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"The theorem of Grassmann in a space of n dimensions."

By LUCIEN GODEAUX.

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Mathematics. — "*The theorem of* GRASSMANN *in a space of n dimensions.*" By LUCIEN GODEAUX, at Morlanwelz (Hainault). (Communicated by Prof. P. H. SCHOUTE).

We shall designate by the letter S a linear space and the number of dimensions of this space shall be the index.

The notation V_i^j represents a variety, the locus of ∞^i elements and of order j.

The order of a variety, locus of spaces S_k occurring in an $(n-k)(k+1)-1$ times infinite number in a space S_n, is the number of S_k of an S_{k+1} through an S_{k-1} of this S_{k+1} and belonging to the variety.

1. In an S_2 the theorem of GRASSMANN can be read thus:

The locus of S_0 for which the S_1 which unite it to three fixed S_0 meet three fixed S_1 in three S_0 of the same S_1 is a variety V_1^3.

In an S_3 it has been given it the two following forms:

The locus of an S_0 for which the S_2 which unite it to four fixed S_1 meet four fixed S_1 in four S_0 of a same S_1 is a V_2^4. (LE PAIGE, *Sur la génération de certaines surfaces par des faisceaux quadrilinéaires, Bul. de Belgique*, 1884, 3e *série, tome VIII*).

The locus of an S_0 for which the S_1 which unite it to four fixed S_0 meet four fixed S_2 in four S_0 of a same S_2 is a V_2^4.

2. Let there be in an S_n k S_{r_i} which we shall designate by A_i and k S_{s_i} which we shall designate by B_i, $(i=1,\dots k)$.

Let p be a number satisfying the $2k$ inequalities

$$r_i + p + 1 \leqq n - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$r_i + s_i + p + 1 \geqq n, \quad (i = 1, \dots k) \quad . \quad . \quad (2)$$

A space S_p determines with the k spaces A_i k spaces S_{r_i+p+1}. These spaces meet the corresponding spaces B_i in k spaces $S_{r_i+s_i+p-n+1}$.

If these k spaces belong to an $S_{\sum\limits_{i=1}^{i=n}(r_i+s_i)+k(p-n+2)-1}$, the space S_p describes a variety $V_{(n-p)(p+1)-1}$ the order of which is to be found.

Let us suppose we have

$$\Sigma (r_i + s_i) + k(p - n + 2) = n + 1 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let C be an S_{p+1} and D an S_{p-1} of C.

Let us designate by Δ an S_p passing through D and situated in C.

Let us take $k-1$ spaces Δ and let us number them $1, \ldots j-1, j+1, \ldots k$.

These $k-1$ spaces Δ determine with $k-1$ spaces A_i suitably chosen $k-1$ spaces S_{r_i+p+1}. These spaces meet the corresponding spaces B_i in $k-1$ spaces $S_{r_i+s_i+p-n+1}$, $(i=1, \ldots, j-1, j+1, \ldots k)$.

These spaces determine an $S_{\sum_{i=1}^{i=j-1}(r_i+s_i)+\sum_{i=j+1}^{i=n}(r_i+s_i)+(k-1)(p-n+2)-1}$

This space has in common with B_j a space

$$S_{\sum_{i=1}^{i=j-1} r_i+\sum_{i=j+1}^{i=n} r_i+\sum_{i=1}^{i=n} s_i+(k-1)(p+2)-kn-1}.$$

In its turn this space determines with A_i a space

$$S_{\sum_{i=1}^{i=j+1}(r_i+s_i)+(k-1)(p+2)-kn}.$$

On account of the equality (3) the latter meets C in a single point, which determines with D a space Δ_j.

When j varies from 1 to k, one obtains k series of spaces Δ between which exists a $(1, 1, \ldots 1)$ correspondence. There are k coincidences.

The variety described by the space S_p is $V^{k}_{(n-p)(p+1)-1}$.

The locus of a space S_p for which the S_{r_i+p+1} which unite it to k fixed spaces S_{r_i} meet k S_{s_i} in k $S_{r_i+s_i+p-n+1}$ of a same $S_{\Sigma(r_i+s_i)+k(p-n+2)-1}$, $(i=1, \ldots k)$, is a variety $V^{k}_{(n-p)(p+1)-1}$.

The spaces A_i are evidently principal spaces of the locus of S_p, principal space having the same meaning as principal point or plane of a complex of rays.

In S_3 we find the following theorem:

The locus of an S_1 for which the S_2 which join it to four S_0 meet four S_1 in four S_0 of a same space S_2 is a variety V^4_3 (complex of order four).

3. If we regard the ordinary space as if generated by right lines we have a geometry of four dimensions. We shall now show two generalizations of the theory of Grassmann in this geometry.

Let us imagine k linear congruences $G_1, \ldots G_k$, and k plane pencils $(P_1, \pi_1), \ldots (P_k, \pi_k)$. Let us imagine moreover to be given a linear system C of linear complexes to the amount in number of ∞^{6-k}.

An arbitrary right line g determines k linear complexes with the

k congruences G. These have in common with the k corresponding plane pencils k lines $p_1, \ldots p_k$.

Let us now find the locus of the line g when the k lines p belong to a same complex of the system C.

Let (A, α) be any plane pencil. Let us take $k-1$ lines of this pencil and let us number them $1, \ldots i-1, i+1, \ldots k$.

Each of these lines determines with the corresponding congruence G a linear complex, which has in common with the corresponding plane pencil (P, π) a line p. The $k-1$ lines p found in this way determine a complex of the system C. This complex has a line p_i in common with the plane pencil (P_i, π_i). This line determines with G_i a complex having a line a_i in common with (A, α). When i varies from 1 to k we have k series of lines a between which exists a $(1, 1, \ldots 1)$ correspondence. There are k coincidences.

The locus of a right line for which the linear complexes that it determines with k fixed linear congruences meet k fixed plane pencils in k lines of a linear complex of a system of $6-k$ terms is a complex of degree k (order and class) to which belong the given k linear congruences.

If $k=6$, we have a theorem of GRASSMANN.

4. Let us suppose five groups of three lines $H_1, \ldots H_5$ and five nets of lines $R_1, \ldots R_5$.

An arbitrary line g determines with $H_1, \ldots H_5$ five linear congruences which meet the five corresponding nets in five lines. If these five lines belong to a selfsame linear congruence the line g describes a congruence.

Let π be a plane. Let us consider in this plane five series of lines $p_1, \ldots p_5$. Between the lines of these series it is easy to see that there is such a correspondence that to four right lines corresponds a fifth.

Let us suppose that three right lines are fixed, whilst the fourth describes a pencil. It is then easy to verify that the fifth also describes a pencil. According to an extension of the principle of ZEUTHEN there are fifteen coincidences.

The locus of a right line taken in such a way that the linear congruences which it determines with five systems of three lines have in common with five nets five lines of a same linear congruence is a congruence of the fifteenth class.

In the same way we can verify that this congruence is also of order fifteen and that it contains the generatrices of the same kind as the given lines of the five quadratic surfaces determined by these lines.