained by showing that their Fourier transforms are equal almost everywhere in $\mathbb{R}^d$. Then combining it with (2.3) and the previous equality, we obtain, almost surely, that

$$X_{\mathbf{h},J}^{(d)}(t) = 2^{-J(h_1+\cdots+h_d-d)} \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\int_0^t \prod_{\ell=1}^d \Phi_\Delta^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds \right) \sigma_{J,\mathbf{k}}^{(\mathbf{h})}.$$

Finally, in the fourth step we show that the series in the right-hand side of the last equality is almost surely uniformly convergent in t on each compact interval of $\mathbb{R}_+$.

3.1 First step of the proof of Theorem 2.8

Definition 3.1. Recall that $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis of the subspace V_0^1 of the multiresolution analysis of $L^2(\mathbb{R})$ associated with the univariate Meyer scaling function ϕ. Let $\delta \in (0, \frac{1}{2})$ be arbitrary and fixed. The function $\Phi^{(-\delta)} \in V_0^1$ is defined as

$$\Phi^{(-\delta)}(x) = \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} \phi(x+p).$$

The latter series of functions is convergent in $L^2(\mathbb{R})$ since the sequence of coefficients $(\gamma_p^{(\delta)})_{p\in\mathbb{N}_0}$ belongs to $\ell^2(\mathbb{N}_0)$ (see (1.9)).

The following proposition easily results from Definition 3.1, the isometry property of multiple Wiener integral and (2.14).

Proposition 3.2. *One has almost surely, for all $J \in \mathbb{N}$ and $\boldsymbol{k} \in \mathbb{Z}^d$,*

$$\sigma_{J,\boldsymbol{k}}^{(\mathbf{h})} = \int_{\mathbb{R}^d}' 2^{J\frac{d}{2}} \prod_{\ell=1}^d \Phi^{(1/2-h_\ell)}(2^J u_\ell - k_\ell)\, dB(u_1)\cdots dB(u_d). \tag{3.1}$$

For later purposes, one needs to determine the Fourier transform of the function $\Phi^{(-\delta)}$. The following lemma provides it.

Lemma 3.3. *For all $\delta \in (0, \frac{1}{2})$, the Fourier transform of the function $\Phi^{(-\delta)}$ is given, for almost all $\xi \in \mathbb{R}$, by*

$$\widehat{\Phi}^{(-\delta)}(\xi) = \left(1 - e^{i\xi}\right)^{-\delta} \widehat{\phi}(\xi).$$

From now on, for the sake of convenience, for all $p \in \mathbb{N}$, we set $\gamma_{-p}^{(\delta)} = 0$. In fact, Lemma 3.3 is mainly a consequence of the following lemma showing that the sequence $(\gamma_p^{(\delta)})_{p\in\mathbb{Z}}$ is nothing else than the sequence of the Fourier coefficient of the function $\xi \mapsto \left(1 - e^{i\xi}\right)^{-\delta}$ which belongs to $L^2([0, 2\pi])$. Recall that $L^2([0, 2\pi])$ is the space of the complex-valued functions defined on the real line which are 2π-periodic and Lebesgue square-integrable on the interval $[0, 2\pi]$.

Lemma 3.4. *For all $\delta \in (0, 1/2)$ and $p \in \mathbb{Z}$, we have*

$$\gamma_p^{(\delta)} = \frac{1}{2\pi} \int_0^{2\pi} e^{-ip\xi}(1 - e^{i\xi})^{-\delta}\, d\xi.$$

A straightforward consequence is that

$$\left(1 - e^{i\xi}\right)^{-\delta} = \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} e^{ip\xi},$$

where the series is convergent in $L^2([0, 2\pi])$.

Proof. Let $\mathbb{C}$ be set of the complex numbers and let $\mathbb{C} \setminus [1, +\infty)$ be the open subset of $\mathbb{C}$ formed by the complex numbers which are not real numbers greater than or equal to 1, that is $\mathbb{C} \setminus [1, +\infty) := \left\{ z \in \mathbb{C} : z \notin [1, +\infty) \right\}$. We denote by F_δ the continuous function on $\mathbb{C} \setminus [1, +\infty)$, defined for all $z \in \mathbb{C} \setminus [1, +\infty)$ as

$$F_\delta(z) := (1 - z)^{-\delta}.$$

Recall that F_δ is analytic on the open unit disk $\left\{ z \in \mathbb{C} : |z| < 1 \right\}$ with Taylor expansion given by

$$F_\delta(z) = \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} z^p, \tag{3.2}$$

where the series is uniformly convergent on each closed disk $\left\{ z \in \mathbb{C} : |z| \leq \rho \right\}$, with $\rho \in (0, 1)$.

Next, observe that using the continuity property of the function F_δ, for any $\xi \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, the quantity $(1 - e^{i\xi})^{-\delta} = F_\delta(e^{i\xi})$ can be expressed as:

$$(1 - e^{i\xi})^{-\delta} = F_\delta(e^{i\xi}) = \lim_{r \in \mathbb{R},\, r \to 1^-} F_\delta(re^{i\xi}) = \lim_{r \in \mathbb{R},\, r \to 1^-} (1 - re^{i\xi})^{-\delta}.$$

Thus, denoting by $(r_j)_{j \in \mathbb{N}}$ an arbitrary increasing sequence of real numbers in the open interval $(0, 1)$ which converges to 1, one has, for all $p \in \mathbb{Z}$, that

$$\int_0^{2\pi} e^{-ip\xi} (1 - e^{i\xi})^{-\delta} \, d\xi = \int_0^{2\pi} e^{-ip\xi} \lim_{j \to +\infty} (1 - r_j e^{i\xi})^{-\delta} \, d\xi.$$

Let us now show that one can interchange the limit and integration symbols. To this end, we need to introduce, for all $j \in \mathbb{N}$, the subset A_j of $[0, 2\pi]$ defined as

$$A_j := \left\{ \xi \in [0, 2\pi] \,:\, |1 - e^{i\xi}| \leq 2(1 - r_j) \right\}.$$

Note that, for all j large enough, if $\xi \in A_j$, then

$$\xi \in [0, 4(1 - r_j)] \cup [2\pi - 4(1 - r_j), 2\pi].$$

Therefore, we can derive from the inequality $|1 - r_j e^{i\xi}| \geq (1 - r_j)$ that

$$\begin{aligned} \left| \int_{A_j} e^{-ip\xi} (1 - r_j e^{i\xi})^{-\delta} \, d\xi \right| &\leq (1 - r_j)^{-\delta} \int_{A_j} \mathbb{1}_{A_j}(\xi) \, d\xi \\ &\leq 8(1 - r_j)^{1-\delta}. \end{aligned}$$

Since $1 - \delta > 0$, the latter inequality entails that

$$\lim_{j \to +\infty} \int_{A_j} e^{-ip\xi} (1 - r_j e^{i\xi})^{-\delta} \, d\xi = 0.$$

On another hand if $\xi \in A_j^c := [0,2\pi] \setminus A_j$, then we have that

$$|1 - r_j e^{i\xi}| = |1 - e^{i\xi} + (1-r_j)e^{i\xi}| \geq |1 - e^{i\xi}| - (1-r_j) > \frac{1}{2}|1 - e^{i\xi}|,$$

which implies that

$$\left| e^{-ip\xi}(1 - r_j e^{i\xi})^{-\delta} \right| < 2^\delta |1 - e^{i\xi}|^{-\delta}.$$

As $\delta \in (0,1/2)$, the function $\xi \mapsto |1 - e^{i\xi}|^{-\delta}$ is integrable on $[0,2\pi]$, and since, for all $\xi \in (0,2\pi)$, $\mathbb{1}_{A_j^c}(\xi) \to 1$, we conclude, by dominated convergence theorem, that, for all $p \in \mathbb{Z}$,

$$\int_0^{2\pi} e^{-ip\xi}(1 - e^{i\xi})^{-\delta}\, d\xi = \lim_{j\to+\infty} \int_0^{2\pi} e^{-ip\xi}(1 - r_j e^{i\xi})^{-\delta}\, d\xi.$$

Moreover, for any arbitrary fixed $j \in \mathbb{N}$, using the uniform convergence property of the series in (3.2), we have

$$\int_0^{2\pi} e^{-ip\xi}(1 - r_j e^{i\xi})^{-\delta}\, d\xi = \sum_{m=0}^{+\infty} \gamma_m^{(\delta)} r_j^m \int_0^{2\pi} e^{-ip\xi} e^{im\xi}\, d\xi = 2\pi r_j^p \gamma_p^{(\delta)}.$$

The conclusion follows immediately. □

We are now in position to prove Lemma 3.3.

Proof of Lemma 3.3. On one hand, it follows from Definition 3.1 and classical properties of Fourier transform that

$$\lim_{q\to+\infty} \int_{\mathbb{R}} \left| \widehat{\Phi}^{(-\delta)}(\xi) - \Big(\sum_{p=0}^{q} \gamma_p^{(\delta)} e^{ip\xi} \Big) \widehat{\phi}(\xi) \right|^2 d\xi = 0. \tag{3.3}$$

On the other hand, using Remark 1.1 and the fact that the two functions $\xi \mapsto (1 - e^{i\xi})^{-\delta}$ and $\xi \mapsto \sum_{p=0}^{q} \gamma_p^{(\delta)} e^{ip\xi}$ are 2π-periodic, one has, for all $q \in \mathbb{N}$, that

$$\begin{aligned}
&\int_{\mathbb{R}} \left| (1 - e^{i\xi})^{-\delta} \widehat{\phi}(\xi) - \Big(\sum_{p=0}^{q} \gamma_p^{(\delta)} e^{ip\xi} \Big) \widehat{\phi}(\xi) \right|^2 d\xi \\
&= \int_{-4\pi/3}^{4\pi/3} \left| (1 - e^{i\xi})^{-\delta} - \Big(\sum_{p=0}^{q} \gamma_p^{(\delta)} e^{ip\xi} \Big) \right|^2 \big|\widehat{\phi}(\xi)\big|^2 d\xi \\
&\leq 2 \left\|\widehat{\phi}\right\|_{L^\infty(\mathbb{R})}^2 \int_0^{2\pi} \left| (1 - e^{i\xi})^{-\delta} - \Big(\sum_{p=0}^{q} \gamma_p^{(\delta)} e^{ip\xi} \Big) \right|^2 d\xi.
\end{aligned}$$

Thus, one can derive from Lemma 3.4 that

$$\lim_{q\to+\infty} \int_{\mathbb{R}} \left| (1 - e^{i\xi})^{-\delta} \widehat{\phi}(\xi) - \Big(\sum_{p=0}^{q} \gamma_p^{(\delta)} e^{ip\xi} \Big) \widehat{\phi}(\xi) \right|^2 d\xi = 0. \tag{3.4}$$

Finally, combining (3.3) and (3.4), one obtains that $\widehat{\Phi}^{(-\delta)}(\xi) = (1 - e^{i\xi})^{-\delta}\, \widehat{\phi}(\xi)$, for almost all $\xi \in \mathbb{R}^d$. □

Before ending the present subsection, let us make the following remark, which is interesting in its own right even though it plays no role in the proof of Theorem 2.8.

Remark 3.5. Lemma 3.3 shows that the expectations involved in the expression (2.15) of the random variables $\sigma^{(\mathbf{h})}_{J,\mathbf{k}}$, $(J,\mathbf{k}) \in \mathbb{Z}\times\mathbb{Z}^d$, are rather easily computable. Indeed, for all $J \in \mathbb{Z}$ and $k, p, k', p' \in \mathbb{Z}$, the expectation $\mathbb{E}[g^{\phi}_{J,k-p}, g^{\phi}_{J,k'-p'}]$ does not vanish only when $k - p = k' - p'$ and, in this case, it is equal to 1. Thus, we can write

$$\mathbb{E}[g^{\phi}_{J,k-p}, g^{\phi}_{J,k'-p'}] = \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-p)\xi} e^{-i(k'-p')\xi}\, d\xi.$$

Then, using Definition 1.6 and Remark 1.7, we obtain, for all $\delta, \delta' \in (0, 1/2)$, $J \in \mathbb{N}$ and $k, k' \in \mathbb{Z}$, that

$$\begin{aligned}
\mathbb{E}[Z^{(\delta)}_{j,k} Z^{(\delta')}_{j,k'}] &= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{p\in\mathbb{Z}} \gamma^{(\delta)}_p e^{i(k-p)\xi}\right)\left(\sum_{p'\in\mathbb{Z}} \gamma^{(\delta')}_{p'} e^{-i(k'-p')\xi}\right) d\xi \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-k')\xi} \left(\sum_{p\in\mathbb{Z}} \gamma^{(\delta)}_p e^{-ip\xi}\right)\left(\sum_{p'\in\mathbb{Z}} \gamma^{(\delta')}_{p'} e^{ip'\xi}\right) d\xi \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-k')\xi} (1 - e^{-i\xi})^{-\delta} (1 - e^{i\xi})^{-\delta'}\, d\xi.
\end{aligned}$$

In particular, if $\delta = \delta'$, a fact that always occurs when we restrict to usual Hermite processes, we get that

$$\begin{aligned}
\mathbb{E}[Z^{(\delta)}_{j,k} Z^{(\delta)}_{j,k'}] &= \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-k')\xi} |1 - e^{-i\xi}|^{-2\delta}\, d\xi \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-k')\xi} \left|2\sin\left(\frac{\xi}{2}\right)\right|^{-2\delta} d\xi.
\end{aligned}$$

3.2 Second step of the proof of Theorem 2.8

Definition 3.6. Recall that $L^2\big([0,2\pi]^d\big)$ is the space of the functions from $\mathbb{R}^d$ to $\mathbb{C}$ which are 2π-periodic with respect to each one of their d variables and Lebesgue square-integrable on the cube $[0,2\pi]^d$. The Fourier transform of any arbitrary sequence $\theta = (\theta_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ is the function of $L^2\big([0,2\pi]^d\big)$ denoted by $\widehat{\theta}$ and defined as

$$\widehat{\theta}(\eta) := \sum_{\mathbf{k}\in\mathbb{Z}^d} \theta_{\mathbf{k}} e^{-i\langle \mathbf{k}, \eta\rangle},$$

where the series is convergent in $L^2\big([0,2\pi]^d\big)$ and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^d$.

Definition 3.7. For any fixed $\mathbf{h} = (h_1, \dots, h_d)$ satisfying (1.11), the sequence $\Upsilon = (\Upsilon_{\mathbf{q}})_{\mathbf{q}\in\mathbb{Z}^d}$, which depends on $\mathbf{h}$ and belongs to $\ell^2(\mathbb{Z}^d)$, is defined, for all $\mathbf{q} \in \mathbb{Z}^d$, as

$$\Upsilon_{\mathbf{q}} := \prod_{\ell=1}^{d} \gamma^{(h_\ell - 1/2)}_{-q_\ell}. \tag{3.5}$$

Recall that, one knows from Lemma 3.4 that $(\gamma_p^{(h_\ell-1/2)})_{p\in\mathbb{Z}}$ is the sequence of the Fourier coefficients of the function $\lambda \mapsto \big(1-e^{i\lambda}\big)^{1/2-h_\ell}$ which belongs to $L^2([0,2\pi])$.

Remark 3.8. One knows from (3.5), Definition 3.6 and Lemma 3.4 that the Fourier transform of the sequence Υ satisfies, for almost all $\eta \in \mathbb{R}^d$,

$$\widehat{\Upsilon}(\eta) = \prod_{\ell=1}^{d} \big(1-e^{i\eta_l}\big)^{1/2-h_\ell}. \tag{3.6}$$

Lemma 3.9. *For any fixed $\mathbf{h}=(h_1,\ldots,h_d)$ satisfying (1.11), and for each fixed $t\in\mathbb{R}_+$ and $J\in\mathbb{N}$, the series of deterministic functions of the variable $\mathbf{u}=(u_1,\ldots,u_d)\in\mathbb{R}^d$*

$$\mathcal{V}_{\mathbf{h},J}^{(d)}(t,\mathbf{u}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\int_0^t \prod_{\ell=1}^{d} \Phi_{\Delta}^{(h_\ell-1/2)}(2^J s-k_\ell)\,ds\right) 2^{J\frac{d}{2}} \prod_{\ell=1}^{d} \Phi^{(1/2-h_\ell)}(2^J u_\ell - k_\ell) \tag{3.7}$$

is normally convergent in $L^2_{\mathbf{u}}(\mathbb{R}^d)$. Thus, the function[2] *$\mathcal{V} := \mathcal{V}_{\mathbf{h},J}^{(d)}(t,\bullet)$ belongs to $L^2(\mathbb{R}^d)$. Moreover, its Fourier transform satisfies, for almost every $\xi\in\mathbb{R}^d$,*

$$\begin{aligned} \widehat{\mathcal{V}}(\xi) &= \widehat{\beta}(2^{-J}\xi)\,2^{-J\frac{d}{2}} \prod_{\ell=1}^{d} \widehat{\Phi}^{(1/2-h_\ell)}(2^{-J}\xi_\ell) \\ &= \Big(\widehat{\beta}(2^{-J}\xi)\widehat{\Upsilon}(2^{-J}\xi)\Big)\,2^{-J\frac{d}{2}} \prod_{\ell=1}^{d} \widehat{\phi}(2^{-J}\xi_\ell), \end{aligned} \tag{3.8}$$

where $\widehat{\beta}$ is the Fourier transform of the sequence $\beta=(\beta_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ defined, for all $\mathbf{k}\in\mathbb{Z}^d$, as

$$\beta_{\mathbf{k}} := \int_0^t \prod_{\ell=1}^{d} \Phi_{\Delta}^{(h_\ell-1/2)}(2^J s-k_\ell)\,ds. \tag{3.9}$$

Notice that the sequence β depends on t, J and $\mathbf{h}=(h_1,\ldots,h_d)$. Also, notice that the second equality in (3.8) results from Lemma 3.3 and (3.6).

Remark 3.10. Since the functions $\Phi_{\Delta}^{(h_\ell-1/2)}$, $\ell\in[\![1,d]\!]$, belong to Schwartz class $\mathcal{S}(\mathbb{R})$, one can easily derive from (3.9) that, for any fixed arbitrarily large positive real number μ, one has

$$\sup_{\mathbf{k}\in\mathbb{Z}^d}\left(|\beta_{\mathbf{k}}|\prod_{\ell=1}^{d}\big(1+|k_\ell|\big)^{\mu}\right) < \infty. \tag{3.10}$$

The latter fact implies that the sequence $\beta=(\beta_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ belongs to $\ell^1(\mathbb{Z}^d)\subset\ell^2(\mathbb{Z}^d)$. Therefore, its Fourier transform $\widehat{\beta}$ is a well-defined function of $L^2\big([0,2\pi]^d\big)$ which is continuous and bounded on $\mathbb{R}^d$ and satisfies

$$\lim_{N\to+\infty}\sup_{\xi\in\mathbb{R}^d}\left|\widehat{\beta}(\xi)-\sum_{|\mathbf{k}|\le N}\beta_{\mathbf{k}}e^{-i2^{-J}\langle\mathbf{k},\xi\rangle}\right| = 0. \tag{3.11}$$

[2] We denote the function $\mathcal{V}_{\mathbf{h},J}^{(d)}(t,\bullet)$ by $\mathcal{V}$ for the sake of simplicity.

Proof of Lemma 3.9. The normal convergence in $L^2_{\mathbf{u}}(\mathbb{R}^d)$ of the series in (3.7) easily follows from (3.10), (3.9) and the straightforward equality, for all $\mathbf{k} \in \mathbb{Z}^d$,

$$\left\| \beta_{\mathbf{k}} 2^{J\frac{d}{2}} \prod_{\ell=1}^{d} \Phi^{(1/2-h_\ell)}(2^J u_\ell - k_\ell) \right\|_{L^2(\mathbb{R}^d)} = |\beta_{\mathbf{k}}| \prod_{\ell=1}^{d} \|\Phi^{(1/2-h_\ell)}\|_{L^2(\mathbb{R}^d)}.$$

Let us now show that (3.8) holds. For all $N \in \mathbb{N}$, we denote by $\mathcal{V}_N$ the finite sum, defined, for each $\mathbf{u} \in \mathbb{R}^d$, as:

$$\mathcal{V}_N(\mathbf{u}) := \sum_{|\mathbf{k}| \le N} \beta_{\mathbf{k}} 2^{J\frac{d}{2}} \prod_{\ell=1}^{d} \Phi^{(1/2-h_\ell)}(2^J u_\ell - k_\ell). \tag{3.12}$$

We already know that $\mathcal{V}_N \to \mathcal{V}$ in $L^2(\mathbb{R}^d)$ as $N \to +\infty$. Therefore, the isometry property of Fourier transform entails that $\widehat{\mathcal{V}}_N \to \widehat{\mathcal{V}}$ in $L^2(\mathbb{R}^d)$ as $N \to +\infty$. One can derive from the latter fact that there exists a subsequence $(N_r)_{r\in\mathbb{N}}$ such that one has

$$\lim_{r\to+\infty} \widehat{\mathcal{V}}_{N_r}(\xi) = \widehat{\mathcal{V}}(\xi), \quad \text{for almost every } \xi \in \mathbb{R}^d. \tag{3.13}$$

Moreover, it follows from (3.12) and basic properties of Fourier transform that, for all $N \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$,

$$\widehat{\mathcal{V}}_N(\xi) = \left(\sum_{|\mathbf{k}| \le N} \beta_{\mathbf{k}} e^{-i2^{-J}\langle \mathbf{k}, \xi \rangle} \right) 2^{-J\frac{d}{2}} \prod_{\ell=1}^{d} \widehat{\Phi}^{(1/2-h_\ell)}(2^{-J}\xi_\ell). \tag{3.14}$$

Finally, putting together (3.11), (3.13) and (3.14), one obtains (3.8). □

Before ending this subsection, let us point out that:

Remark 3.11. Using Proposition 3.2, Lemma 3.9 and the isometry property of multiple Wiener integral, it follows that the random series in the right-hand side of (2.16) is, for each fixed $t \in \mathbb{R}_+$ and $J \in \mathbb{N}$, convergent in $L^2(\Omega)$ and satisfies almost surely

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\int_0^t \prod_{\ell=1}^{d} \Phi_\Delta^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds \right) \sigma_{J,\mathbf{k}}^{(\mathbf{h})} = \int'_{\mathbb{R}^d} \mathcal{V}_{\mathbf{h},J}^{(d)}(t,\mathbf{u})\, dB(u_1)\cdots dB(u_d). \tag{3.15}$$

Thus, in view of (2.3) and (3.15), in order to show that, for each fixed $t \in \mathbb{R}_+$ and $J \in \mathbb{N}$, the equality (2.16) holds almost surely, it is enough to prove that the two functions $K_{\mathbf{h},J}^{(d)}(t,\bullet) : \mathbf{u} \mapsto K_{\mathbf{h},J}^{(d)}(t,\mathbf{u})$ and $2^{-J(h_1+\cdots+h_d-d)}\mathcal{V}_{\mathbf{h},J}^{(d)}(t,\bullet) : \mathbf{u} \mapsto 2^{-J(h_1+\cdots+h_d-d)}\mathcal{V}_{\mathbf{h},J}^{(d)}(t,\mathbf{u})$ are equal for almost all $\mathbf{u} \in \mathbb{R}^d$, which amounts to proving that their Fourier transforms are equal almost everywhere in $\mathbb{R}^d$.

3.3 Third step of the proof of Theorem 2.8

The goal of this subsection is to show that the following lemma holds.

Lemma 3.12. *For any fixed* $\mathbf{h} = (h_1, \ldots, h_d)$ *satisfying* (1.11), *and for each fixed* $t \in \mathbb{R}_+$ *and* $J \in \mathbb{N}$, *the Fourier transforms of the two functions, of* $L^2(\mathbb{R}^d)$, $K_{\mathbf{h},J}^{(d)}(t,\bullet)$ *and* $2^{-J(h_1+\cdots+h_d-d)}\mathcal{V}_{\mathbf{h},J}^{(d)}(t,\bullet)$ *are equal almost everywhere in* $\mathbb{R}^d$.

For proving Lemma 3.12 we need some preliminary results.

Lemma 3.13. *For any fixed* $\mathbf{h} = (h_1, \ldots, h_d)$ *satisfying* (1.11)*, and for each fixed* $t \in \mathbb{R}_+$ *and* $J \in \mathbb{N}$*, the Fourier transform of the function* $K_{\mathbf{h},J}^{(d)}(t, \bullet)$ *can be expressed, for almost all* $\xi \in \mathbb{R}^d$*, as*

$$2^{-J(h_1+\cdots+h_d-d)}\, \widehat{\alpha}(2^{-J}\xi)\, 2^{-J\frac{d}{2}} \prod_{\ell=1}^{d} \widehat{\phi}(2^{-J}\xi_\ell),$$

where $\widehat{\alpha}$ *denotes the Fourier transform of the sequence* $\alpha = (\alpha_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$ *of* $\ell^2(\mathbb{Z}^d)$ *defined, for all* $\boldsymbol{k} \in \mathbb{Z}^d$*, as*

$$\alpha_{\boldsymbol{k}} := \int_0^t \prod_{\ell=1}^{d} \phi_{h_\ell}(2^J s - k_\ell)\, ds. \tag{3.16}$$

Recall that ϕ_{h_l} *is the fractional primitive of order* $h_l - 1/2$ *of the Meyer univariate scaling function* ϕ*. Notice that the sequence* α *depends on* t*,* J *and* $\mathbf{h} = (h_1, \ldots, h_d)$*.*

Proof. Since $K_{\mathbf{h},J}^{(d)}(t, \bullet)$ is the orthogonal projection $K_{\mathbf{h}}^{(d)}(t, \bullet)$ on the space V_J^d (see the beginning of Section 2), and $\big(2^{J\frac{d}{2}}\Phi(2^J \cdot -\mathbf{k})\big)_{\mathbf{k}\in\mathbb{Z}^d}$ is an orthonormal basis of this space, one has that

$$K_{\mathbf{h},J}^{(d)}(t, \mathbf{u}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \mathfrak{K}_{J,\mathbf{k}}^{(d,\mathbf{h})}(t)\, 2^{J\frac{d}{2}} \prod_{\ell=1}^{d} \phi(2^J u_\ell - k_\ell),$$

where $\mathfrak{K}_{J,\mathbf{k}}^{(d,\mathbf{h})}(t)$ is as in (2.7) and the convergence of the series holds in $L^2_{\mathbf{u}}(\mathbb{R}^d)$. Then combining the last equality with (2.12) and basic properties of Fourier transform, one obtains the lemma. □

Remark 3.14. It follows from Lemmas 3.9 and 3.13 that for proving Lemma 3.12 it is enough to show that

$$\widehat{\alpha}(\eta) = \widehat{\beta}(\eta)\widehat{\Upsilon}(\eta), \quad \text{for almost all } \eta \in \mathbb{R}^d. \tag{3.17}$$

In fact, since the sequence β belongs to $\ell^1(\mathbb{Z}^d)$ (see Remark 3.10), the function $\eta \mapsto \widehat{\beta}(\eta)\widehat{\Upsilon}(\eta)$, which belongs to $L^2([0, 2\pi]^d)$, is nothing else than the Fourier transform of the convolution product $\beta * \Upsilon$. The latter sequence $\beta * \Upsilon = \big((\beta * \Upsilon)_{\mathbf{k}}\big)_{\mathbf{k}\in\mathbb{Z}^d}$ of $\ell^2(\mathbb{Z}^d)$ is defined, for all $\mathbf{k} \in \mathbb{Z}^d$, as

$$(\beta * \Upsilon)_{\mathbf{k}} := \sum_{\mathbf{q}\in\mathbb{Z}^d} \Upsilon_{\mathbf{q}}\, \beta_{\mathbf{k}-\mathbf{q}}. \tag{3.18}$$

Thus, in view of (3.18), it turns out that for proving (3.17), it is enough to show that, for all $\mathbf{k} \in \mathbb{Z}^d$, one has

$$\alpha_{\mathbf{k}} = \sum_{\mathbf{q}\in\mathbb{Z}^d} \Upsilon_{\mathbf{q}}\, \beta_{\mathbf{k}-\mathbf{q}}. \tag{3.19}$$

Lemma 3.15. *The equality* (3.19) *holds for all* $\boldsymbol{k} \in \mathbb{Z}^d$*.*

Proof. First notice that one can derive from Remark 1.1, Definition 1.3 and (2.13) that the two functions $\eta \mapsto \prod_{\ell=1}^d \widehat{\Phi}_\Delta^{(h_\ell-1/2)}(\eta_\ell)$ and $\eta \mapsto \prod_{\ell=1}^d \widehat{\phi}_{h_\ell}(\eta_\ell)$ belong to $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Therefore, using inverse Fourier transform, one gets, for all $(v_1, \ldots, v_d) \in \mathbb{R}^d$, that

$$\prod_{\ell=1}^d \Phi_\Delta^{(h_\ell-1/2)}(v_\ell) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp\Big(i \sum_{\ell=1}^d v_\ell \eta_\ell\Big) \prod_{\ell=1}^d \widehat{\Phi}_\Delta^{(h_\ell-1/2)}(\eta_\ell) d\eta \tag{3.20}$$

and

$$\begin{aligned} \prod_{\ell=1}^d \phi_{h_\ell}(v_\ell) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp\Big(i \sum_{\ell=1}^d v_\ell \eta_\ell\Big) \prod_{\ell=1}^d \widehat{\phi}_{h_\ell}(\eta_\ell) d\eta \\ &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp\Big(i \sum_{\ell=1}^d v_\ell \eta_\ell\Big) \widehat{\Upsilon}(-\eta) \prod_{\ell=1}^d \widehat{\Phi}_\Delta^{(h_\ell-1/2)}(\eta_\ell) d\eta, \end{aligned} \tag{3.21}$$

where the last equality follows (3.6), (2.13) and Definition 1.3. Next, let N be an arbitrary positive integer. One can derive from (3.16), (3.9), (3.20), (3.21), standard calculations, (1.3) and Cauchy-Schwarz inequality that, for all $\mathbf{k} \in \mathbb{Z}^d$, one has

$$\begin{aligned} &\Big| \alpha_{\mathbf{k}} - \sum_{|\mathbf{q}| \le N} \Upsilon_{\mathbf{q}} \, \beta_{\mathbf{k}-\mathbf{q}} \Big| \\ &\le \int_0^t \Big| \int_{\mathbb{R}^d} \exp\Big(i \sum_{\ell=1}^d (2^j s - k_\ell)\eta_\ell\Big) \Big(\widehat{\Upsilon}(-\eta) - \sum_{|\mathbf{q}| \le N} \Upsilon_{\mathbf{q}} \, e^{i\langle \mathbf{q}, \eta \rangle} \Big) \times \ldots \\ &\qquad\qquad \ldots \times \prod_{\ell=1}^d \widehat{\Phi}_\Delta^{(h_\ell-1/2)}(\eta_\ell) d\eta \Big| ds \\ &\le t \int_{[-\frac{4\pi}{3}, \frac{4\pi}{3}]^d} \Big| \widehat{\Upsilon}(-\eta) - \sum_{|\mathbf{q}| \le N} \Upsilon_{\mathbf{q}} \, e^{i\langle \mathbf{q}, \eta \rangle} \Big| \prod_{\ell=1}^d \big| \widehat{\Phi}_\Delta^{(h_\ell-1/2)}(\eta_\ell) \big| d\eta \\ &\le t \prod_{\ell=1}^d \| \widehat{\Phi}_\Delta^{(h_\ell-1/2)} \|_{L^2(\mathbb{R}^d)} \bigg(\int_{[-\frac{4\pi}{3}, \frac{4\pi}{3}]^d} \Big| \widehat{\Upsilon}(\eta) - \sum_{|\mathbf{q}| \le N} \Upsilon_{\mathbf{q}} \, e^{-i\langle \mathbf{q}, \eta \rangle} \Big|^2 d\eta \bigg)^{\frac{1}{2}} \\ &\le 2^{\frac{d}{2}} \, t \prod_{\ell=1}^d \| \widehat{\Phi}_\Delta^{(h_\ell-1/2)} \|_{L^2(\mathbb{R}^d)} \bigg(\int_{[0,2\pi]^d} \Big| \widehat{\Upsilon}(\eta) - \sum_{|\mathbf{q}| \le N} \Upsilon_{\mathbf{q}} \, e^{-i\langle \mathbf{q}, \eta \rangle} \Big|^2 d\eta \bigg)^{\frac{1}{2}}, \end{aligned}$$

where the last inequality results from the fact that the function $\eta \mapsto \widehat{\Upsilon}(\eta) - \sum_{|\mathbf{q}| \le N} \Upsilon_{\mathbf{q}} \, e^{-i\langle \mathbf{q}, \eta \rangle}$ is 2π-periodic in each one of its d variables. Moreover, since the function $\widehat{\Upsilon}$ is the Fourier transform of the sequence $\Upsilon = (\Upsilon_{\mathbf{q}})_{\mathbf{q} \in \mathbb{Z}^d}$, one has that

$$\lim_{N \to +\infty} \int_{[0,2\pi]^d} \Big| \widehat{\Upsilon}(\eta) - \sum_{|\mathbf{q}| \le N} \Upsilon_{\mathbf{q}} \, e^{-i\langle \mathbf{q}, \eta \rangle} \Big|^2 d\eta = 0.$$

Therefore, using the previous bound for $\Big| \alpha_{\mathbf{k}} - \sum_{|\mathbf{q}| \le N} \Upsilon_{\mathbf{q}} \, \beta_{\mathbf{k}-\mathbf{q}} \Big|$, one gets that

$$\Big| \alpha_{\mathbf{k}} - \sum_{\mathbf{q} \in \mathbb{Z}^d} \Upsilon_{\mathbf{q}} \, \beta_{\mathbf{k}-\mathbf{q}} \Big| = \lim_{N \to +\infty} \Big| \alpha_{\mathbf{k}} - \sum_{|\mathbf{q}| \le N} \Upsilon_{\mathbf{q}} \, \beta_{\mathbf{k}-\mathbf{q}} \Big| = 0,$$

which shows that the equality (3.19) holds for all $\mathbf{k} \in \mathbb{Z}^d$. □

Proof of Lemma 3.12. This lemma is a straightforward of Remark 3.14 and Lemma 3.15.

□

3.4 Fourth step of the proof of Theorem 2.8

So far, we have shown that, for each fixed $t \in \mathbb{R}_+$ and $J \in \mathbb{N}$, the equality (2.16) holds almost surely, and that the random series in its right-hand side is convergent in $L^2(\Omega)$. The goal of the present subsection is to complete the proof of Theorem 2.8 by showing that the latter random series is almost surely convergent uniformly in t on any compact interval of $\mathbb{R}_+$. To this end, we need to bound, for all $J \in \mathbb{N}$, the generalized FARIMA sequence $(\sigma_{J,\mathbf{k}}^{(\mathbf{h})})_{\mathbf{k}\in\mathbb{Z}^d}$ in a convenient way.

Remark 3.16. Thanks to the representation (3.1), using Theorem 6.7 in [13], the isometry property of multiple Wiener integral and arguments similar to those in the proofs of Lemmas 1 and 2 in [3], it can be shown that there exist $\widetilde{C}$ a positive finite random variable and $\widetilde{\Omega}$ an event of probability 1, such that, one has on $\widetilde{\Omega}$, for all $J \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{Z}^d$,

$$\left|\sigma_{J,\mathbf{k}}^{(\mathbf{h})}\right| \le \widetilde{C}\left(\log\left(3 + J + |\mathbf{k}|\right)\right)^{d/2}. \tag{3.22}$$

End of the Proof of Theorem 2.8. For showing that the random series in the right-hand side of the equality (2.16) is almost surely convergent uniformly in t on any compact interval of $\mathbb{R}_+$, it is enough to prove that on the event $\widetilde{\Omega}$ of probability 1, introduced in Remark 3.16, one has, for each fixed $J \in \mathbb{N}$ and positive real number T,

$$\sum_{\mathbf{k}\in\mathbb{Z}^d}\left(\sup_{t\in[0,T]}\left|\int_0^t \prod_{\ell=1}^{d} \Phi_{\Delta}^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds\right|\right)\left|\sigma_{J,\mathbf{k}}^{(\mathbf{h})}\right| < +\infty. \tag{3.23}$$

In fact, using (3.22) and easy calculations, it turns out that (3.23) can be obtained by showing that

$$\sum_{\mathbf{k}\in\mathbb{Z}^d}\left(\sup_{x\in[0,2^J T]}\prod_{\ell=1}^{d}\left|\Phi_{\Delta}^{(h_\ell-1/2)}(x-k_\ell)\right|\right)\left(\log\left(3+J+|\mathbf{k}|\right)\right)^{d/2} < +\infty. \tag{3.24}$$

Finally, since the functions $\Phi_{\Delta}^{(h_\ell-1/2)}$, $\ell \in [\![1,d]\!]$, belong to Schwartz class $\mathcal{S}(\mathbb{R})$, it is clear that (3.24) holds. □

4 Proof of Theorem 2.12

In this section, we aim at proving Theorem 2.12. We will need a number of intermediary results which mainly consist in bounding in convenient ways well-chosen parts of the random series in (2.8). We mention in passing that the event Ω^* of probability 1 (see Lemmata 2.15 and 2.16) will appear in the statements of many of them. Also, we mention that we will frequently use the fact that (see (2.11)) the deterministic coefficients $\mathcal{K}_{\mathbf{j},\mathbf{k}}^{(d,\mathbf{h})}(t)$ in (2.8) can be expressed as

$$\mathcal{K}_{\mathbf{j},\mathbf{k}}^{(d,\mathbf{h})}(t) = 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)}\mathcal{A}_{\mathbf{j},\mathbf{k}}(t), \tag{4.1}$$

where, for all $(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^d)^2$ and $t \in \mathbb{R}_+$,

$$\mathcal{A}_{\mathbf{j},\mathbf{k}}(t) := \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell}s - k_\ell)\, ds. \tag{4.2}$$

Recall that each function ψ_{h_ℓ} is the fractional primitive of order $h_\ell - 1/2$ of the univariate Meyer mother wavelet ψ (see Remark 1.1 and Definition 1.2). Also, recall that ψ_{h_ℓ} satisfies the very nice localization property (1.4) which implies that, for any fixed arbitrarily large positive real number L, one has, for some finite constant c (only depending on L) and for all $(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^d)^2$ and $t \in \mathbb{R}_+$,

$$\big|\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)\big| \le c\int_0^t \prod_{\ell=1}^d \big(3 + |2^{j_\ell}s - k_\ell|\big)^{-L}\, ds. \tag{4.3}$$

Our study of the random series in (2.8) is to certain extent inspired by the methodology which was introduced in [2] in the framework of the generalized Rosenblatt process. In this respect, the first thing to do is to express the maximum $\max_{\ell\in[\![1,d]\!]} j_\ell$ (see (2.8)) in a way which is convenient to handle. To this end, for each $n \in [\![1,d]\!]$ and $J \in \mathbb{N}$, we introduce the infinite subset $\aleph_{n,J} \subset \mathbb{Z}^d$. defined as

$$\aleph_{n,J} := \left\{\mathbf{j} \in \mathbb{Z}^d : j_n \ge J \text{ and } \max_{\ell\in[\![1,d]\!]} j_\ell = j_n\right\}. \tag{4.4}$$

Observe that the indexation set in the sum in (2.8), can then be expressed as the union $\bigcup_{n=1}^d \aleph_{n,J} \times \mathbb{Z}^d$. Thus, it results from (2.8), the triangle inequality and (4.1) that, for any fixed positive real number T,

$$\|X_{\mathbf{h},J}^{(d,\perp)}\|_{[0,T],\infty} \le \sum_{n=1}^d \Delta_{n,J},$$

where, for all $n \in [\![1,d]\!]$,

$$\Delta_{n,J} := \sum_{\mathbf{j}\in\aleph_{n,J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sup_{t\in[0,T]} \Big|\sum_{\mathbf{k}\in\mathbb{Z}^d} \varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)\Big|. \tag{4.5}$$

From now on, we focus on the positive random series $\Delta_{n,J}$, for any arbitrary and fixed $n \in [\![1,d]\!]$. We will show that it is formed by a main part $\Delta_{n,J}^1$, and three other parts $\Delta_{n,J}^0$, $\Delta_{n,J}^2$ and $\Delta_{n,J}^3$ which are negligible for our purposes; namely, on the event Ω^* of probability 1, when J goes to $+\infty$, they converge to zero more quickly than the rate $J^{\frac{d}{2}}2^{-J(h_1+\cdots+h_d-d+1/2)}$ targeted in Theorem 2.12. The definitions of $\Delta_{n,J}^0, \ldots, \Delta_{n,J}^3$ are closely connected with the nth axis of $\mathbb{Z}^d$. We are now going to give them and to motivate them.

The negligible part $\Delta_{n,J}^0$ is defined by replacing in (4.5) the sum $\sum_{\mathbf{k}\in\mathbb{Z}^d}$ by the sum $\sum_{\mathbf{k}\in\beth_{n,T}^{j_n}}$ where

$$\beth_{n,T}^{j_n} := \big\{\mathbf{k} \in \mathbb{Z}^d : k_n \in \mathbb{Z};\quad \exists\, \ell \in [\![1,d]\!] \setminus \{n\},\quad |k_\ell| > 2^{j_n+1}T\big\}. \tag{4.6}$$

Thus, one can derive from the triangle inequality that

$$\Delta^0_{n,J} \le \mathcal{H}^0_{n,J}, \tag{4.7}$$

where

$$\mathcal{H}^0_{n,J} := \sum_{\mathbf{j}\in\aleph_{n,J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sum_{\mathbf{k}\in\beth^{j_n}_{n,T}} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)|. \tag{4.8}$$

For understanding the motivation behind the definition of $\beth^{j_n}_{n,T}$, one has to relate it to the inequality (4.3). Indeed, when $|k_\ell| > 2^{j_n+1}T$, then using the inequality $j_l \le j_n$ (since $j_n = \max_{\ell\in[\![1,d]\!]} j_\ell$), it can easily be shown that the quantity $\big(3+|2^{j_\ell}s-k_\ell|\big)^{-L}$ in the right-hand side of (4.3) satisfies $\big(3+|2^{j_\ell}s-k_\ell|\big)^{-L} \le 2^L\big(3+|k_\ell|\big)^{-L}$. The latter inequality, combined with (4.8) and (2.31), allows to show that, on the event Ω^*, when J goes to $+\infty$, $\mathcal{H}^0_{n,J}$ (and consequently $\Delta^0_{n,J}$, see (4.7)) converges to zero at a very fast rate, see Lemma 4.2 in Subsection 4.1 below.

For defining the main part $\Delta^1_{n,J}$ as well as the two other negligible parts $\Delta^2_{n,J}$ and $\Delta^3_{n,J}$ of the random series in (4.5), we need to introduce some additional sets of indices.

Definition 4.1. Let a be a fixed real number satisfying $1/2 < a < 1$. For all $(j,k)\in\mathbb{Z}_+\times\mathbb{Z}$, we denote by $B_{j,k}$ the compact interval of the real line $\mathbb{R}$

$$B_{j,k} := [k2^{-j} - 2^{-ja}, k2^{-j} + 2^{-ja}]. \tag{4.9}$$

Then, for all $j\in\mathbb{N}$ and $t\in\mathbb{R}_+$, the three disjoint subsets $D^1_j(t)$, $D^2_j(t)$, and $D^3_j(t)$ of $\mathbb{Z}$, which depend on j, t and a, are defined as

$$D^1_j(t) := \{k\in\mathbb{Z} : B_{j,k}\subseteq[0,t]\}, \tag{4.10}$$

$$D^2_j(t) := \{k\in\mathbb{Z}\setminus D^1_j(t) : B_{j,k}\cap[0,t]\neq\emptyset\}, \tag{4.11}$$

$$D^3_j(t) := \{k\in\mathbb{Z} : B_{j,k}\cap[0,t]=\emptyset\}. \tag{4.12}$$

They clearly form a partition of $\mathbb{Z}$, that is $\mathbb{Z} = \bigcup_{\ell=1}^3 D^\ell_j(t)$.

We denote by $\widetilde{\beth}^{j_n}_{n,T} := \mathbb{Z}^d\setminus\beth^{j_n}_{n,T}$ the complement in $\mathbb{Z}^d$ of the set $\beth^{j_n}_{n,T}$ introduced in (4.6). For each $t\in\mathbb{R}_+$ and $\ell\in\{1,2,3\}$, the subset $\widetilde{\beth}^{j_n,\ell}_{n,T}(t)\subset\widetilde{\beth}^{j_n}_{n,T}$ is defined as $\widetilde{\beth}^{j_n,\ell}_{n,T}(t) := \big\{\mathbf{k}\in\widetilde{\beth}^{j_n}_{n,T} : k_n\in D^\ell_{j_n}(t)\big\}$, that is

$$\widetilde{\beth}^{j_n,\ell}_{n,T}(t) := \big\{\mathbf{k}\in\mathbb{Z}^d : k_n\in D^\ell_{j_n}(t);\quad \forall\,\ell\in[\![1,d]\!]\setminus\{n\},\quad |k_\ell|\le 2^{j_n+1}T\big\}. \tag{4.13}$$

Then, for every $\ell\in\{1,2,3\}$, we set

$$\Delta^\ell_{n,J} := \sum_{\mathbf{j}\in\aleph_{n,J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sup_{t\in[0,T]} \Big|\sum_{\mathbf{k}\in\widetilde{\beth}^{j_n,\ell}_{n,T}(t)} \varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)\Big|. \tag{4.14}$$

Observe that, one can derive from (4.14), (4.13) and the triangle inequality that

$$\Delta^2_{n,J}\le\mathcal{H}^2_{n,J} \quad\text{and}\quad \Delta^3_{n,J}\le\mathcal{H}^3_{n,J}, \tag{4.15}$$

where

$$\mathcal{H}^2_{n,J} := \sum_{\mathbf{j}\in\aleph_{n,J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \times \ldots$$

$$\ldots \times \sup_{t\in[0,T]} \left\{ \sum_{k_n\in D^2_{j_n}(t)} \sum_{\substack{k_\ell\in\mathbb{Z}\\ \ell\in[\![1,d]\!]\setminus\{n\}}} |\varepsilon_{\mathbf{j},\mathbf{k}}|\,|\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \right\} \tag{4.16}$$

and

$$\mathcal{H}^3_{n,J} := \sum_{\mathbf{j}\in\aleph_{n,J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \times \ldots$$

$$\ldots \times \sup_{t\in[0,T]} \left\{ \sum_{k_n\in D^3_{j_n}(t)} \sum_{\substack{k_\ell\in\mathbb{Z}\\ \ell\in[\![1,d]\!]\setminus\{n\}}} |\varepsilon_{\mathbf{j},\mathbf{k}}|\,|\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \right\}. \tag{4.17}$$

On one hand using (4.11), (4.10) and (4.9), it can be shown $D^2_{j_n}(t)$ is a rather small finite set with cardinality bounded from above by $c'\,2^{j_n(1-a)}$, for some finite constant c' not depending on $t\in[0,T]$ and j_n. The latter fact, combined with (4.3) and (2.31), is the main ingredient for proving, on the event Ω^*, that, when J goes to $+\infty$, $\mathcal{H}^2_{n,J}$ (and consequently $\Delta^2_{n,J}$, see (4.15)) converges to zero more quickly than the rate $J^{\frac{d}{2}}2^{-J(h_1+\cdots+h_d-d+1/2)}$ targeted in Theorem 2.12, see Lemma 4.5 in Subsection 4.1 below.

On another hand, Lemma B.3 in Appendix B, combined with (4.3) and (2.31), is the main ingredient for proving, on the event Ω^*, that, when J goes to $+\infty$, $\mathcal{H}^3_{n,J}$ (and consequently $\Delta^3_{n,J}$, see (4.15)) converges to zero more quickly than the rate $J^{\frac{d}{2}}2^{-J(h_1+\cdots+h_d-d+1/2)}$ targeted in Theorem 2.12, see Lemma 4.4 in Subsection 4.1 below.

So far, we have reached the conclusion that the main part of the random series in (4.5) is $\Delta^1_{n,J}$ defined through (4.14) with $\ell=1$. For the sake of simplicity in notation, we set

$$\daleth^{j_n}_n(t) := \widetilde{\beth}^{j_n,1}_{n,T}(t) := \{\mathbf{k}\in\mathbb{Z}^d \,:\, k_n\in D^1_{j_n}(t);\, \forall\ell\in[\![1,d]\!]\setminus\{n\},\, |k_\ell|\le 2^{j_n+1}T\}. \tag{4.18}$$

We are now going to introduce a simplified version of $\Delta^1_{n,J}$, denoted by $\mathcal{M}_{n,J}$, in which the coefficients $\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)$ (see (4.2)) are replaced by the coefficients $F_{\mathbf{j},\mathbf{k}}$, not depending on t, defined, for all $(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2$, as

$$F_{\mathbf{j},\mathbf{k}} := \int_{\mathbb{R}} \prod_{\ell=1}^{d} \psi_{h_\ell}(2^{j_\ell}s-k_\ell)\,ds. \tag{4.19}$$

Thus, in view of (4.18) and (4.14) (with $\ell=1$), $\mathcal{M}_{n,J}$ can be expressed as

$$\mathcal{M}_{n,J} := \sum_{\mathbf{j}\in\aleph_{n,J}} 2^{j_1(1-h_1)+\cdots j_d(1-h_d)} \sup_{t\in[0,T]} \Big| \sum_{\mathbf{k}\in\daleth^{j_n}_n(t)} F_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{j},\mathbf{k}} \Big|. \tag{4.20}$$

The advantage in working with $\mathcal{M}_{n,J}$ instead of $\Delta^1_{n,J}$ is that the random function $\widetilde{\mathcal{M}}_{n,\mathbf{j}}$ defined, for all $t\in[0,T]$, as

$$\widetilde{\mathcal{M}}_{n,\mathbf{j}}(t) := \sum_{\mathbf{k}\in\daleth^{j_n}_n(t)} F_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{j},\mathbf{k}}, \tag{4.21}$$

is in fact a step function whose jumps occur at the deterministic finite set of points $\{m2^{-j_n} + 2^{-j_n a} : m \in \mathbb{N} \cap (2^{j(1-a)} - 1, 2^j T - 2^{j(1-a)}]\}$. Thus, the supremum in (4.20) reduces to the supremum on this finite set, which makes the study of its asymptotic behavior, when j_n goes $+\infty$, much more accessible and doable thanks to Borel-Cantelli Lemma, see the Subsection 4.3 below.

Yet, for showing that it is possible to approximate $\Delta^1_{n,J}$ by $\mathcal{M}_{n,J}$ without altering the rate of convergence $J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}$ targeted in Theorem 2.12, one has to prove that, on the event Ω^*, the error of approximation $\big|\Delta^1_{n,J} - \mathcal{M}_{n,J}\big|$ converges to zero at a faster rate, when J goes $+\infty$. Notice that, it can be derived from (4.14) with $\ell = 1$, (4.18), (4.20) and the triangle inequality that

$$\big|\Delta^1_{n,J} - \mathcal{M}_{n,J}\big| \le \mathcal{H}^1_{n,J}, \tag{4.22}$$

where

$$\mathcal{H}^1_{n,J} := \sum_{\mathbf{j} \in \aleph_{n,J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \times \ldots$$

$$\ldots \times \sup_{t \in [0,T]} \left\{ \sum_{k_n \in D^1_{j_n}(t)} \sum_{\substack{k_\ell \in \mathbb{Z} \\ \ell \in [\![1,d]\!] \setminus \{n\}}} |\varepsilon_{\mathbf{j},\mathbf{k}}| \, |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t) - F_{\mathbf{j},\mathbf{k}}| \right\}. \tag{4.23}$$

Also notice that one knows from (4.10) and (4.9) that, for some finite constant c'' not depending on $t \in [0,T]$ and j_n, the cardinality of the finite set $D^1_{j_n}(t)$ is bounded from above by $c''\, 2^{j_n}$, and that any $k_n \in D^1_{j_n}(t)$ satisfies $2^{j_n(1-a)} \le k_n \le 2^{j_n} t - 2^{j_n(1-a)}$. These two facts, combined with (4.3), (2.31), Lemma B.3 and (B.1) in Appendix B, are the main ingredients for proving that, when J goes to $+\infty$, $\mathcal{H}^1_{n,J}$ (and consequently $\big|\Delta^1_{n,J} - \mathcal{M}_{n,J}\big|$, see (4.22)) converges to zero more quickly than the rate $J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}$ targeted in Theorem 2.12, see Lemma 4.6 in Subsection 4.2 below.

We complete the proof of Theorem 2.12 in Subsection 4.4 below.

4.1 Rates of convergence to zero of $\mathcal{H}^0_{n,J}$, $\mathcal{H}^2_{n,J}$ and $\mathcal{H}^3_{n,J}$

The goal of this subsection is to show that, on the event Ω^* of probability 1 (see Lemmata 2.15 and 2.16), when J goes to $+\infty$, the three random variables $\mathcal{H}^0_{n,J}$, $\mathcal{H}^2_{n,J}$ and $\mathcal{H}^3_{n,J}$ converge to zero more quickly than the rate $J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}$ targeted in Theorem 2.12.

Lemma 4.2. *Let $T > 2$ and $L \ge 3/2$ be two fixed real numbers. There exits a positive almost surely finite random variables C such that, for all $n \in [\![1,d]\!]$ and $J \in \mathbb{N}$, on Ω^*, the random variable $\mathcal{H}^0_{n,J}$, defined in (4.8), is bounded from above by $CJ^{\frac{d}{2}} \left(\log(3+J)\right)^{\frac{d-1}{2}} 2^{-J(h_1+\cdots+h_d+L-d-1)}$.*

Proof. The set $\mathcal{B}_n$ of d-dimensional boolean vectors is defined as

$$\mathcal{B}_n := \Big\{ v = (v_l)_{\ell \in [\![1,d]\!]} \in \{0,1\}^d \,:\, v_n = 1 \text{ and } \exists\, \ell' \neq n \,:\, v_{\ell'} = 0 \Big\}.$$

Moreover, for all $v \in \mathcal{B}_n$ and $\mathbf{j} \in \aleph_{n,J}$ (recall that j_n denotes the nth coordinate of $\mathbf{j}$), the set $\beth^{j_n,v}_{n,T}$ is defined as

$$\beth^{j_n,v}_{n,T} := \left\{ \mathbf{k} \in \mathbb{Z}^d : k_\ell \in \mathbb{Z} \text{ if } v_\ell = 1 \text{ and } |k_\ell| > 2^{j_n+1} T \text{ otherwise} \right\}.$$

Then, it can easily be derived from (4.6) that

$$\sum_{\mathbf{k}\in\beth_{n,T}^{j_n}} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \leq \sum_{v\in\mathcal{B}_n} \sum_{\mathbf{k}\in\beth_{n,T}^{j_n,v}} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)|\,. \tag{4.24}$$

Using, the inequality (B.1), the triangular inequality, the fact that the function $y \mapsto (2+y)^{-L}\sqrt{\log(2+y)}$ is decreasing on $\mathbb{R}_+$ and the inequality (B.2), we have, for all $v \in \mathcal{B}_n$, $\mathbf{j} \in \aleph_{n,J}$ and $s \in [0,T]$,

$$\begin{aligned}
&\sum_{\mathbf{k}\in\beth_{n,T}^{j_n,v}} \prod_{\ell=1}^{d} \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(3+|2^{j_\ell}s-k_\ell|)^L} \\
&= \left(\prod_{\ell:v_\ell=1} \sum_{k_\ell\in\mathbb{Z}} \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(3+|2^{j_\ell}s-k_\ell|)^L}\right) \times \dots \\
&\qquad \dots \times \left(\prod_{\ell':v_{\ell'}=0} \sum_{|k_{\ell'}|>2^{j_n+1}T} \frac{\sqrt{\log(3+|j_{\ell'}|+|k_{\ell'}|)}}{(3+|2^{j_{\ell'}}s-k_{\ell'}|)^L}\right) \\
&\leq c_0 2^L \left(\prod_{\ell:v_\ell=1} \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)}\right) \times \dots \\
&\qquad \dots \times \left(\prod_{\ell':v_{\ell'}=0} \sum_{|k_{\ell'}|>2^{j_n+1}T} \frac{\sqrt{\log(3+|j_{\ell'}|)}\sqrt{\log(3+|k_{\ell'}|)}}{(3+|k_{\ell'}|)^L}\right) \\
&\leq c_1 \left(\prod_{\ell:v_\ell=1} \sqrt{\log(3+|j_\ell|+2^{j_n}T)}\right) \times \dots \\
&\qquad \dots \times \left(\prod_{\ell':v_{\ell'}=0} \sqrt{\log(3+|j_{\ell'}|)} \int_{2^{j_n+1}T}^{+\infty} \frac{\sqrt{\log(2+y)}}{(2+y)^L} dy\right) \\
&\leq c_2 \sqrt{\prod_{\ell=1,\ell\neq n}^{d} \log(3+|j_\ell|) \times j_n^{\frac{d}{2}} \times 2^{-j_n(L-1)(\#\{\ell' : v_{\ell'}=0\})}} \\
&\leq c_2 \sqrt{\prod_{\ell=1,\ell\neq n}^{d} \log(3+|j_\ell|) \times j_n^{\frac{d}{2}} \times 2^{-j_n(L-1)}},
\end{aligned} \tag{4.25}$$

with c_0, c_1 and c_2 positive deterministic constants not depending on n, $\mathbf{j}$, v nor J. Then, the expression (4.2), the bound (2.31), the inequality (4.3) and inequality (4.25) give

$$\sum_{\mathbf{k}\in\beth_{n,T}^{j_n,v}} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \leq C_1 \sqrt{\prod_{\ell=1,\ell\neq n}^{d} \log(3+|j_\ell|) \times j_n^{\frac{d}{2}} \times 2^{-j_n(L-1)}}, \tag{4.26}$$

with C_1 a positive almost surely finite random variable not depending on n, $\mathbf{j}$,

v nor J. Then, using (4.8), (4.24), (4.26) and the triangular inequality, we get

$$\mathcal{H}^0_{n,J} \leq C_2 \sum_{j_n \geq J} \sum_{\substack{j_\ell \leq j_n \\ \ell \in [\![1,d]\!] \setminus \{n\}}} \sqrt{\prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \log(3+|j_\ell|) \times j_n^{\frac{d}{2}} \times 2^{-j_n(L-1)} \prod_{\ell=1}^{d} 2^{j_\ell(1-h_\ell)}}$$

$$= C_2 \sum_{j_n \geq J} \sum_{\substack{j_\ell \leq j_n \\ \ell \in [\![1,d]\!] \setminus \{n\}}} \left(\prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sqrt{\log(3+|j_\ell|)} 2^{j_\ell(1-h_\ell)} \right) \times j_n^{\frac{d}{2}} \times 2^{-j_n(h_n+L-2)}$$

$$= C_2 \sum_{j_n \geq J} j_n^{\frac{d}{2}} 2^{-j_n(h_n+L-2)} \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sum_{j_\ell=-\infty}^{j_n} \sqrt{\log(3+|j_\ell|)} 2^{j_\ell(1-h_\ell)}$$

and since, by inequality (B.1), we have

$$\prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sum_{j_\ell=-\infty}^{j_n} \sqrt{\log(3+|j_\ell|)} 2^{j_\ell(1-h_\ell)}$$

$$= \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sum_{p=0}^{+\infty} \sqrt{\log(3+|j_n-p|)} 2^{(j_n-p)(1-h_\ell)}$$

$$= \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sqrt{\log(3+|j_n|)} 2^{j_n(1-h_\ell)} \left(\sum_{p=0}^{+\infty} \sqrt{\log(3+|p|)} 2^{-p(1-h_n)} \right)$$

$$\leq c \left(\log(3+|j_n|)\right)^{\frac{d-1}{2}} \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} 2^{j_n(1-h_\ell)},$$

for a deterministic constant $c > 0$, we conclude that

$$\mathcal{H}^0_{n,J} \leq C_2 \sum_{j_n \geq J} j_n^{\frac{d}{2}} \left(\log(3+|j_n|)\right)^{\frac{d-1}{2}} 2^{-j_n(h_n+L-2)} \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} 2^{j_n(1-h_\ell)}$$

$$= C_2 \sum_{j_n \geq J} j_n^{\frac{d}{2}} \left(\log(3+|j_n|)\right)^{\frac{d-1}{2}} 2^{-j_n(h_1+\cdots+h_d+L-d-1)}$$

$$\leq C_3 J^{\frac{d}{2}} \left(\log(3+J)\right)^{\frac{d-1}{2}} 2^{-J(h_1+\cdots+h_d+L-d-1)},$$

where C_2 and C_3 are positive almost surely finite random variables. □

Remark 4.3. One knows from Definition 4.1, that $D^3_j(t)$ is always an infinite countable set while $D^2_j(t)$ and $D^1_j(t)$ are two finite sets, possibly empty. Moreover, for all strictly positive real number T and all $j \in \mathbb{Z}_+$, we have

$$\sup_{t \in [0,T]} \left\{ \mathrm{Card}(D^2_j(t)) \right\} \leq c' 2^{j(1-a)}, \tag{4.27}$$

$$\sup_{t \in [0,T]} \left\{ \mathrm{Card}(D^1_j(t)) \right\} \leq c'' 2^{j}, \tag{4.28}$$

where $c' \geq 1$ and $c'' \geq 1$ are two finite positive constants not depending j. One mentions in passing that c' does not even depend on T.

Lemma 4.4. *Let $T > 2$ and $L \geq 2^{-1}(1-a)^{-1}+1$ be two fixed real numbers, where $a \in (1/2,1)$ is as in Definition 4.1. There exists a positive almost surely finite random variable C such that, for all $n \in [\![1,d]\!]$ and $J \in \mathbb{N}$, on Ω^*, the random variable $\mathcal{H}^3_{n,J}$, defined in (4.17), is bounded from above by $CJ^{\frac{d+1}{2}}\left(\log(3+J)\right)^{\frac{d-1}{2}} 2^{-J(h_1+\cdots+h_d+(L-1)(1-a)-d)}$.*

Proof. Let $t \in [0,T]$ and $\mathbf{j} \in \aleph_{n,J}$. Using together the expression (4.2), the inequality (4.3), the bound (2.31) as well as Lemmata B.2 and B.3, we get

$$\begin{aligned}
&\sum_{k_n \in D^3_{j_n}(t)} \sum_{\substack{k_\ell \in \mathbb{Z}\\ \ell \in [\![1,d]\!]\setminus\{n\}}} |\varepsilon_{\mathbf{j},\mathbf{k}}|\,|\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \\
&\leq C_0 \sum_{k_n \in D^3_{j_n}(t)} \sum_{\substack{k_\ell \in \mathbb{Z}\\ \ell \in [\![1,d]\!]\setminus\{n\}}} \int_0^T \prod_{\ell=1}^d \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(3+|2^{j_\ell}s-k_\ell|)^L}\,ds \\
&= C_0 \int_0^T \left(\sum_{k_n \in D^3_{j_n}(t)} \frac{\sqrt{\log(3+|j_n|+|k_n|)}}{(3+|2^{j_n}s-k_n|)^L}\right) \prod_{\substack{\ell=1\\ \ell\neq n}}^d \sum_{k_\ell \in \mathbb{Z}} \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(3+|2^{j_\ell}s-k_\ell|)^L}\,ds \\
&\leq C_1 \prod_{\substack{\ell=1\\ \ell\neq n}}^d \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)} \int_0^T \sum_{k_n \in D^3_{j_n}(t)} \frac{\sqrt{\log(3+|j_n|+|k_n|)}}{(3+|2^{j_n}s-k_n|)^L}\,ds \\
&\leq C_2 \prod_{\substack{\ell=1\\ \ell\neq n}}^d \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)}\,(1+j_n)2^{-j_n(L-1)(1-a)}. \qquad (4.29)
\end{aligned}$$

Next, notice that using the triangular inequality and (B.1), we obtain that

$$\begin{aligned}
&\sum_{\substack{j_\ell \leq j_n\\ \ell \in [\![1,d]\!]\setminus\{n\}}} \prod_{\substack{\ell=1\\ \ell\neq n}}^d \left(\sqrt{\log(3+|j_\ell|+2^{j_\ell}T)}\,2^{j_\ell(1-h_\ell)}\right) \\
&\quad= \prod_{\substack{\ell=1\\ \ell\neq n}}^d \left(\sum_{j_\ell=-\infty}^{j_n} \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)}\,2^{j_\ell(1-h_\ell)}\right) \\
&\quad= \prod_{\substack{\ell=1\\ \ell\neq n}}^d \left(\sum_{p=0}^{+\infty} \sqrt{\log(3+|j_n-p|+2^{j_n-p}T)}\,2^{(j_n-p)(1-h_\ell)}\right) \\
&\quad\leq \left(\sqrt{\log(3+2^{j_n}T)\log(3+|j_n|)}\right)^{d-1} \times \ldots \\
&\qquad\quad \ldots \times \prod_{\substack{\ell=1\\ \ell\neq n}}^d 2^{j_n(1-h_\ell)} \left(\sum_{p=0}^{+\infty} \sqrt{\log(3+p)}\,2^{-p(1-h_\ell)}\right) \\
&\quad\leq c\left(\sqrt{\log(3+2^{j_n}T)\log(3+|j_n|)}\right)^{d-1} \prod_{\substack{\ell=1\\ \ell\neq n}}^d 2^{j_n(1-h_\ell)} \qquad (4.30)
\end{aligned}$$

where $c > 0$ is a deterministic constant. Next, putting together (4.17), (4.4), (4.29) and (4.30), it follows that

$$\begin{aligned}\mathcal{H}^3_{n,J} &\le C_3 \sum_{j_n \ge J} (1+j_n) 2^{-j_n(h_1+\cdots+h_d+(L-1)(1-a)-d)} \times \dots \\ &\qquad \dots \times \left(\sqrt{\log(3+2^{j_n}T)\log(3+|j_n|)}\right)^{d-1} \\ &\le C_4 \sum_{j_n \ge J} j_n^{\frac{d+1}{2}} (\log(3+|j_n|))^{\frac{d-1}{2}} 2^{-j_n(h_1+\cdots+h_d+(L-1)(1-a)-d)} \\ &\le C_5 J^{\frac{d+1}{2}} (\log(3+|J|))^{\frac{d-1}{2}} 2^{-J(h_1+\cdots+h_d+(L-1)(1-a)-d)}\end{aligned}$$

where C_3, C_4 and C_5 are positive almost surely finite random variables. □

Lemma 4.5. *Let $a \in (1/2, 1)$ be as in Definition 4.1 and let $T > 2$ be a fixed real number. There exists a positive almost surely finite random variable C such that, for all $n \in [\![1, d]\!]$ and $J \in \mathbb{N}$, on Ω^*, the random variable $\mathcal{H}^2_{n,J}$, defined in* (4.16)*, is bounded from above by $CJ^{\frac{d}{2}} (\log(3+J))^{\frac{d-1}{2}} 2^{-J(h_1+\cdots+h_d+a-d)}$.*

Proof. let $L > 1$ be a fixed real number, $t \in [0, T]$ and $\mathbf{j} \in \aleph_{n,J}$. Using the definition (4.2), the inequalities (4.3) and (2.31), Lemma B.2, the inequality $|k_n| \le 2^{j_n(1-a)} + 2^{j_n}T$, for all $k_n \in D^2_{j_n}(t)$, the change of variable $z = 2^{j_n}s - k_n$, the bound (4.27) and the inequality (B.2), we have

$$
\begin{aligned}
&\sum_{k_n \in D^2_{j_n}(t)} \sum_{\substack{k_\ell \in \mathbb{Z} \\ \ell \in [\![1,d]\!] \setminus \{n\}}} |\varepsilon_{\mathbf{j},\mathbf{k}}|\, |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \\
&\leq C_0 \int_0^T \sum_{k_n \in D^2_{j_n}(t)} \sum_{\substack{k_\ell \in \mathbb{Z} \\ \ell \in [\![1,d]\!] \setminus \{n\}}} \prod_{\ell=1}^{d} \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(3+|2^{j_\ell}s-k_\ell|)^L} ds \\
&= C_0 \int_0^T \sum_{k_n \in D^2_{j_n}(t)} \frac{\sqrt{\log(3+|j_n|+|k_n|)}}{(3+|2^{j_n}s-k_n|)^L} \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sum_{k_\ell \in \mathbb{Z}} \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(3+|2^{j_\ell}s-k_\ell|)^L} ds \\
&\leq C_1 \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)} \int_0^T \sum_{k_n \in D^2_{j_n}(t)} \frac{\sqrt{\log(3+|j_n|+|k_n|)}}{(3+|2^{j_n}s-k_n|)^L} ds \\
&\leq C_1 \sqrt{\log(3+|j_n|+2^{j_n(1-a)}+2^{j_n}T)} \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)} \times \dots \\
&\qquad \dots \times \sum_{k_n \in D^2_{j_n}(t)} \int_0^T \frac{ds}{(3+|2^{j_n}s-k_n|)^L} \\
&= C_2 \left(\int_{\mathbb{R}} \frac{dz}{(3+|z|)^L} \right) \sqrt{\log(3+|j_n|+2^{j_n(1-a)}+2^{j_n}T)} \times \dots \\
&\qquad \dots \times \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)}\, \mathrm{Card}(D^2_{j_n}(t)) 2^{-j_n} \\
&\leq C_3 2^{-j_n a} \sqrt{\log(3+|j_n|+2^{j_n(1-a)}+2^{j_n}T)} \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)} \\
&\leq C_4 2^{-j_n a} \sqrt{1+j_n} \prod_{\substack{\ell=1 \\ \ell \neq n}}^{d} \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)},
\end{aligned}
$$

where C_0, C_1, C_2, C_3 and C_4 are positive almost surely finite random variables not depending on n, t, $\mathbf{j}$ and J. Then, combining the (4.16) and (4.4) with the inequalities (4.30), $a > 1/2$ and $\sum_{\ell=1}^{d} h_\ell > d - 1/2$, we get

$$\begin{aligned}
\mathcal{H}_{n,J}^2 &\le C_4 \sum_{\mathbf{j}\in\aleph_{n,J}} \left(\prod_{\ell=1}^{d} 2^{j_\ell(1-h_\ell)}\right) 2^{-j_n a}\sqrt{1+j_n}\prod_{\substack{\ell=1\\ \ell\neq n}}^{d}\sqrt{\log(3+|j_\ell|+2^{j_\ell}T)} \\
&\le C_5 \sum_{j_n\ge J} 2^{-j_n(h_n+a-1)}\sqrt{1+j_n}\big(\sqrt{\log(3+2^{j_n}T)\log(3+|j_n|)}\big)^{d-1}\times\dots \\
&\qquad \dots\times\prod_{\substack{\ell=1\\ \ell\neq n}}^{d} 2^{j_n(1-h_\ell)} \\
&\le C_6 \sum_{j_n\ge J} 2^{-j_n(h_1+\dots+h_n+a-d)} j_n^{\frac{d}{2}}\left(\log(3+|j_n|)\right)^{\frac{d-1}{2}} \\
&\le C_7 2^{-J(h_1+\dots+h_n+a-d)} J^{\frac{d}{2}}\left(\log(3+J)\right)^{\frac{d-1}{2}},
\end{aligned}$$

where C_5, C_6 and C_7 are positive almost surely finite random variables. □

4.2 Rate of convergence to zero of $\mathcal{H}_{n,J}^1$

The goal of this subsection is to show that, on the event Ω^* of probability 1 (see Lemmata 2.15 and 2.16), when J goes to $+\infty$, the random variable $\mathcal{H}_{n,J}^1$ converges to zero more quickly than the rate $J^{\frac{d}{2}}2^{-J(h_1+\dots+h_d-d+1/2)}$ targeted in Theorem 2.12.

Lemma 4.6. *Let $T > 2$ and $L > d+2$ be two fixed real numbers. There exits a positive almost surely finite random variable C such that, for all $n \in [\![1,d]\!]$ and $J \in \mathbb{N}$, on Ω^*, the random variable $\mathcal{H}_{n,J}^1$, defined in* (4.23)*, is bounded from above by $CJ^{d-1}\,2^{-J((L-2)(1-a)+h_1+\dots+h_d-d+1)}$.*

Proof. Let us fix $t \in [0,T]$ and $\mathbf{j} \in \aleph_{n,J}$. Using the definitions (4.2), (4.19), (4.9) and (4.10), the inequality (4.3), the bound (2.31), the inequality $|k_n| \le 2^{j_n}T$, for all $k_n \in D_{j_n}^1(t)$, the inequality $2^{j_n}T \ge j_n$, Lemma B.2, the fact that

$j_n = \max_{\ell \in [\![1,d]\!]} j_\ell$, the triangular inequality and finally inequality (B.1), it comes

$$\begin{aligned}
&\sum_{k_n \in D^1_{j_n}(t)} \sum_{\substack{k_\ell \in \mathbb{Z} \\ \ell \in [\![1,d]\!] \setminus \{n\}}} |\varepsilon_{\mathbf{j},\mathbf{k}}| \, |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t) - F_{\mathbf{j},\mathbf{k}}| \\
&\le C_0 \sqrt{\log(3 + 2^{j_n+1}T)} \int_{\mathbb{R}\setminus[0,t]} \left(\sum_{k_n \in D^1_{j_n}(t)} \frac{1}{(3 + |2^{j_n}s - k_n|)^L} \right) \times \ldots \\
&\qquad \ldots \times \prod_{\ell=1;\, \ell \neq n}^{d} \left(\sum_{k_\ell \in \mathbb{Z}} \frac{\sqrt{\log(3 + |j_\ell| + |k_\ell|)}}{(3 + |2^{j_\ell}s - k_\ell|)^L} \right) ds \\
&\le C_1 \sqrt{\log(3 + 2^{j_n+1}T)} \int_{\mathbb{R}\setminus[0,t]} \sum_{k_n \in D^1_{j_n}(t)} \frac{\prod_{\ell=1;\, \ell \neq n}^{d} \sqrt{\log(3 + |j_\ell| + |2^{j_\ell}s|)}}{(3 + |2^{j_n}s - k_n|)^L} ds \\
&\le C_1 \sqrt{\log(3 + 2^{j_n+1}T)} \int_{\mathbb{R}\setminus[0,t]} \sum_{k_n \in D^1_{j_n}(t)} \frac{\prod_{\ell=1;\, \ell \neq n}^{d} \sqrt{\log(3 + |j_\ell| + |2^{j_n}s|)}}{(3 + |2^{j_n}s - k_n|)^L} ds \\
&\le C_1 \int_{\mathbb{R}\setminus[0,t]} \sum_{k_n \in D^1_{j_n}(t)} \frac{\prod_{\ell=1;\, \ell \neq n}^{d} \sqrt{\log(3 + |j_\ell| + |2^{j_n}s - k_n|)}}{(3 + |2^{j_n}s - k_n|)^L} ds \times \ldots \\
&\qquad \ldots \times (\log(3 + 2^{j_n+1}T))^{\frac{d}{2}}, \qquad (4.31)
\end{aligned}$$

where C_0 and C_1 are positive almost surely finite random variables not depending on n, t, $\mathbf{j}$ and J. Let us estimate the last integral in (4.31). First we bound it by the sum of the two integrals $I^1_{\mathbf{j},\mathbf{k}}(t)$ and $I^2_{\mathbf{j},\mathbf{k}}$ where

$$I^1_{\mathbf{j},\mathbf{k}}(t) := \int_t^{+\infty} \sum_{k_n \le 2^{j_n}t - 2^{j_n(1-a)}} \frac{\prod_{\ell=1;\, \ell \neq n}^{d} \sqrt{\log(3 + |j_\ell| + 2^{j_n}s - k_n)}}{(3 + 2^{j_n}s - k_n)^L} ds$$

and

$$I^2_{\mathbf{j},\mathbf{k}} := \int_{-\infty}^{0} \sum_{k_n \ge 2^{j_n(1-a)}} \frac{\prod_{\ell=1;\, \ell \neq n}^{d} \sqrt{\log(3 + |j_\ell| + k_n - 2^{j_n}s)}}{(3 + k_n - 2^{j_n}s)^L} ds.$$

Next, for bounding $I^1_{\mathbf{j},\mathbf{k}}(t)$, we use the change of variable $y = 2^{j_n}(s-t)$ and the fact that, for all $j \in \mathbb{Z}$, the function $y \mapsto (2+y)^{-L/(d-1)} \sqrt{\log(2 + |j| + y)}$ is decreasing on $\mathbb{R}_+$. By this way we get that

$$\begin{aligned}
I^1_{\mathbf{j},\mathbf{k}}(t) &= 2^{-j_n}\int_0^{+\infty}\sum_{k_n\le 2^{j_n}t-2^{j_n(1-a)}}\prod_{\substack{\ell=1\\ \ell\neq n}}^{d}\frac{\sqrt{\log(3+|j_\ell|+y+2^{j_n}t-k_n)}}{(3+y+2^{j_n}t-k_n)^{L/(d-1)}}\,dy\\
&\le 2^{-j_n}\int_0^{+\infty}\sum_{p=0}^{+\infty}\prod_{\substack{\ell=1\\ \ell\neq n}}^{d}\frac{\sqrt{\log(3+|j_\ell|+y+2^{j_n(1-a)}+p)}}{(3+y+2^{j_n(1-a)}+p)^{L/(d-1)}}\,dy\\
&\le 2^{-j_n}\int_0^{+\infty}\left(\int_0^{+\infty}\prod_{\substack{\ell=1\\ \ell\neq n}}^{d}\frac{\sqrt{\log(2+|j_\ell|+y+2^{j_n(1-a)}+z)}}{(2+y+2^{j_n(1-a)}+z)^{L/(d-1)}}\,dz\right)dy\\
&\le 2^{-j_n}\int_0^{+\infty}\left(\int_0^{+\infty}\frac{\prod_{\ell=1;\,\ell\neq n}^{d}\sqrt{\log(2+|j_\ell|+y+2^{j_n(1-a)}+z)}}{(2+y+2^{j_n(1-a)}+z)^{L}}\,dz\right)dy.
\end{aligned}\tag{4.32}$$

Now, we are going to estimate the integral over z in (4.32). To this end, we will make an integration by parts. Notice that there is no restriction to assume that J is large enough so that the inequality $d-1\le\log(2+2^{J(1-a)})$ holds. Then, it follows from the inequality $j_n\ge J$ that, for all $(y,z)\in\mathbb{R}^2_+$ and for every $j_\ell\in\mathbb{Z}$ (with $l\neq n$), one has

$$\frac{d-1}{\sqrt{\log(2+|j_\ell|+y+2^{j_n(1-a)}+z)})}\le\sqrt{\log(2+|j_\ell|+y+2^{j_n(1-a)}+z)}.$$

Thus, denoting by D_z the partial derivative operator with respect of the variable z, one can derive from the latter inequality that

$$\begin{aligned}
&D_z\prod_{\ell=1;\,\ell\neq n}^{d}\sqrt{\log(2+|j_\ell|+y+2^{j_n(1-a)}+z)}\\
&=\sum_{\ell=1;\,\ell\neq n}^{d}\frac{\prod_{i=1;\,i\neq n,\ell}^{d}\sqrt{\log(2+|j_i|+y+2^{j_i(1-a)}+z)}}{2\sqrt{\log(2+|j_\ell|+y+2^{j_n(1-a)}+z)}(2+|j_\ell|+y+2^{j_n(1-a)}+z)}\\
&\le\frac{\prod_{\ell=1;\,\ell\neq n}^{d}\sqrt{\log(2+|j_\ell|+y+2^{j_n(1-a)}+z)}}{2(2+y+2^{j_n(1-a)}+z)}
\end{aligned}$$

and consequently that

$$\begin{aligned}
&\int_0^{+\infty}\frac{\prod_{\ell=1;\,\ell\neq n}^{d}\sqrt{\log(2+|j_\ell|+y+2^{j_n(1-a)}+z)}}{(2+y+2^{j_n(1-a)}+z)^{L}}\,dz\\
&\le 2\times\frac{\prod_{\ell=1;\,\ell\neq n}^{d}\sqrt{\log(2+|j_\ell|+y+2^{j_n(1-a)})}}{(2+y+2^{j_n(1-a)})^{L-1}}.
\end{aligned}$$

This leads to

$$\begin{aligned}
I^1_{\mathbf{j},\mathbf{k}}(t) &\le 2^{1-j_n}\int_0^{+\infty}\frac{\prod_{\ell=1;\,\ell\neq n}^{d}\sqrt{\log(2+|j_\ell|+y+2^{j_n(1-a)})}}{(2+y+2^{j_n(1-a)})^{L-1}}\,dy\\
&\le 2^{2-j_n}\times\frac{\prod_{\ell=1;\,\ell\neq n}^{d}\sqrt{\log(2+|j_\ell|+2^{j_n(1-a)})}}{(2+2^{j_n(1-a)})^{L-2}},
\end{aligned}\tag{4.33}$$

where the last inequality is obtained through an integration by parts and the same arguments as before. Observe that, by using the definition of $I^2_{\mathbf{j},\mathbf{k}}$ one can show, as we already did it for deriving (4.33), that

$$I^2_{\mathbf{j},\mathbf{k}} \le 2^{2-j_n} \times \frac{\prod_{\ell=1;\,\ell\neq n}^{d} \sqrt{\log(2+|j_\ell|+2^{j_n(1-a)})}}{\big(2+2^{j_n(1-a)}\big)^{L-2}}. \tag{4.34}$$

Next, it follows from the definition (4.4), the inequalities (4.31), (4.33) and (4.34), the triangle inequality, the inequalities (B.1) and (B.2) and the assumptions (1.11) that

$$\begin{aligned}
\mathcal{H}^1_{n,J} &:= \sum_{\mathbf{j}\in\aleph_{n,J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \times \ldots \\
&\qquad \ldots \times \sup_{t\in[0,T]} \left\{ \sum_{k_n\in D^1_{j_n}(t)} \sum_{\substack{k_\ell\in\mathbb{Z}\\ \ell\in[\![1,d]\!]\setminus\{n\}}} |\varepsilon_{\mathbf{j},\mathbf{k}}|\, |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)-F_{\mathbf{j},\mathbf{k}}| \right\} \\
&\le C_2 \sum_{j_n=J}^{+\infty} (j_n+1)^{\frac{d}{2}}\, 2^{-j_n((L-2)(1-a)+h_n)} \times \ldots \\
&\qquad \ldots \times \prod_{\ell=1;\,\ell\neq n}^{d} \left(\sum_{j_\ell=-\infty}^{j_n} 2^{j_\ell(1-h_\ell)} \sqrt{\log(2+|j_\ell|+2^{j_n(1-a)})} \right) \\
&\le C_2 \sum_{j_n=J}^{+\infty} (j_n+1)^{\frac{d}{2}}\, 2^{-j_n((L-2)(1-a)+h_1+\cdots+h_d-d+1)} \times \ldots \\
&\qquad \ldots \times \prod_{\ell=1;\,\ell\neq n}^{d} \left(\sum_{p=0}^{+\infty} 2^{-p(1-h_\ell)} \sqrt{\log(2+j_n+2^{j_n(1-a)}+p)} \right) \\
&\le C_3 \sum_{j_n=J}^{+\infty} j_n^{d-1}\, 2^{-j_n((L-2)(1-a)+h_1+\cdots+h_d-d+1)} \\
&\le C_4 J^{d-1}\, 2^{-J((L-2)(1-a)+h_1+\cdots+h_d-d+1)}
\end{aligned}$$

where C_2, C_3 and C_4 are positive almost surely finite random variables not depending on n, t and J large enough. □

4.3 Rate of convergence to zero of $\mathcal{M}_{n,J}$

The goal of this subsection is to show, by making use of Borel-Cantelli Lemma, that, on some event Ω^{**} of probability 1, when J goes to $+\infty$, the random variable $\mathcal{M}_{n,J}$, defined in (4.20), converges to zero at the rate $J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}$ targeted in Theorem 2.12.

We start by giving a useful upper bound for $(F_{\mathbf{j},\mathbf{k}})^2$, the square of deterministic integral $F_{\mathbf{j},\mathbf{k}}$ defined in (4.19).

Lemma 4.7. *There exists a deterministic constant $c_\psi > 0$ such that for all*

$(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^d)^2$, *with* $j_n = \max_{\ell\in[\![1,d]\!]} j_\ell$, *the following inequality holds:*

$$(F_{\mathbf{j},\mathbf{k}})^2 \le c_\psi\, 2^{-j_n} \int_{\mathbb{R}} \big|\psi_{h_n}(2^{j_n}s-k_n)\big| \prod_{\substack{\ell=1\\ \ell\neq n}}^{d} \big|\psi_{h_\ell}(2^{j_\ell}s-k_\ell)\big|^2\, ds.$$

Proof. Let $\mathbb{P}_{j_n,k_n}$ be the absolutely continuous probability measure on the Borel σ-algebra of $\mathbb{R}$ whose density is the function $s\mapsto 2^{j_n}\|\psi_{h_n}\|^{-1}_{L^1(\mathbb{R})}|\psi_{h_n}(2^{j_n}s-k_n)|$. We clearly have that

$$\begin{aligned}(F_{\mathbf{j},\mathbf{k}})^2 &\le \left(\int_{\mathbb{R}} \prod_{\ell=1}^{d} \big|\psi_{h_\ell}(2^{j_\ell}s-k_\ell)\big|\, ds\right)^2\\ &= 2^{-2j_n}\|\psi_{h_n}\|^2_{L^1(\mathbb{R})}\left(\int_{\mathbb{R}} \prod_{\substack{\ell=1\\ \ell\neq n}}^{d} \big|\psi_{h_\ell}(2^{j_\ell}s-k_\ell)\big|\, \mathrm{d}\mathbb{P}_{j_n,k_n}(s)\right)^2.\end{aligned}$$

Then, it results from Jensen's inequality that

$$\begin{aligned}(F_{\mathbf{j},\mathbf{k}})^2 &\le 2^{-2j_n}\|\psi_{h_n}\|^2_{L^1(\mathbb{R})}\int_{\mathbb{R}} \prod_{\substack{\ell=1\\ \ell\neq n}}^{d} \big|\psi_{h_\ell}(2^{j_\ell}s-k_\ell)\big|^2\, \mathrm{d}\mathbb{P}_{j_n,k_n}(s)\\ &= 2^{-j_n}\|\psi_{h_n}\|_{L^1(\mathbb{R})}\int_{\mathbb{R}} \big|\psi_{h_n}(2^{j_n}s-k_n)\big| \prod_{\substack{\ell=1\\ \ell\neq n}}^{d} \big|\psi_{h_\ell}(2^{j_\ell}s-k_\ell)\big|^2\, ds.\end{aligned}$$

Then, setting $c_\psi := \max_{\ell\in[\![1,d]\!]}\|\psi_{h_\ell}\|_{L^1(\mathbb{R})}$ one obtains the lemma. □

In order to bound in a convenient way the random variables $\widetilde{\mathcal{M}}_{n,\mathbf{j}}(t)$, defined in (4.21), we will combine some Borel-Cantelli arguments with the following fundamental result [13, Theorem 6.7].

Lemma 4.8. *For any fixed integer* $d\ge 1$, *there exists a (strictly) positive finite universal deterministic constant* c_d *such that, for every random variable* X *belonging to the Wiener chaos of order* d *and for each real number* $y\ge 2$, *one has*

$$\mathbb{P}(X \ge y\|X\|_{L^2(\Omega)}) \le \exp\big(-c_d\, y^{2/d}\big).$$

We will apply Lemma 4.8 to the random variable $\widetilde{\mathcal{M}}_{n,\mathbf{j}}(t)$. This is why it is useful to control its $L^2(\Omega)$ norm uniformly in $t\in[0,T]$.

Lemma 4.9. *There exists a finite constant* $c>0$, *depending on* T, *such that, for all* $n\in[\![1,d]\!]$ *and* $\mathbf{j}\in\aleph_{n,1}$, *we have*

$$\sup_{t\in[0,T]}\|\widetilde{\mathcal{M}}_{n,\mathbf{j}}(t)\|_{L^2(\Omega)} \le c\, 2^{-j_n/2}. \tag{4.35}$$

Proof. Throughout the proof $t\in[0,T]$ and $\mathbf{j}\in\aleph_{n,J}$ (recall that j_n denotes the nth coordinate of $\mathbf{j}$) are arbitrary and fixed. The equivalence relation $\sim$ on the set $\urcorner_n^{j_n}(t)$ is defined as:

$$\forall\,(\mathbf{k},\mathbf{k}')\in \urcorner_n^{j_n}(t)\times \urcorner_n^{j_n}(t),\quad \mathbf{k}\sim\mathbf{k}' \Longleftrightarrow \varepsilon_{\mathbf{j},\mathbf{k}} = \varepsilon_{\mathbf{j},\mathbf{k}'}.$$

Let us emphasize that, we know from Remark 2.18 and Proposition 2.19 that

$$\forall\,(\mathbf{k},\mathbf{k}') \in \daleth_n^{j_n}(t) \times \daleth_n^{j_n}(t), \quad \mathbf{k} \sim \mathbf{k}' \Longleftrightarrow \mathbb{E}[\varepsilon_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{j},\mathbf{k}'}] \neq 0. \tag{4.36}$$

Since $\daleth_n^{j_n}(t)$ is a finite set, the equivalence classes for the equivalence relation $\sim$ are in finite number denoted by M. Let us then denote them by $\daleth_{n,1}^{j_n}(t), \ldots, \daleth_{n,M}^{j_n}(t)$. Then using a well-known result on equivalence relations, the set $\daleth_n^{j_n}(t)$ can be expressed as:

$$\daleth_n^{j_n}(t) = \bigcup_{i=1}^{M} \daleth_{n,i}^{j_n}(t) \quad \text{(disjoint union)}. \tag{4.37}$$

Also we mention that, we know from Remark 2.18 that, for each $i \in [\![1, M]\!]$, we have

$$\operatorname{Card}(\daleth_{n,i}^{j_n}(t)) \leq d\,! \tag{4.38}$$

Next, observe that it follows from (4.36) that

$$\operatorname{Cov}\Bigg(\sum_{\mathbf{k}\in\daleth_{n,i'}^{j_n}(t)} F_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{j},\mathbf{k}}\,, \sum_{\mathbf{k}\in\daleth_{n,i''}^{j_n}(t)} F_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{j},\mathbf{k}} \Bigg) = 0, \quad \text{when } i' \neq i''. \tag{4.39}$$

Then, one can derive from (4.21), (4.37), (4.39), Proposition 2.19 and the triangular inequality, that

$$\begin{aligned}
\|\widetilde{\mathcal{M}}_{n,\mathbf{j}}(t)\|_{L^2(\Omega)}^2 &= \Bigg\| \sum_{i=1}^{M} \sum_{\mathbf{k}\in\daleth_{n,i}^{j_n}(t)} F_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{j},\mathbf{k}} \Bigg\|_{L^2(\Omega)}^2 = \sum_{i=1}^{M} \Bigg\| \sum_{\mathbf{k}\in\daleth_{n,i}^{j_n}(t)} F_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{j},\mathbf{k}} \Bigg\|_{L^2(\Omega)}^2 \\
&= \sum_{i=1}^{M} \sum_{\mathbf{k}\in\daleth_{n,i}^{j_n}(t)} \sum_{\mathbf{k}'\in\daleth_{n,i}^{j_n}(t)} F_{\mathbf{j},\mathbf{k}}F_{\mathbf{j},\mathbf{k}'}\mathbb{E}\left[\varepsilon_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{j},\mathbf{k}'}\right] \\
&\leq d\,!\sum_{i=1}^{M} \sum_{\mathbf{k}\in\daleth_{n,i}^{j_n}(t)} \sum_{\mathbf{k}'\in\daleth_{n,i}^{j_n}(t)} \big|F_{\mathbf{j},\mathbf{k}}\big|\big|F_{\mathbf{j},\mathbf{k}'}\big| \\
&= d\,!\sum_{i=1}^{M} \Bigg(\sum_{\mathbf{k}\in\daleth_{n,i}^{j_n}(t)} \big|F_{\mathbf{j},\mathbf{k}}\big| \Bigg)^2 .
\end{aligned}$$

Then using the convexity of the function $x \mapsto x^2$, the inequality (4.38) and the equality (4.37), we get that

$$\|\widetilde{\mathcal{M}}_{n,\mathbf{j}}(t)\|_{L^2(\Omega)}^2 \leq (d\,!)^2 \sum_{i=1}^{M} \sum_{\mathbf{k}\in\daleth_{n,i}^{j_n}(t)} F_{\mathbf{j},\mathbf{k}}^2 = (d\,!)^2 \sum_{\mathbf{k}\in\daleth_n^{j_n}(t)} F_{\mathbf{j},\mathbf{k}}^2\,. \tag{4.40}$$

Moreover, putting together Lemma 4.7, (4.18), the fast decay property (1.4) with $L > 1$, and the inequality $\sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \big(3 + |x-k|\big)^{-L} < \infty$ we get that

$$
\begin{aligned}
\sum_{\mathbf{k}\in\daleth_n^{j_n}(t)} F_{\mathbf{j},\mathbf{k}}^2 &\leq c_\psi\, 2^{-j_n} \sum_{k_n\in D^1_{j_n}(t)} \int_{\mathbb{R}} \big|\psi_{h_n}(2^{j_n}s - k_n)\big| \prod_{\substack{\ell=1\\ \ell\neq n}}^{d} \left(\sum_{k_\ell\in\mathbb{Z}} \big|\psi_{h_\ell}(2^{j_\ell}s - k_\ell)\big|^2 \right) ds \\
&\leq c_1 2^{-j_n} \sum_{k_n\in D^1_{j_n}(t)} \int_{\mathbb{R}} \big|\psi_{h_n}(2^{j_n}s - k_n)\big| \prod_{\substack{\ell=1\\ \ell\neq n}}^{d} \left(\sum_{k_\ell\in\mathbb{Z}} \big(3 + |2^{j_\ell}s - k_\ell|\big)^{-L} \right) ds \\
&\leq c_2 2^{-j_n} \sum_{k_n\in D^1_{j_n}(t)} \int_{\mathbb{R}} \big|\psi_{h_n}(2^{j_n}s - k_n)\big|\, ds \\
&= c_2 \|\psi_{h_n}\|_{L^1(\mathbb{R})}\, 2^{-2j_n}\, \mathrm{Card}(D^1_{j_n}(t))\,, \qquad (4.41)
\end{aligned}
$$

where c_1 and c_2 are finite positive constants not depending on n, t and $\mathbf{j}$. Then, (4.40), (4.41), and (4.28) entail that (4.35) is satisfied. □

The following remark shows that the supremum in (4.20) is in fact a supremum on a well-chosen finite set.

Remark 4.10. For each fixed $j \in \mathbb{N}$ and $t \in \mathbb{R}_+$, we denote by $m_{j,t}$ the integer part of the real number $2^j t - 2^{j(1-a)}$, that is $m_{j,t} := \lfloor 2^j t - 2^{j(1-a)} \rfloor$. Thus, in view of the definition (4.10) of the set $D^1_j(t)$, it turns out that

$$
D^1_j(t) = \begin{cases} \emptyset & \text{if } t \in [0, 2^{1-ja}) \\ D^1_j(m_{j,t}2^{-j} + 2^{-ja}) & \text{if } t \in [2^{1-ja}, \infty). \end{cases} \qquad (4.42)
$$

Then, we can derive from (4.42) that, for all $n \in [\![1, d]\!]$ and $\mathbf{j} \in \aleph_{n,1}$,

$$
\sup_{t\in[0,T]} |\widetilde{\mathcal{M}}_{n,\mathbf{j}}(t)| = \sup_{m\in\mathcal{I}_{j_n}} |\widetilde{\mathcal{M}}_{n,\mathbf{j}}(m2^{-j_n} + 2^{-j_n a})|, \qquad (4.43)
$$

where the arbitrary real number $T > 2$ is fixed and $\mathcal{I}_j$ stands for the finite set

$$
\mathcal{I}_j := \mathbb{N} \cap (2^{j(1-a)} - 1, 2^j T - 2^{j(1-a)}]. \qquad (4.44)
$$

Lemma 4.11. *Let $T > 2$ be a fixed real number. There exist Ω^{**} an event of probability 1 and a positive almost surely finite random variable C^{**} such that, for all $n \in [\![1, d]\!]$ and $\mathbf{j} \in \aleph_{n,1}$, on Ω^{**}, we have*

$$
\sup_{t\in[0,T]} |\widetilde{\mathcal{M}}_{n,\mathbf{j}}(t)| \leq C^{**} 2^{-j_n/2} \log(3 + |\mathbf{j}| + 2^{j_n}T)^{\frac{d}{2}}. \qquad (4.45)
$$

Proof. Let us first show that if $(X_j)_{j\in\mathbb{N}}$ is an arbitrary sequence of random variables in the Wiener chaos of order d, there exist Ω_1, an event of probability 1, and a positive almost surely finite random variable C_1 such that, for all $j \in \mathbb{N}$, on Ω_1, we have

$$
|X_j| \leq C_1 \log(3 + j)^{\frac{d}{2}} \|X_j\|_{L^2(\Omega)}. \qquad (4.46)
$$

Let $\kappa \geq 2$ be a constant which will be precisely defined later. Applying, for any $j \in \mathbb{N}$, Lemma 4.8 to the random variable X_j, we get that

$$
\mathbb{P}\left(|X_j| \geq \kappa \log(3 + j)^{\frac{d}{2}} \|X_j\|_{L^2(\Omega)} \right) \leq \exp\big(- c_d \kappa^{\frac{2}{d}} \log(3 + j) \big),
$$

where c_d is the same universal positive constant as in Lemma 4.8. Thus, assuming that the constant κ satisfies $\kappa > c_d^{-\frac{d}{2}}$, it turns out that the series

$$\sum_{j\in\mathbb{N}} \mathbb{P}\left(|X_j| \geq \kappa \log(3+j)^{\frac{d}{2}} \|X_j\|_{L^2(\Omega)}\right)$$

is convergent; then, the existence of Ω_1 and C_1 follows from Borel-Cantelli Lemma. Next, notice that, thanks to an indexation argument, the result obtained in (4.46) can be applied to the sequence of random variables

$$\left\{\widetilde{\mathcal{M}}_{n,\mathbf{j}}(m2^{-j_n} + 2^{-j_n a}) \,:\, n \in [\![1,d]\!], \mathbf{j} \in \aleph_{n,1}, m \in \mathcal{I}_{j_n}\right\}.$$

By this way, we can show that there are Ω^{**} an event of probability 1 and a positive almost surely finite random variable C_2 (depending on T) such that, on Ω^{**}, we have, for all $n \in [\![1,d]\!]$, $\mathbf{j} \in \aleph_{n,1}$ and $m \in \mathcal{I}_{j_n}$, that

$$|\widetilde{\mathcal{M}}_{n,\mathbf{j}}(m2^{-j_n} + 2^{-j_n a})| \leq C_2 \log(3 + |\mathbf{j}| + m)^{\frac{d}{2}} \|\widetilde{\mathcal{M}}_{n,\mathbf{j}}(m2^{-j_n} + 2^{-j_n a})\|_{L^2(\Omega)}. \tag{4.47}$$

Then, putting together (4.47), (4.35), (4.44) and (4.43), we obtain (4.45). □

Lemma 4.12. *Let $T > 2$ be a fixed real number. There exists a positive almost surely finite random variable C such that, for all $n \in [\![1,d]\!]$ and $J \in \mathbb{N}$, on Ω^{**} (see Lemma 4.11), the random variable $\mathcal{M}_{n,J}$ (see (4.20)) is bounded from above by $CJ^{\frac{d}{2}}2^{-J(h_1+\dots+h_d-d+\frac{1}{2})}$.*

Proof. Let us fix $J \in \mathbb{N}$, using (4.20), (4.4), (4.45), (B.1), the triangular inequality, (B.2) and (1.11), we obtain that

$$\begin{aligned}
\mathcal{M}_{n,J} &\leq C_0 \sum_{j_n=J}^{+\infty} 2^{j_n(\frac{1}{2}-h_n)} \log(3 + d\,j_n + 2^{j_n}T)^{\frac{d}{2}} \times \dots \\
&\dots \times \prod_{\substack{\ell=1\\ \ell\neq n}}^{d} \left(\sum_{j_\ell=-\infty}^{j_n} 2^{j_\ell(1-h_\ell)} \log(3 + |j_n - j_\ell|)^{\frac{d}{2}}\right) \\
&\leq C_1 \sum_{j_n=J}^{+\infty} 2^{-j_n(h_1+\dots+h_d-d+\frac{1}{2})} \log(3 + d\,j_n + 2^{j_n}T)^{\frac{d}{2}} \times \dots \\
&\dots \times \prod_{\substack{\ell=1\\ \ell\neq n}}^{d} \left(\sum_{p=0}^{+\infty} 2^{-p(1-h_\ell)} \log(3 + p + 2^{j_n}T)^{\frac{d}{2}}\right) \\
&\leq C_2 \sum_{j_n=J}^{+\infty} 2^{-j_n(h_1+\dots+h_d-d+\frac{1}{2})} j_n^{\frac{d}{2}} \\
&\leq C_3 2^{-J(h_1+\dots+h_d-d+\frac{1}{2})} J^{\frac{d}{2}},
\end{aligned}$$

where C_0, C_1, C_2 and C_3 are positive almost surely finite random variables not depending on n and J. □

4.4 End of the proof of Theorem 2.12

We are now in position to complete the proof of Theorem 2.12.

End of the Proof of Theorem 2.12. Without loss of generality, one can assume that the compact interval I in the statement of the theorem is of the form $I = [0, T]$ for a fixed real number $T > 2$. Let $\widetilde{\Omega}$ be the event of probability 1 defined as: $\widetilde{\Omega} := \Omega^* \cap \Omega^{**}$, where Ω^* and Ω^{**} are as in Lemmata 2.16 and 4.11.

First, we will show that, for each fixed $\omega \in \widetilde{\Omega}$ and $\mathbf{j} \in \mathbb{Z}^d$, the series of continuous function $\sum_{\mathbf{k}\in\mathbb{Z}^d} \mathcal{A}_{\mathbf{j},\mathbf{k}}\varepsilon_{\mathbf{j},\mathbf{k}}(\omega)$ is normally convergent with respect to the uniform norm $\|\cdot\|_{I,\infty}$. Using the inequality (4.3), the bound (2.31) and the triangular inequality, one gets, for some positive finite random variable C_1, depending on T and $\mathbf{j} \in \mathbb{Z}^d$, that

$$\begin{aligned}
&\sum_{\mathbf{k}\in\mathbb{Z}^d} \|\mathcal{A}_{\mathbf{j},\mathbf{k}}\|_{I,\infty} |\varepsilon_{\mathbf{j},\mathbf{k}}(\omega)| \\
&\le C_1(\omega) \sum_{\mathbf{k}\in\mathbb{Z}^d} \int_0^T \prod_{\ell=1}^d \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(1+2^{j_\ell}T + |2^{j_\ell}s - k_\ell|)^2}\, ds \\
&\le C_1(\omega) \sum_{\mathbf{k}\in\mathbb{Z}^d} \int_0^T \prod_{\ell=1}^d \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(1+2^{j_\ell}T + |k_\ell| - |2^{j_\ell}s|)^2}\, ds \\
&\le C_1(\omega) T \sum_{\mathbf{k}\in\mathbb{Z}^d} \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(1+|k_\ell|)^2} < \infty,
\end{aligned}$$

which shows that the normal convergence holds.

Next, for each $\mathbf{j} \in \mathbb{Z}^d$, we denote by $\{X_{\mathbf{j}}(t)\}_{t\in I}$ the stochastic process with continuous paths vanishing outside of $\widetilde{\Omega}$ and defined on $I \times \widetilde{\Omega}$ as

$$X_{\mathbf{j}}(t,\omega) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \mathcal{A}_{\mathbf{j},\mathbf{k}}(t)\varepsilon_{\mathbf{j},\mathbf{k}}(\omega). \tag{4.48}$$

Observe that in order to complete the proof of the theorem, it is enough to show that there exists a positive finite random variable $\widetilde{C}$ such that, for every $J \in \mathbb{N}$, the following inequality holds on $\widetilde{\Omega}$:

$$\sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \\ \max_{\ell\in[\![1,d]\!]} j_\ell \ge J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \|X_{\mathbf{j}}\|_{I,\infty} \le \widetilde{C} J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+\frac{1}{2})}. \tag{4.49}$$

Indeed, assuming that (4.49) is true, then it clearly entails that, for all fixed $J \in \mathbb{N}$ and every $\omega \in \widetilde{\Omega}$, one has

$$\sum_{\substack{\mathbf{j}\in\mathbb{Z}^d \\ \max_{\ell\in[\![1,d]\!]} j_\ell \ge J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \|X_{\mathbf{j}}(\cdot,\omega)\|_{I,\infty} <, \infty,$$

which means that the series of continuous function

$$X_J(\cdot,\omega) := \sum_{\substack{\mathbf{j}\in\mathbb{Z}^d \\ \max_{\ell\in[\![1,d]\!]} j_\ell \ge J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} X_{\mathbf{j}}(\cdot,\omega) \tag{4.50}$$

is normally convergent with respect to the uniform norm $\|\cdot\|_{I,\infty}$ and thus $X_J(\cdot,\omega)$ is a continuous function on I. Then, we denote by $\{X_J(t)\}_{t\in I}$ the stochastic process with continuous paths vanishing outside of $\widetilde{\Omega}$ and defined on $I\times\widetilde{\Omega}$ by (4.50). Thus, (4.49) and the triangular inequality imply that, for all $J\in\mathbb{N}$, the following inequality holds on $\widetilde{\Omega}$:

$$\|X_J\|_{I,\infty} \le \widetilde{C} J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+\frac{1}{2})}. \tag{4.51}$$

On another hand, we know from the equality (2.8) that, for all fixed $J\in\mathbb{N}$ and $t\in I$, the random series

$$\sum_{\substack{\mathbf{j}\in\mathbb{Z}^d\\ \max_{\ell\in[\![1,d]\!]} j_\ell\ge J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} X_{\mathbf{j}}(t)$$

converges to $X^{(d,\perp)}_{\mathbf{h},J}(t) := X^{(d)}_{\mathbf{h}}(t) - X^{(d)}_{\mathbf{h},J}(t)$ in $L^2(\Omega)$. Combining this fact with (4.50) one concludes that, for all $t\in I$, almost surely,

$$X_J(t) = X^{(d)}_{\mathbf{h}}(t) - X^{(d)}_{\mathbf{h},J}(t).$$

This latter equality and the fact that the two stochastic processes $\{X_J(t)\}_{t\in I}$ and $\{X^{(d)}_{\mathbf{h}}(t) - X^{(d)}_{\mathbf{h},J}(t)\}_{t\in I}$ have continuous paths imply that these two processes are indistinguishable. Thus, the inequality (4.51) is nothing else than the inequality (2.25).

It remains us to show that (4.49) holds. In fact, it results from Lemmata 4.2, 4.4, 4.5, 4.6, 4.12 and the inequality:

$$\sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2\\ \max_{\ell\in[\![1,d]\!]} j_\ell\ge J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \|X_{\mathbf{j}}\|_{I,\infty} \le \sum_{n=1}^{d}\left(\mathcal{M}_{n,J} + \sum_{m=0}^{3}\mathcal{H}^m_{n,J}\right),$$

which is obtained by using (4.48), the triangular inequality, standard computations, and the definitions of the random variables $\mathcal{M}_{n,J}$, and $\mathcal{H}^m_{n,J}$ with $m\in[\![0,3]\!]$. □

5 Proof of Theorem 2.14

The real number $T>2$ is arbitrary and fixed. Let $\widetilde{\Omega}$ be the same event of probability 1 as in the proof of Theorem 2.12; recall that it is defined as $\widetilde{\Omega} := \Omega^*\cap\Omega^{**}$, where the two events Ω^* and Ω^{**} of probability 1 are as in Lemmata 2.16 and 4.11. Next, observe that for proving Theorem 2.14 it is enough to show that there exists a positive finite random variable C such that, on $\widetilde{\Omega}$, we have, for all $N,P\in\mathbb{N}$,

$$\big\|\widetilde{X}^{(d)}_{\mathbf{h},N+P} - \widetilde{X}^{(d)}_{\mathbf{h},N}\big\|_{[0,T],\infty} \le C N^{\frac{d}{2}} 2^{-N(h_1+\cdots+h_d-d+1/2)}, \tag{5.1}$$

where, for all fixed $\omega\in\Omega$, the continuous function $\widetilde{X}^{(d)}_{\mathbf{h},N}(\cdot,\omega)$ is defined through (2.27). Indeed, assuming that (5.1) is true, then it turns out that, for each fixed $\omega\in\widetilde{\Omega}$, the sequence of functions $\big(\widetilde{X}^{(d)}_{\mathbf{h},N}(\cdot,\omega)\big)_{N\in\mathbb{N}}$ is a Cauchy sequence in the Banach space of the real-valued continuous functions over $[0,T]$, equipped with the uniform norm $\|\cdot\|_{[0,T],\infty}$. Therefore, it converges, for this norm, to a continuous function over $[0,T]$ denoted by $\widetilde{X}_{\mathbf{h}}^{(d)}(\cdot,\omega)$. On another hand, when $\omega\in\Omega\setminus\widetilde{\Omega}$ we set $\widetilde{X}_{\mathbf{h}}^{(d)}(t,\omega)=0$, for all $t\in[0,T]$. Next, observe that, in view of the previous definition of the stochastic process $\{\widetilde{X}_{\mathbf{h}}^{(d)}(t)\}_{t\in[0,T]}$ we have, for all $t\in[0,T]$, almost surely,

$$X_{\mathbf{h}}^{(d)}(t)=\widetilde{X}_{\mathbf{h}}^{(d)}(t), \tag{5.2}$$

since we know from (2.27) and (2.26), that, for each fixed $t\in\mathbb{R}_+$ (and in particular for $t\in[0,T]$), the sequence of random variables $(\widetilde{X}_{\mathbf{h},N}^{(d)}(t))_{N\in\mathbb{N}}$ converges to $X_{\mathbf{h}}^{(d)}(t)$ in $L^2(\Omega)$. Next, using the fact that the two stochastic processes $\{X_{\mathbf{h}}^{(d)}(t)\}_{t\in[0,T]}$ and $\{\widetilde{X}_{\mathbf{h}}^{(d)}(t)\}_{t\in[0,T]}$ have continuous paths, we can derive from the almost sure equality (5.2) that these two processes are indistinguishable. Thus, letting P in (5.1) tends to $+\infty$, we obtain (2.28).

From now on, we focus on the proof of the inequality (5.1). Let us explain its main lines. Observe that we know from Definition 2.13 that the two sequences of subsets of $\mathbb{Z}^d$ $(\mathcal{S}_N^+)_{N\in\mathbb{N}}$ and $(\mathcal{S}_N^-)_{N\in\mathbb{N}}$, which are related to $\big(\widetilde{X}_{\mathbf{h},N}^{(d)}(\cdot,\omega)\big)_{N\in\mathbb{N}}$ (see (2.27)), are increasing in the sense of the inclusion, and one has, for all $N',N''\in\mathbb{N}$, that $\mathcal{S}_{N'}^+\cap\mathcal{S}_{N''}^-=\emptyset$. Thus, one can derive from (2.27) that, for every $N\in\mathbb{N}$, $P\in\mathbb{N}$ and $t\in[0,T]$,

$$\widetilde{X}_{\mathbf{h},N+P}^{(d)}(t)-\widetilde{X}_{\mathbf{h},N}^{(d)}(t)=\sum_{(\mathbf{j},\mathbf{k})\in\mathcal{S}_{N+P}^+\setminus\mathcal{S}_N^+}\varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{K}_{\mathbf{j},\mathbf{k}}^{(d,\mathbf{h})}(t)+\sum_{(\mathbf{j},\mathbf{k})\in\mathcal{S}_{N+P}^-\setminus\mathcal{S}_N^-}\varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{K}_{\mathbf{j},\mathbf{k}}^{(d,\mathbf{h})}(t). \tag{5.3}$$

The first step of the proof of (5.1) consists in showing that, on $\widetilde{\Omega}$, one has, for some positive finite random variable denoted by C^+ and for all $N,P\in\mathbb{N}$,

$$\Big\|\sum_{(\mathbf{j},\mathbf{k})\in\mathcal{S}_{N+P}^+\setminus\mathcal{S}_N^+}\varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{K}_{\mathbf{j},\mathbf{k}}^{(d,\mathbf{h})}\Big\|_{[0,T],\infty}\le C^+N^{\frac{d}{2}}2^{-N(h_1+\cdots+h_d-d+1/2)}. \tag{5.4}$$

Observe that, in view of Definition 2.13, the the set $\mathcal{S}_{N+P}^+\setminus\mathcal{S}_N^+$ can be expressed as

$$\mathcal{S}_{N+P}^+\setminus\mathcal{S}_N^+=(\aleph_{N,P}^0\times\beth_{N,P,T}^{(i)})\cup(\aleph_{N,P}^1\times\beth_{N,P,T}^{(ii)})\cup(\aleph_{N,P}^2\times\beth_{N,P,T}^{(i)}), \tag{5.5}$$

where

$$\aleph_{N,P}^0:=\Big\{\mathbf{j}\in\mathbb{Z}^d\,:\,-2^{(N+P)b}\le\min_{\ell\in[\![1,d]\!]}j_\ell\ \text{ and }\ N\le\max_{\ell\in[\![1,d]\!]}j_\ell<N+P\Big\}, \tag{5.6}$$

$$\aleph_{N,P}^1:=\Big\{\mathbf{j}\in\mathbb{Z}^d\,:\,-2^{Nb}\le\min_{\ell\in[\![1,d]\!]}j_\ell\ \text{ and }\ 0\le\max_{\ell\in[\![1,d]\!]}j_\ell<N\Big\}, \tag{5.7}$$

$$\aleph_{N,P}^2:=\Big\{\mathbf{j}\in\mathbb{Z}^d\,:\,-2^{(N+P)b}\le\min_{\ell\in[\![1,d]\!]}j_\ell<-2^{Nb}\ \text{ and }\ 0\le\max_{\ell\in[\![1,d]\!]}j_\ell<N\Big\} \tag{5.8}$$

and

$$\beth_{N,P,T}^{(i)}:=\Big\{\mathbf{k}\in\mathbb{Z}^d\,:\,\max_{\ell\in[\![1,d]\!]}|k_\ell|\le 2^{N+P+1}T\Big\}, \tag{5.9}$$

$$\beth_{N,P,T}^{(ii)}:=\Big\{\mathbf{k}\in\mathbb{Z}^d\,:\,2^{N+1}T<\max_{\ell\in[\![1,d]\!]}|k_\ell|\le 2^{N+P+1}T\Big\}. \tag{5.10}$$

Then, it results from (5.5), the triangular inequality and (4.1) that

$$\Big\| \sum_{(\mathbf{j},\mathbf{k})\in\mathcal{S}^+_{N+P}\setminus\mathcal{S}^+_N} \varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{K}^{(d,\mathbf{h})}_{\mathbf{j},\mathbf{k}} \Big\|_{[0,T],\infty} \le \sum_{\ell=0}^{2} \nabla^{\ell}_{N,P}, \tag{5.11}$$

where

$$\nabla^0_{N,P} := \sum_{\mathbf{j}\in\aleph^0_{N,P}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sup_{t\in[0,T]} \Big| \sum_{\mathbf{k}\in\beth^{(i)}_{N,P,T}} \varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)\Big|,$$

$$\nabla^1_{N,P} := \sum_{\mathbf{j}\in\aleph^1_{N,P}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sup_{t\in[0,T]} \Big| \sum_{\mathbf{k}\in\beth^{(ii)}_{N,P,T}} \varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)\Big|,$$

and

$$\nabla^2_{N,P} := \sum_{\mathbf{j}\in\aleph^2_{N,P}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sup_{t\in[0,T]} \Big| \sum_{\mathbf{k}\in\beth^{(i)}_{N,P,T}} \varepsilon_{\mathbf{j},\mathbf{k}}\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)\Big|.$$

Observe that, we know from (4.10) that the inclusion $D^1_j(t) \subset \{k\in\mathbb{Z} : |k| \le 2^{N+P+1}T\}$ holds for all $t\in[0,T]$ and $1\le j\le N+P+1$. Thus, putting together (5.6), (5.9), (4.4), (4.23), (4.16), (4.17), (4.8) and (4.20), we obtain that

$$\nabla^0_{N,P} \le \sum_{n=1}^{d} \left(\mathcal{M}_{n,N} + \sum_{m=0}^{3} \mathcal{H}^m_{n,N} \right).$$

Then, it results from Lemmata 4.2, 4.4, 4.5, 4.6, and 4.12 that, on the event $\widetilde{\Omega} = \Omega^*\cap\Omega^{**}$ of probability 1, one has, for some positive finite random variable C^+_0 and for all $N,P\in\mathbb{N}$,

$$\nabla^0_{N,P} \le C^+_0 N^{\frac{d}{2}} 2^{-N(h_1+\cdots+h_d-d+1/2)}. \tag{5.12}$$

On another hand, we can derive from (5.7) and (5.10) and the triangle inequality that

$$\nabla^1_{N,P} \le \sum_{n=1}^{d} \mathcal{L}^1_{n,N}, \tag{5.13}$$

where, for all $n\in[\![1,d]\!]$,

$$\mathcal{L}^1_{n,N} := \sum_{\substack{j_\ell<N\\ \ell\in[\![1,d]\!]}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \times \ldots$$

$$\ldots \times \sum_{|k_n|>2^{N+1}T} \sum_{\substack{k_\ell\in\mathbb{Z}\\ \ell\in[\![1,d]\!]\setminus\{n\}}} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} \{|\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)|\}. \tag{5.14}$$

Moreover, we can derive from (5.8) and (5.9) and the triangle inequality that

$$\nabla^2_{N,P} \le \sum_{n=1}^{d} \mathcal{L}^2_{n,N}, \tag{5.15}$$

where, for all $n \in [\![1,d]\!]$,

$$\mathcal{L}^2_{n,N} := \sum_{j_n \le -2^{Nb}} \sum_{\substack{j_\ell < N \\ \ell \in [\![1,d]\!] \setminus \{n\}}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sum_{\mathbf{k} \in \mathbb{Z}^d} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t \in [0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| . \tag{5.16}$$

In Subsection 5.1, it is shown that, on the event Ω^* of probability 1, for some positive finite random variables C_1 and C_2, one has, for all $n \in [\![1,d]\!]$ and $N \in \mathbb{N}$, that

$$\mathcal{L}^1_{n,N} \le C_1 2^{-N(h_1+\cdots+h_d+L-d-1)} N^{\frac{d}{2}} \log(3+N)^{\frac{d}{2}} = o\big(N^{\frac{d}{2}} 2^{-N(h_1+\cdots+h_d-d+1/2)}\big), \tag{5.17}$$

the real number $L > 3/2$ being arbitrary and fixed, and that

$$\mathcal{L}^2_{n,N} \le C_2 N^{\frac{d}{2}} 2^{N\left(\sum_{\ell \ne n}(1-h_\ell)\right) - 2^{Nb}(1-h_n)} = o\big(N^{\frac{d}{2}} 2^{-N(h_1+\cdots+h_d-d+1/2)}\big), \tag{5.18}$$

where $b > 0$ is as in Definition 2.13. Then, putting together (5.11), (5.12), (5.13), (5.15), (5.17) and (5.18), we obtain (5.4). We mention in passing that the inequalities (4.3) and (2.31), as well as the inequalities (B.1) (B.2) and (B.3) in Appendix B, are the main ingredients for proving (5.17) and (5.18).

The second step of the proof of (5.1) consists in showing that, on Ω^*, one has, for some positive finite random variable denoted by C^- and for all $N, P \in \mathbb{N}$,

$$\begin{aligned} &\Big\| \sum_{(\mathbf{j},\mathbf{k}) \in \mathcal{S}^-_{N+P} \setminus \mathcal{S}^-_N} \varepsilon_{\mathbf{j},\mathbf{k}} \mathcal{K}^{(d,\mathbf{h})}_{\mathbf{j},\mathbf{k}} \Big\|_{[0,T],\infty} \\ &\le C^- \Big(2^{-N(L-1)g} \sqrt{N} + N^{\frac{d}{2}} 2^{-2^{Nb'}(1-h_n)} \Big) = o\big(N^{\frac{d}{2}} 2^{-N(h_1+\cdots+h_d-d+1/2)}\big), \end{aligned} \tag{5.19}$$

where $g > 0$ and $b' > 0$ are as in Definition 2.13 and the real number $L > 1+2g^{-1}$ is arbitrary and fixed. Observe that, in view of Definition 2.13, the the set $\mathcal{S}^-_{N+P} \setminus \mathcal{S}^-_N$ can be expressed as

$$\mathcal{S}^-_{N+P} \setminus \mathcal{S}^-_N = (\aleph^3_{N,P} \times \beth^{(iii)}_{N,P,T}) \cup (\aleph^4_{N,P} \times \beth^{(iv)}_{N,P,T}) \tag{5.20}$$

where

$$\aleph^3_{N,P} := \{\mathbf{j} \in -\mathbb{N}^d \,:\, -2^{Nb'} \le \min_{\ell \in [\![1,d]\!]} j_\ell\}, \tag{5.21}$$

$$\aleph^4_{N,P} := \{\mathbf{j} \in \mathbb{Z}^d \,:\, \mathbf{j} \in -\mathbb{N}^d \,:\, -2^{(N+P)b'} \le \min_{\ell \in [\![1,d]\!]} j_\ell < -2^{Nb'}\}, \tag{5.22}$$

and

$$\beth^{(iii)}_{N,P,T} := \{\mathbf{k} \in \mathbb{Z}^d \,:\, 2^{Ng} < \max_{\ell \in [\![1,d]\!]} |k_\ell| \le 2^{(N+P)g}\}, \tag{5.23}$$

$$\beth^{(iv)}_{N,P,T} := \{\mathbf{k} \in \mathbb{Z}^d \,:\, \max_{\ell \in [\![1,d]\!]} |k_\ell| \le 2^{(N+P)g}\}. \tag{5.24}$$

It results from (5.20), the triangular inequality and (4.1) that

$$\Big\| \sum_{(\mathbf{j},\mathbf{k}) \in \mathcal{S}^-_{N+P} \setminus \mathcal{S}^-_N} \varepsilon_{\mathbf{j},\mathbf{k}} \mathcal{K}^{(d,\mathbf{h})}_{\mathbf{j},\mathbf{k}} \Big\|_{[0,T],\infty} \le \nabla^3_{N,P} + \nabla^4_{N,P}, \tag{5.25}$$

where

$$\nabla^3_{N,P} := \sum_{\mathbf{j}\in\aleph^3_{N,P}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sup_{t\in[0,T]} \Big| \sum_{\mathbf{k}\in\beth^{(iii)}_{N,P,T}} \varepsilon_{\mathbf{j},\mathbf{k}} \mathcal{A}_{\mathbf{j},\mathbf{k}}(t) \Big|$$

and

$$\nabla^4_{N,P} := \sum_{\mathbf{j}\in\aleph^4_{N,P}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sup_{t\in[0,T]} \Big| \sum_{\mathbf{k}\in\beth^{(iv)}_{N,P,T}} \varepsilon_{\mathbf{j},\mathbf{k}} \mathcal{A}_{\mathbf{j},\mathbf{k}}(t) \Big|.$$

Then, notice that (5.21), (5.23) and the triangle inequality imply that

$$\nabla^3_{N,P} \le \sum_{n=1}^{d} \mathcal{L}^3_{n,N}, \tag{5.26}$$

where, for all $n \in [\![1, d]\!]$,

$$\mathcal{L}^3_{n,N} := \sum_{\mathbf{j}\in-\mathbb{N}^d} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sum_{|k_n|>2^{Ng}} \sum_{\substack{k_\ell\in\mathbb{Z}\\ \ell\in[\![1,d]\!],\ell\neq n}} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \,. \tag{5.27}$$

Also, notice that (5.22), (5.24) and the triangle inequality entail that

$$\nabla^4_{N,P} \le \sum_{n=1}^{d} \mathcal{L}^4_{n,N}, \tag{5.28}$$

where, for all $n \in [\![1, d]\!]$,

$$\mathcal{L}^4_{n,N} := \sum_{j_n\le -2^{Nb'}} \sum_{\substack{j_\ell<0\\ \ell\in[\![1,d]\!]\setminus\{n\}}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sum_{\mathbf{k}\in\mathbb{Z}^d} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \,. \tag{5.29}$$

In Subsection 5.2, it is shown that, on the event Ω^* of probability 1, for some positive finite random variables C_3 and C_4, one has, for all $n \in [\![1, d]\!]$ and $N \in \mathbb{N}$, that

$$\mathcal{L}^3_{n,N} \le C_3 2^{-N(L-1)g} \sqrt{N}, \tag{5.30}$$

and that

$$\mathcal{L}^4_{n,N} \le C_4 N^{\frac{d}{2}} 2^{-2^{Nb'}(1-h_n)}. \tag{5.31}$$

Then, putting together (5.25), (5.26), (5.28), (5.30) and (5.31), we obtain (5.19). We mention in passing that the inequalities (4.3), (2.31) and (B.1) in Appendix B, are the main ingredients for proving (5.30) and (5.31).

Finally, combining (5.3) with (5.4) and (5.19), it follows that (5.1) is satisfied.

5.1 Rates of convergence to zero of $\mathcal{L}^1_{n,N}$ and $\mathcal{L}^2_{n,N}$

Lemma 5.1. *Let $T > 2$ and $L > 3/2$ be two fixed real numbers. There exits a positive almost surely finite random variables C such that, for all $n \in [\![1, d]\!]$ and $N \in \mathbb{N}$, on Ω^*, the random variable $\mathcal{L}^1_{n,N}$, defined by (5.14), is bounded from above by $C 2^{-N(h_1+\cdots+h_d+L-d-1)} N^{\frac{d}{2}} \log(3+N)^{\frac{d}{2}}$.*

Proof. Let us fix $n \in [\![1,d]\!]$, $N \in \mathbb{N}$ and $\mathbf{j} \in (-\infty, N)^d$. Using the definition (4.2), the inequality (4.3), the bound (2.31), the triangular inequality, inequalities (B.1) and (B.3) and the fact that the function $y \mapsto (2+y)^{-L}\sqrt{\log(2+y)}$ is decreasing on $\mathbb{R}_+$, we get

$$\begin{aligned}
&\sum_{|k_n|>2^{N+1}T} \sum_{\substack{k_\ell \in \mathbb{Z} \\ \ell \in [\![1,d]\!]\setminus\{n\}}} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \\
&\leq C_0 \int_0^T \sum_{|k_n|>2^{N+1}T} \frac{\sqrt{\log(3+|j_n|+|k_n|)}}{(3+|2^{j_n}s-k_n|)^L} \times \dots \\
&\qquad \dots \times \prod_{\substack{\ell=1\\ \ell\neq n}}^d \left(\sum_{k_\ell\in\mathbb{Z}} \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(3+|2^{j_\ell}s-k_\ell|)^L} \right) ds \\
&\leq C_1 \int_0^T \sum_{|k_n|>2^{N+1}T} \frac{\sqrt{\log(3+|j_n|+|k_n|)}}{(3+|k_n|-2^{j_n}s)^L} \prod_{\substack{\ell=1\\ \ell\neq n}}^d \left(\sqrt{\log(3+|j_\ell|+2^{j_\ell}s)} \right) ds \\
&\leq C_1 T 2^L \sqrt{\log(3+|j_n|)} \left(\prod_{\substack{\ell=1\\ \ell\neq n}}^d \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)} \right) \times \dots \\
&\qquad \dots \times \left(\sum_{|k_n|>2^{N+1}T} \frac{\sqrt{\log(3+|k_n|)}}{(3+|k_n|)^L} \right) \\
&\leq C_1 T 2^{L+1} \sqrt{\log(3+|j_n|)} \left(\prod_{\substack{\ell=1\\ \ell\neq n}}^d \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)} \right) \times \dots \\
&\qquad \dots \times \left(\int_{2^{N+1}T}^{+\infty} \frac{\sqrt{\log(2+y)}}{(2+y)^L} \right) \\
&\leq C_2 \sqrt{\prod_{\ell=1}^d \log(3+|j_\ell|)} N^{\frac{d}{2}} 2^{-N(L-1)},
\end{aligned} \tag{5.32}$$

where C_0, C_1 and C_2 are positive almost surely finite random variables not depending on $\mathbf{j}$ and N. It follows from (5.32) that

$$\begin{aligned}
\mathcal{L}^1_{n,N} &\leq C_2 N^{\frac{d}{2}} 2^{-N(L-1)} \sum_{j_1=-\infty}^{N-1} \cdots \sum_{j_d=-\infty}^{N-1} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \sqrt{\prod_{\ell=1}^d \log(3+|j_\ell|)} \\
&\leq C_3 N^{\frac{d}{2}} \log(3+N)^{\frac{d}{2}} 2^{-N(h_1+\cdots+h_d+L-d-1)},
\end{aligned}$$

where C_3 is a positive almost surely finite random variable not depending on N. ☐

Lemma 5.2. *Let $T>2$ and $b>0$ be two fixed real numbers. There exits a positive almost surely finite random variable C such that, for all $n \in [\![1,d]\!]$ and $N \in \mathbb{N}$, on Ω^*, the random variable $\mathcal{L}^2_{n,N}$, defined in* (5.16)*, is bounded from above by $CN^{\frac{d}{2}} 2^{N\left(\sum_{\ell\neq n}(1-h_\ell)\right) - 2^{Nb}(1-h_n)}$.*

Proof. Let us fix $n \in [\![1, d]\!]$ and $N \in \mathbb{N}$ and $\mathbf{j} \in \mathbb{Z}^d$ such that $j_n \leq -2^{Nb}$ and, for $\ell \in [\![1, d]\!] \setminus \{n\}$, $j_\ell < N$. Using the definition (4.2), the inequality (4.3), the bound (2.31), the inequality (B.1), the triangular inequality and the inequalities (B.3) and (B.2), we get

$$
\begin{aligned}
&\sum_{\mathbf{k}\in\mathbb{Z}^d} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \\
&\leq C_0 \int_0^T \left(\sum_{k_n\in\mathbb{Z}} \frac{\sqrt{\log(3+|j_n|+|k_n|)}}{(3+T+|2^{j_n}s-k_n|)^2} \right) \times \dots \\
&\qquad \dots \times \prod_{\ell=1,\ell\neq n}^{d} \left(\sum_{\ell\in\mathbb{Z}^d} \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(3+|2^{j_\ell}s-k_\ell|)^2} \right) ds \\
&\leq C_0 \sqrt{\log(3+|j_n|)} \int_0^T \left(\sum_{k_n\in\mathbb{Z}} \frac{\sqrt{\log(3+|k_n|)}}{(3+|k_n|)^2} \right) \times \dots \\
&\qquad \dots \times \prod_{\ell=1,\ell\neq n}^{d} \sqrt{\log(3+|j_\ell|+2^{j_\ell}s)}\, ds \\
&\leq C_1 \sqrt{\log(3+|j_n|)} \prod_{\ell=1,\ell\neq n}^{d} \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)}, \qquad (5.33)
\end{aligned}
$$

where C_0 and C_1 are positive almost surely finite random variables not depending on $\mathbf{j}$ and N. It follows from (5.33) that

$$
\begin{aligned}
\mathcal{L}^2_{n,N} &\leq C_1 \left(\sum_{j_n=-\infty}^{\lfloor -2^{Nb} \rfloor} 2^{j_n(1-h_n)} \sqrt{\log(3+|j_n|)} \right) \times \cdots \\
&\qquad \cdots \times \prod_{\ell=1,\ell\neq n}^{d} \left(\sum_{j_\ell=-\infty}^{N} 2^{j_\ell(1-h_\ell)} \sqrt{\log(3+|j_\ell|+2^{j_\ell}T)} \right) \\
&\qquad \dots \times \prod_{\ell=1,\ell\neq n}^{d} \\
&\leq C_2 N^{\frac{d}{2}} 2^{N\left(\sum_{\ell\neq n}(1-h_\ell)\right)-2^{Nb}(1-h_n)},
\end{aligned}
$$

where C_2 is a positive almost surely finite random variable not depending on N. □

5.2 Rates of convergence to zero of $\mathcal{L}^3_{n,N}$ and $\mathcal{L}^4_{n,N}$

Lemma 5.3. *Let $T > 2$, $g > 0$ and $L > 1 + 2g^{-1}$ be three fixed real numbers. There exits a positive almost surely finite random variable C such that, for all $n \in [\![1, d]\!]$ and $N \in \mathbb{N}$, on Ω^*, the random variable $\mathcal{L}^2_{n,N}$, defined by (5.27), is bounded from above by $C 2^{-N(L-1)g}\sqrt{N}$.*

Proof. Let us fix $n \in [\![1, d]\!]$ and $N \in \mathbb{N}$ and $\mathbf{j} \in -\mathbb{N}^d$. Using the definition (4.2), the inequality (4.3), the bound (2.31), the inequality (B.1), the triangular inequality and the fact that the function $y \mapsto (2+y)^{-L}\sqrt{\log(2+y)}$ is decreasing on $\mathbb{R}_+$, we get

$$
\begin{aligned}
&\sum_{|k_n|>2^{Ng}} \sum_{\substack{k_\ell\in\mathbb{Z} \\ \ell\in[\![1,d]\!]\setminus\{n\}}} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \\
&\le C_0 \int_0^T \sum_{|k_n|>2^{Ng}} \frac{\sqrt{\log(3+|j_n|+|k_n|)}}{(3+T+|2^{j_n}s-k_n|)^L} \times \dots \\
&\qquad \dots \times \prod_{\substack{\ell=1\\ \ell\neq n}}^d \left(\sum_{k_\ell\in\mathbb{Z}} \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(3+T+|2^{j_\ell}s-k_\ell|)^L} \right) ds \\
&\le C_0 \sqrt{\prod_{\ell=1}^d \log(3+|j_\ell|)} \int_0^T \sum_{|k_n|>2^{Ng}} \frac{\sqrt{\log(3+|k_n|)}}{(3+|k_n|)^L} \times \dots \\
&\qquad \dots \times \prod_{\substack{\ell=1\\ \ell\neq n}}^d \left(\sum_{k_\ell\in\mathbb{Z}} \frac{\sqrt{\log(3+|k_\ell|)}}{(3+|k_\ell|)^L} \right) ds \\
&\le C_1 \sqrt{\prod_{\ell=1}^d \log(3+|j_\ell|)} \sum_{|k_n|>2^{Ng}} \frac{\sqrt{\log(3+|k_n|)}}{(3+|k_n|)^L} \\
&\le 2C_1 \sqrt{\prod_{\ell=1}^d \log(3+|j_\ell|)} \int_{2^{Ng}}^{+\infty} \frac{\sqrt{\log(2+y)}}{(2+y)^L} \\
&\le C_2 \sqrt{\prod_{\ell=1}^d \log(3+|j_\ell|)}\, 2^{-N(L-1)g}\sqrt{N}, \qquad (5.34)
\end{aligned}
$$

where C_0, C_1 and C_2 are positive almost surely finite random variables not depending on $\mathbf{j}$ and N. It follows from (5.34) that

$$
\begin{aligned}
\mathcal{L}^3_{n,N} &\le C_2 2^{-N(L-1)g}\sqrt{N} \sum_{\mathbf{j}\in-\mathbb{N}^d} 2^{j_1(1-h_1)+\dots+j_d(1-h_d)} \sqrt{\prod_{\ell=1}^d \log(3+|j_\ell|)} \\
&\le C_3 2^{-N(L-1)g}\sqrt{N},
\end{aligned}
$$

where C_3 is a positive almost surely finite random variables not depending on N. □

Lemma 5.4. *Let $T>2$ and $b'>0$ be two fixed real numbers. There exits a positive almost surely finite random variable C such that, for all $n\in[\![1,d]\!]$ and $N\in\mathbb{N}$, on Ω^*, the random variable $\mathcal{L}^4_{n,N}$, defined in (5.29), is bounded from above by $CN^{\frac{d}{2}}2^{-2^{Nb'}(1-h_n)}$.*

Proof. Let us fix $n\in[\![1,d]\!]$ and $N\in\mathbb{N}$ and $\mathbf{j}\in-\mathbb{N}^d$ such that $j_n\le -2^{Nb'}$. Using the definition (4.2), the inequality (4.3), the bound (2.31), the inequality

(B.1) and the triangular inequality, we get

$$
\begin{aligned}
&\sum_{\mathbf{k}\in\mathbb{Z}^d} |\varepsilon_{\mathbf{j},\mathbf{k}}| \sup_{t\in[0,T]} |\mathcal{A}_{\mathbf{j},\mathbf{k}}(t)| \le C_0 \int_0^T \prod_{\ell=1}^d \left(\sum_{k_\ell\in\mathbb{Z}} \frac{\sqrt{\log(3+|j_\ell|+|k_\ell|)}}{(3+T+|2^{j_\ell}s-k_\ell|)^2} \right) ds \\
&\le C_0 \int_0^T \prod_{\ell=1}^d \left(\sum_{k_\ell\in\mathbb{Z}} \frac{\sqrt{\log(3+|j_\ell|)}\sqrt{\log(3+|k_\ell|)}}{(3+|k_\ell|)^2} \right) ds \\
&\le C_1 \prod_{\ell=1}^d \sqrt{\log(3+|j_\ell|)}, \qquad (5.35)
\end{aligned}
$$

where C_0 and C_1 are positive almost surely finite random variables not depending on $\mathbf{j}$ and N. It follows from (5.35) that

$$
\begin{aligned}
\mathcal{L}^4_{v,N} &\le C_1 \left(\sum_{j_n=-\infty}^{\lfloor -2^{Nb'} \rfloor} 2^{j_n(1-h_n)} \sqrt{\log(3+|j_n|)} \right) \times \dots \\
&\qquad \dots \times \prod_{\ell=1,\ell\neq n}^d \left(\sum_{j_\ell=-\infty}^{-1} 2^{j_\ell(1-h_\ell)} \sqrt{\log(3+|j_{\ell'}|)} \right) \\
&\le C_2 N^{\frac{d}{2}} 2^{-2^{Nb'}(1-h_n)},
\end{aligned}
$$

where C_2 is a positive almost surely finite random variable not depending on N. □

Acknowledgement. The authors are very grateful to the two anonymous referees for their valuable comments and remarks which have greatly improved the manuscript. The authors are members of the GDR 3475 (Analyse Multifractale et Autosimilarité) which partially supports them. Antoine Ayache is also partially supported by the Labex CEMPI (ANR-11-LABX-0007-01) and the Australian Research Council's Discovery Projects funding scheme (project number DP220101680). A significant portion of the manuscript was written while Laurent Loosveldt was supported by the FNR OPEN grant APOGEe at University of Luxembourg. The first results in it were obtained during a visit of Antoine Ayache at the University of Luxembourg, also funded by the FNR OPEN grant APOGEe.

A Some facts concerning multiple Wiener integrals

In this section, we mainly give the proof of the crucial equality (2.20). This proof relies on some fundamental facts concerning multiple Wiener integrals. We refer to the two books [22, 23] for detailed presentations of such stochastic integrals and many other related topics (Wiener chaoses, Malliavin calculus, and so on). We recall that a function $f \in L^2(\mathbb{R}^n)$ is said to be symmetric if, for all $\sigma \in \mathfrak{S}_n$ (the set of permutations of $[\![1,n]\!] = \{1,\dots,n\}$) and for Lebesgue almost every $(x_1,\dots,x_n) \in \mathbb{R}^n$, one has $f(x_{\sigma(1)},\dots,x_{\sigma(n)}) = f(x_1,\dots,x_n)$. In other words, $f \in L^2(\mathbb{R}^n)$ is symmetric if and only if it is almost everywhere equal to its canonical symmetrization $\widetilde{f}$ defined, for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, as:

$$\widetilde{f}(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \tag{A.1}$$

We point out that a very fundamental property of multiple integrals is that

$$I_n(f) = I_n(\widetilde{f}), \quad \text{for all } f \in L^2(\mathbb{R}^n). \tag{A.2}$$

For proving the equality (2.20), we will make use of the so-called product formula for multiple Wiener integrals [23, Proposition 1.1.3]. In order to give this important formula, first we need the following definition: let m and n be two arbitrary positive integers, if $f \in L^2(\mathbb{R}^m)$ and $g \in L^2(\mathbb{R}^n)$ are symmetric functions and $r \in [\![0, m \wedge n]\!]$, the contraction $f \otimes_r g$ is the $L^2(\mathbb{R}^{m+n-2r})$ function defined, for all $(x_1, \ldots, x_{m+n-2r}) \in \mathbb{R}^{m+n-2r}$, through the Lebesgue integral

$$\begin{aligned}&(f \otimes_r g)(x_1, \ldots, x_{m+n-2r})\\ &:= \int_{\mathbb{R}^r} f(x_1, \ldots, x_{m-r}, s_1, \ldots, s_r) g(x_{m-r+1}, \ldots, x_{m+n-2r}, s_1, \ldots, s_r)\, ds_1 \ldots ds_r\,,\end{aligned}$$

with the convention that $f \otimes_0 g := f \otimes g$, which means that $f \otimes_0 g$ is the usual tensor product of f and g; also notice that when $m = n$, then $f \otimes_n g$ is identified with the Lebesgue integral $\int_{\mathbb{R}^n} fg$. Using, the previous definition, one can write the product formula for multiple Wiener integrals in the following way: for each positive integers m and n, and for every symmetric functions $f \in L^2(\mathbb{R}^m)$ and $g \in L^2(\mathbb{R}^n)$, one has

$$I_m(f) I_n(g) = \sum_{r=0}^{m \wedge n} r! \binom{m}{r} \binom{n}{r} I_{m+n-2r}(f \otimes_r g), \tag{A.3}$$

where, for $p = m$ or $p = n$, the quantity $\binom{p}{r}$ is the usual binomial coefficient

$$\binom{p}{r} := \frac{p!}{r!(p-r)!}\,.$$

For proving the equality (2.20), we will also make use of the following fundamental result, which, for instance corresponds to [22, Theorem 2.7.7].

Theorem A.1. *Let $f \in L^2(\mathbb{R})$ be such that $\|f\|_{L^2(\mathbb{R})} = 1$. For all positive integer n, let H_n the Hermite polynomial of degree n. Then, one has*

$$H_n\left(I_1(f)\right) = I_n(f^{\otimes_n}).$$

We are now in position to prove the equality (2.20)

Proof of the equality (2.20). It follows from Theorem A.1 that

$$\prod_{\ell=1}^{p} H_{n_\ell}\big(I_1(\varphi_\ell)\big) = \prod_{\ell=1}^{p} I_{n_\ell}\left(\varphi_\ell^{\otimes_{n_\ell}}\right),$$

and thus, it remains to show

$$\prod_{\ell=1}^{p} I_{n_\ell}\left(\varphi_\ell^{\otimes n_\ell}\right) = I_{n_1+\cdots+n_p}\left(\bigotimes_{\ell=1}^{p} \varphi_\ell^{\otimes n_\ell}\right). \tag{A.4}$$

We proceed by induction on the positive integer p. It is clear that (A.4) is satisfied when $p = 1$. So from now on, we assume that $p \geq 2$ and that

$$\prod_{\ell=1}^{p-1} I_{n_\ell}\left(\varphi_\ell^{\otimes n_\ell}\right) = I_{n_1+\cdots+n_{p-1}}\left(\bigotimes_{\ell=1}^{p-1} \varphi_\ell^{\otimes n_\ell}\right).$$

Then, setting $n = n_1 + \cdots + n_{p-1}$ and $d = n_1 + \cdots + n_p = n + n_p$, we can derive from the product formula (A.3) that

$$\begin{aligned}
\prod_{\ell=1}^{p} I_{n_\ell}\left(\varphi_\ell^{\otimes n_\ell}\right) &= I_n\left(\bigotimes_{\ell=1}^{p-1} \varphi_\ell^{\otimes n_\ell}\right) I_{n_p}\left(\varphi_p\right) \\
&= \sum_{r=0}^{n\wedge n_p} r! \binom{n}{r}\binom{n_p}{r} I_{n+n_p-2r}\left(\widetilde{\bigotimes_{\ell=1}^{p-1}} \varphi_\ell^{\otimes n_\ell} \otimes_r \varphi_p^{\otimes n_p}\right) \\
&= I_d\left(\widetilde{\bigotimes_{\ell=1}^{p-1}} \varphi_\ell^{\otimes n_\ell} \otimes \varphi_p^{\otimes n_p}\right) = I_d\left(\overbrace{\widetilde{\bigotimes_{\ell=1}^{p-1}} \varphi_\ell^{\otimes n_\ell} \otimes \varphi_p^{\otimes n_p}}\right).
\end{aligned} \tag{A.5}$$

Notice that the third equality in (A.5) results from the equality

$$\widetilde{\bigotimes_{\ell=1}^{p-1}} \varphi_\ell^{\otimes n_\ell} \otimes_r \varphi_p^{\otimes n_p} = 0, \quad \text{for all } r \in [\![1, n]\!],$$

which is a consequence of the orthonormality of the system $(\varphi_\ell)_{\ell=1}^{p}$. Also notice that the last equality in (A.5) results from (A.2). Next observe that, in view of (A.5) and (A.2), in order to show that (A.4) holds, it remains us to prove that

$$\overbrace{\widetilde{\bigotimes_{\ell=1}^{p-1}} \varphi_\ell^{\otimes n_\ell} \otimes \varphi_p^{\otimes n_p}} = \overbrace{\bigotimes_{\ell=1}^{d} \varphi_\ell^{\otimes n_\ell}}.$$

Notice that any arbitrary permutation $\sigma \in \mathfrak{S}_n$ can be extended in a natural way into a permutation $\check{\sigma} \in \mathfrak{S}_d$ defined, for all $i \in \{1, \ldots, n\}$, as $\check{\sigma}(i) = \sigma(i)$, and for, each $i \in \{n+1, \ldots, d\}$, as $\check{\sigma}(i) = i$. Thus, using (A.1), the latter notation and the fact that the composition map $\nu \mapsto \nu \circ \check{\sigma}$ is a bijection from $\mathfrak{S}_d$ to itself, one gets, for all $(x_1, \dots, x_d) \in \mathbb{R}^d$, that

$$\left(\widetilde{\bigotimes_{\ell=1}^{p-1} \varphi_\ell^{\otimes n_\ell} \otimes \varphi_p^{\otimes n_p}}\right)(x_1, \dots, x_d)$$

$$= \frac{1}{d!}\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\nu \in \mathfrak{S}_d} \bigotimes_{\ell=1}^{p-1} \varphi_\ell^{\otimes n_\ell}(x_{\nu(\sigma(1))}, \dots, x_{\nu(\sigma(n))}) \otimes \varphi_p^{\otimes n_p}(x_{\nu(n+1)}, \dots, x_{\nu(d)})$$

$$= \frac{1}{d!}\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\nu \in \mathfrak{S}_d} \bigotimes_{\ell=1}^{p} \varphi_\ell^{\otimes n_\ell}(x_{\nu o \breve{\sigma}(1)}, \dots, x_{\nu o \breve{\sigma}(d)})$$

$$= \frac{1}{d!}\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\nu' \in \mathfrak{S}_d} \bigotimes_{\ell=1}^{p} \varphi_\ell^{\otimes n_\ell}(x_{\nu'(1)}, \dots, x_{\nu'(d)})$$

$$= \frac{1}{d!} \sum_{\nu' \in \mathfrak{S}_d} \bigotimes_{\ell=1}^{p} \varphi_\ell^{\otimes n_\ell}(x_{\nu'(1)}, \dots, x_{\nu'(d)})$$

$$= \left(\widetilde{\bigotimes_{\ell=1}^{p} \varphi_\ell^{\otimes n_\ell}}\right)(x_1, \dots, x_d).$$

□

B Some useful lemmata

The proofs of the following lemmata, which are extensively used in our articles, can be found in [2].

Lemma B.1. *For all $(x, y) \in \mathbb{R}_+^2$, we have*

$$\log(3 + x + y) \le \log(3 + x)\log(3 + y). \tag{B.1}$$

Moreover, for each fixed positive real number T, there exists a constant $c > 0$ such that, for every $x \in \mathbb{R}_+$, we have

$$\log(3 + x + 2^x T) \le c(1 + x). \tag{B.2}$$

Lemma B.2. *For each fixed real number $L > 1$, there exists a constant $c > 0$ such that, for all $j \in \mathbb{Z}$ and for all $s \in \mathbb{R}$, we have*

$$\sum_{k \in \mathbb{Z}} \frac{\sqrt{\log(3 + |j| + |k|)}}{\big(3 + |2^j s - k|\big)^L} \le c\sqrt{\log\big(3 + |j| + 2^j|s|\big)}. \tag{B.3}$$

Lemma B.3. *For each fixed real number $L > 1$, there exists a constant $c > 0$ such that, for all $t \in \mathbb{R}_+$, for all $s \in [0, t]$ and for all $j \in \mathbb{N}$, we have*

$$\sum_{k \in D_j^3(t)} \frac{\sqrt{\log(3 + |j| + |k|)}}{(3 + |2^j s - k|)^L} \le c(1 + j)2^{-j(L-1)(1-a)}\sqrt{\log(3 + t)},$$

where $D_j^3(t)$ is the infinite subset of $\mathbb{Z}$ defined through (4.12).

References

[1] Patrice Abry and Fabrice Sellan. The wavelet-based synthesis for fractional Brownian motion proposed by F. Sellan and Y. Meyer: remarks and fast implementation. *Appl. Comput. Harmon. Anal.*, 3(4):377–383, 1996.

[2] Antoine Ayache and Yassine Esmili. Wavelet-type expansion of the generalized Rosenblatt process and its rate of convergence. *J. Fourier Anal. Appl.*, 26(3):Paper No. 51, 35, 2020.

[3] Antoine Ayache and Murad S. Taqqu. Rate optimality of wavelet series approximations of fractional Brownian motion. *J. Fourier Anal. Appl.*, 9(5):451–471, 2003.

[4] Shuyang Bai and Murad S. Taqqu. Generalized Hermite processes, discrete chaos and limit theorems. *Stochastic Process. Appl.*, 124(4):1710–1739, 2014.

[5] Karine Bertin, Soledad Torres, and Ciprian A. Tudor. Maximum-likelihood estimators and random walks in long memory models. *Statistics*, 45(4):361–374, 2011.

[6] Alexandra Chronopoulou, Ciprian A. Tudor, and Frederi G. Viens. Self-similarity parameter estimation and reproduction property for non-Gaussian Hermite processes. *Commun. Stoch. Anal.*, 5(1):161–185, 2011.

[7] Alexandra Chronopoulou, Frederi G. Viens, and Ciprian A. Tudor. Variations and Hurst index estimation for a Rosenblatt process using longer filters. *Electron. J. Stat.*, 3:1393–1435, 2009.

[8] Ingrid Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[9] Lara Daw and Laurent Loosveldt. Wavelet methods to study the pointwise regularity of the generalized Rosenblatt process. *Electron. J. Probab.*, 27:1–45, 2022.

[10] Roland Lvovich Dobrushin and Peter Major. Non-central limit theorems for nonlinear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete*, 50(1):27–52, 1979.

[11] Paul Doukhan, George Oppenheim, and Murad S. Taqqu, editors. *Theory and applications of long-range dependence*. Birkhäuser Boston, Inc., Boston, MA, 2003.

[12] Céline Esser and Laurent Loosveldt. Slow, ordinary and rapid points for Gaussian wavelets series and application to fractional Brownian motions. *ALEA Lat. Am. J. Probab. Math. Stat.*, 19(2):1471–1495, 2022.

[13] Svante Janson. *Gaussian Hilbert spaces*, volume 129 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997.

[14] Olav Kallenberg. On an independence criterion for multiple Wiener integrals. *Ann. Probab.*, 19(2):483–485, 1991.

[15] Andreï Nikolaïevitch Kolmogorov. Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 26:115–118, 1940.

[16] Makoto Maejima and Ciprian A. Tudor. Wiener integrals with respect to the Hermite process and a non-central limit theorem. *Stoch. Anal. Appl.*, 25(5):1043–1056, 2007.

[17] Makoto Maejima and Ciprian A. Tudor. Selfsimilar processes with stationary increments in the second Wiener chaos. *Probab. Math. Statist.*, 32(1):167–186, 2012.

[18] Benoit B. Mandelbrot and John W. Van Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Rev.*, 10:422–437, 1968.

[19] Yves Meyer. *Ondelettes et opérateurs. I.* Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1990. Ondelettes. [Wavelets].

[20] Yves Meyer, Fabrice Sellan, and Murad S. Taqqu. Wavelets, generalized white noise and fractional integration: the synthesis of fractional Brownian motion. *J. Fourier Anal. Appl.*, 5(5):465–494, 1999.

[21] Ivan Nourdin. *Selected aspects of fractional Brownian motion*, volume 4 of *Bocconi & Springer Series*. Springer, Milan; Bocconi University Press, Milan, 2012.

[22] Ivan Nourdin and Giovanni Peccati. *Normal approximations with Malliavin calculus*, volume 192 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2012. From Stein's method to universality.

[23] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[24] Vladas Pipiras. Wavelet-type expansion of the Rosenblatt process. *J. Fourier Anal. Appl.*, 10(6):599–634, 2004.

[25] Vladas Pipiras. Wavelet-based simulation of fractional Brownian motion revisited. *Appl. Comput. Harmon. Anal.*, 19(1):49–60, 2005.

[26] Laurent Schwartz. *Théorie des distributions*. Hermann, 1978.

[27] Murad S. Taqqu. Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 31:287–302, 1974/75.

[28] Murad S. Taqqu. Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete*, 50(1):53–83, 1979.

[29] Ciprian A. Tudor and Frederi G. Viens. Variations and estimators for self-similarity parameters via Malliavin calculus. *Ann. Probab.*, 37(6):2093–2134, 2009.

[30] Mark S. Veillette and Murad S. Taqqu. Properties and numerical evaluation of the Rosenblatt distribution. *Bernoulli*, 19(3):982–1005, 2013.