Parametric upper and lower bounds of linear variations of a linear problem's LHS

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LIÈGE université

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Uncertainty

Linear programming

- In LP optimization
  - Formalize problem in terms of
    - Constraints
    - Objective function
  - Get one optimal solution

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Linear programming

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Parameters

Uncertainty

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Parameters

Hypothesis

Uncertainty

Linear programming

- In LP optimization
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    - Objective function

    Parameters ← Hypothesis
  - Get one optimal solution → Further assessment and analysis

Formalization

Parametric uncertainty

- Generic parametric linear optimization problem:

$$\begin{aligned} \min \quad & c^t x \\ \text{s.t} \quad & Ax \leq b \end{aligned}$$

Formalization

Parametric uncertainty

- Generic parametric linear optimization problem:

$$\begin{array}{ll} \min & c^t x + \lambda c_\lambda^t x \\ \text{s.t} & Ax \leq b \end{array}$$

- Modification in the objective coefficients $+\lambda c_\lambda^t x$

Formalization

Parametric uncertainty

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$$\begin{array}{ll} \min & c^t x + \lambda c_\lambda^t x \\ \text{s.t} & Ax \leq b + \lambda b_\lambda \end{array}$$

- Modification in the objective coefficients $+\lambda c_\lambda^t x$
- Modification on the right-hand side $+\lambda b_\lambda$

Formalization

Parametric uncertainty

- Generic parametric linear optimization problem:

$$\begin{array}{ll} \min & c^t x + \lambda c_\lambda^t x \\ \text{s.t} & (A + \lambda D)x \leq b + \lambda b_\lambda \end{array}$$

- Modification in the objective coefficients $+\lambda c_\lambda^t x$
- Modification on the right-hand side $+\lambda b_\lambda$
- Modification on the left-hand side $+\lambda D$

Formalization

Parametric uncertainty

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- Modification in the objective coefficients $+\lambda c_\lambda^t x$
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Discussed a LOT in the literature

Formalization

Parametric uncertainty

- Generic parametric linear optimization problem:

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- Modification on the left-hand side $+\lambda D$ → Not much ?

Formalization

Parametric uncertainty

- Generic parametric linear optimization problem:

$$\begin{array}{ll} \min & c^t x + \lambda c_\lambda^t x \\ \text{s.t} & (A + \lambda D)x \leq b + \lambda b_\lambda \end{array}$$

- Modification in the objective coefficients $+\lambda c_\lambda^t x$
- Modification on the right-hand side $+\lambda b_\lambda$

Discussed a LOT in the literature

- Modification on the left-hand side $+\lambda D$ $\longrightarrow$ Not much ?

Note : The left-hand side modification $+\lambda D$ encapsulates the other modifications

Formalization

Parametric uncertainty

- Our problem formalization

$$
\begin{aligned}
f(\lambda) = \min \quad & c^t x \\
s.t. \quad & A_1 x \leq b_1 \\
& A_2 x + \lambda D x \leq b_2
\end{aligned}
$$

For $\lambda \in [\underline{\lambda}, \overline{\lambda}]$.

- In literature :
  - Usually rely on heavy computation,
  - approximations
  - and/or hypothesis on the matrix D

Naive solution

Heavy computations

- Compute the exact solution of

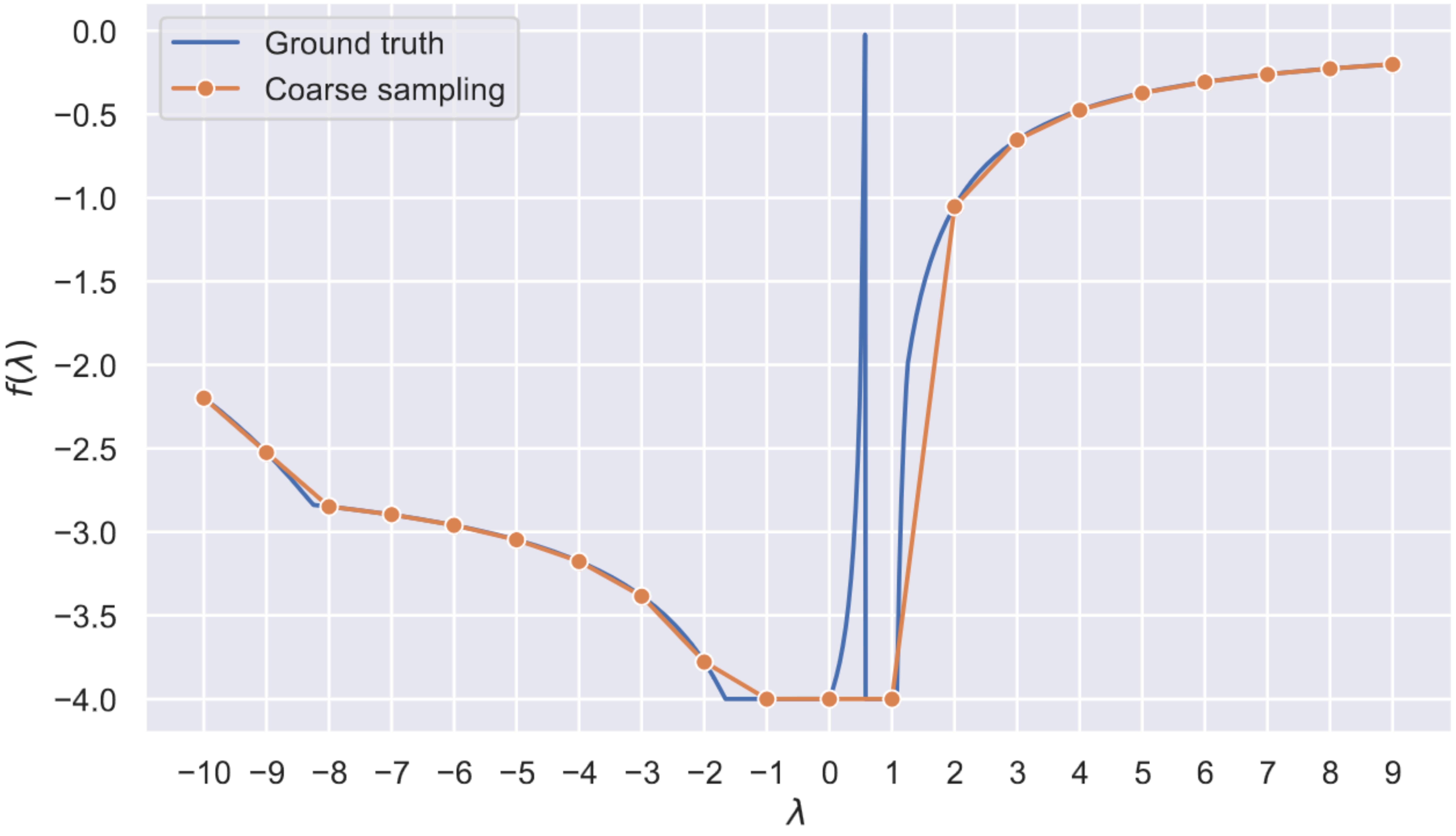
$$
\begin{aligned}
f(\lambda) = \min \quad & c^t x \\
s.t. \quad & A_1 x \leq b_1 \\
& A_2 x + \lambda D x \leq b_2
\end{aligned}
$$

For every $\lambda \in [\underline{\lambda}, \overline{\lambda}]$.

- What is the right level of granularity ?

Naive solution

Heavy computations



Bounds

Idea

$$lb(\underline{\lambda}, \overline{\lambda}) \leq \quad f(\lambda) = \min \quad c^t x \quad \leq ub(\underline{\lambda}, \overline{\lambda})$$

$$s.t. \quad A_1 x \leq b_1$$

$$A_2 x + \lambda D x \leq b_2$$

$$\lambda \in [\underline{\lambda}, \overline{\lambda}]$$

If we find a lower bound (resp. upper bound) to the primal, it becomes an upper (resp. lower) bound on the dual

Optimum reuse

First bound

- Optimize for $\lambda = \underline{\lambda}$ and get a solution $x^\star_\lambda, o^\star_\lambda = f(\lambda)$
    - $x^\star_\lambda$ the optimal solution of $f(\lambda)$
    - $o^\star_\lambda$ the optimal objective of $f(\lambda)$
- If we consider $\lambda' = \underline{\lambda} + \epsilon$, if $x^\star_\lambda$ is a feasible solution of $f(\lambda')$ then we have $o^\star_\lambda \leq o^\star_{\lambda'}$.
- Else, we recompute the optimal solution of $x^\star_{\lambda'}, o^\star_{\lambda'} = f(\lambda')$.

(-) Heavy computations & difficult to chose ϵ

Constant robust solution

Robust bounds

- We try to find a solution $x^\star$ robust to every modification for an interval of λ
- Let us consider $\lambda \in [\lambda_1, \lambda_2]$

$$
\begin{aligned}
f(\lambda) = \min \quad & c^t x \\
s.t. \quad & A_1 x \leq b_1 \\
& A_2 x + \lambda D x \leq b_2
\end{aligned}
\quad \leq \quad
\begin{aligned}
\min \quad & c^t x \\
s.t. \quad & A_1 x \leq b_1 \\
& A_2 x + \lambda_1 D x \leq b_2 \\
& A_2 x + \lambda_2 D x \leq b_2
\end{aligned}
$$

Constant robust solution

Robust bounds

Constant robust solution

Robust bounds

- If we note by n_1 and n_2 the number of constraints in A_1 and A_2 respectively and m the number of variables in $f(\lambda)$
- The constant robust bound reformulates the problem using $n_1 + 2 * n_2$ constraints and m variables
- (+) Not computationally very costly, a lot of redundancies
- (-) Its dual is usually not very tight

Variable robust solution

Robust bounds

- We reformulate all the variables x by a linear function $y + \lambda z$
- Let us consider $\lambda \in [\lambda_1, \lambda_2]$

$$
\begin{aligned}
f(\lambda) = \min \quad & c^t x \\
s.t. \quad & A_1 x \leq b_1 \\
& A_2 x + \lambda D x \leq b_2
\end{aligned}
\quad \leq \quad
\begin{aligned}
\min \quad & c^t(y + \lambda z) \\
s.t. \quad & A_1 y + \lambda_1 A_1 z \leq b_1 \\
& A_1 y + \lambda_2 A_1 z \leq b_1 \\
& (A_2 + \lambda_1 D)(y + \lambda_1 z) \leq b_2 \\
& (A_2 + \lambda_2 D)(y + \lambda_2 z) \leq b_2 \\
& A_2 y + D z \lambda_1 \lambda_2 + (D y + A_2 z)\frac{\lambda_1 + \lambda_2}{2} \leq b_2
\end{aligned}
$$

Two optimization variable robust

Robust bounds

$$
\begin{aligned}
\min \quad & \underline{c^t(y + \lambda z)} \\
s.t. \; & A_1 y + \lambda_1 A_1 z \leq b_1 \\
& A_1 y + \lambda_2 A_1 z \leq b_1 \\
& (A_2 + \lambda_1 D)(y + \lambda_1 z) \leq b_2 \\
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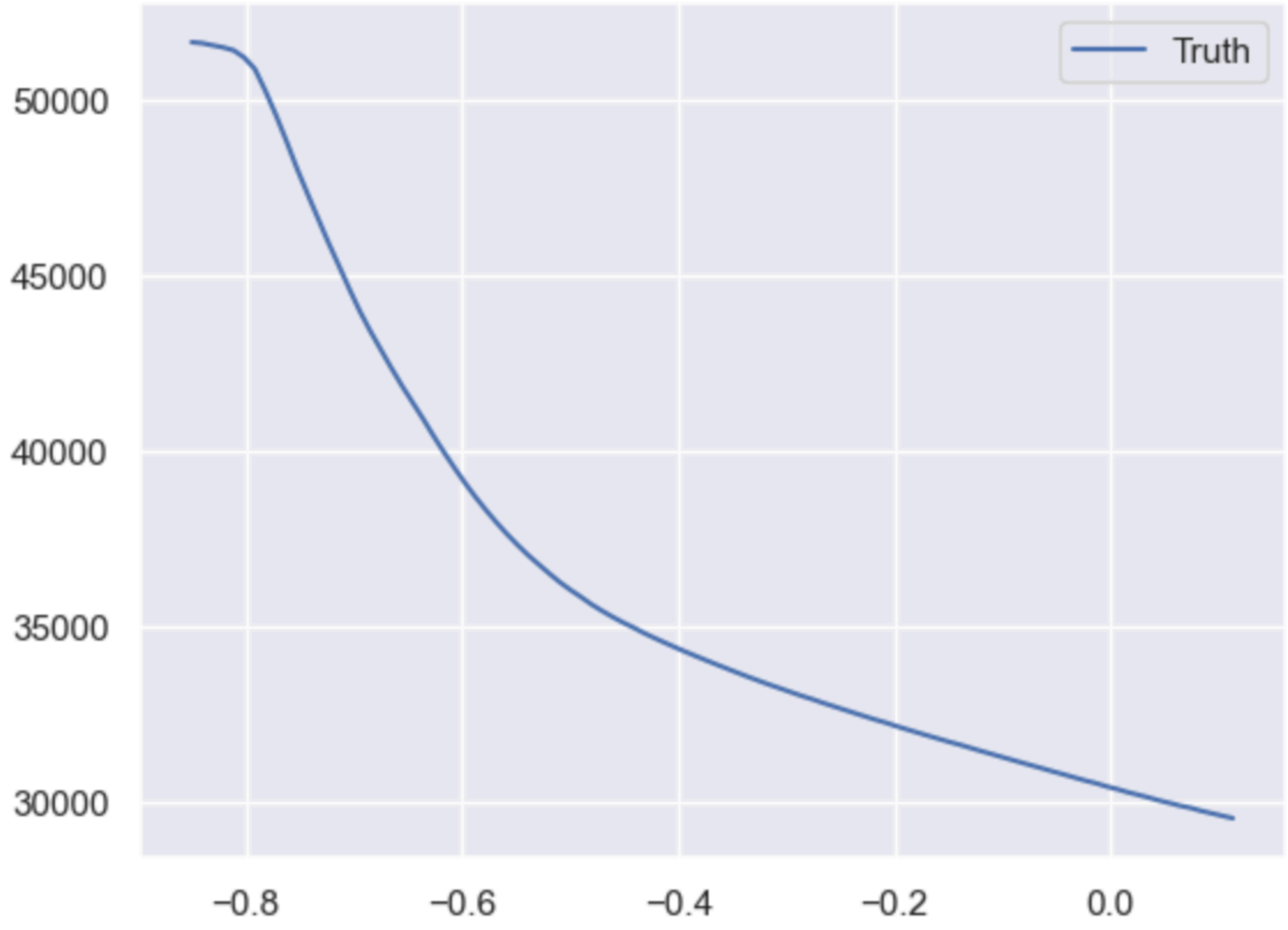
& A_2 y + D z \lambda_1 \lambda_2 + (D y + A_2 z)\frac{\lambda_1 + \lambda_2}{2} \leq b_2
\end{aligned}
$$



Two optimization variable robust

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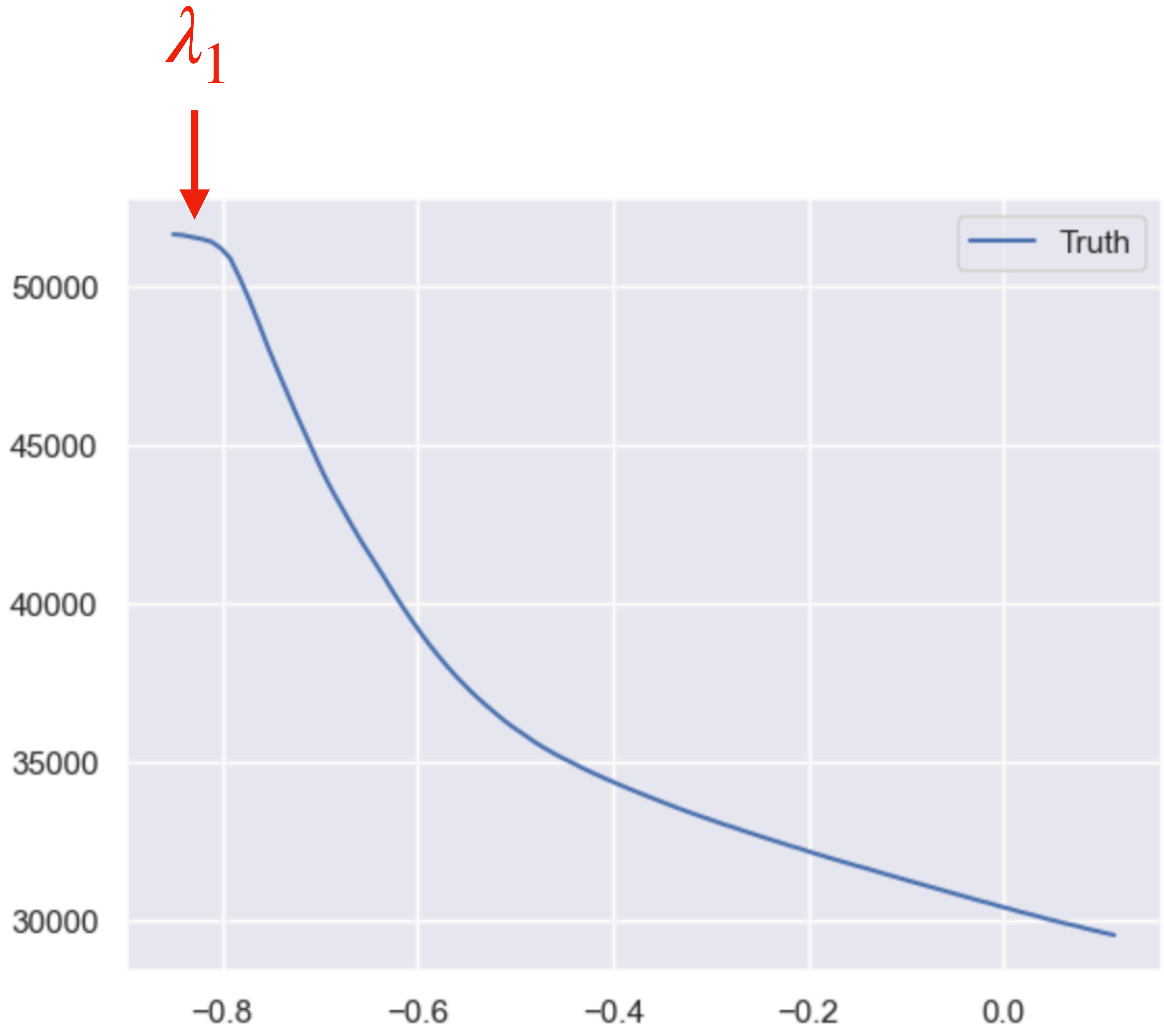
$$
\begin{aligned}
\min \quad & \underline{c^t(y + \lambda z)} \\
s.t. \; & A_1 y + \lambda_1 A_1 z \leq b_1 \\
& A_1 y + \lambda_2 A_1 z \leq b_1 \\
& (A_2 + \lambda_1 D)(y + \lambda_1 z) \leq b_2 \\
& (A_2 + \lambda_2 D)(y + \lambda_2 z) \leq b_2 \\
& A_2 y + D z \lambda_1 \lambda_2 + (Dy + A_2 z)\frac{\lambda_1 + \lambda_2}{2} \leq b_2
\end{aligned}
$$



Two optimization variable robust

Robust bounds

$$\min \quad \underline{c^t(y + \lambda z)}$$

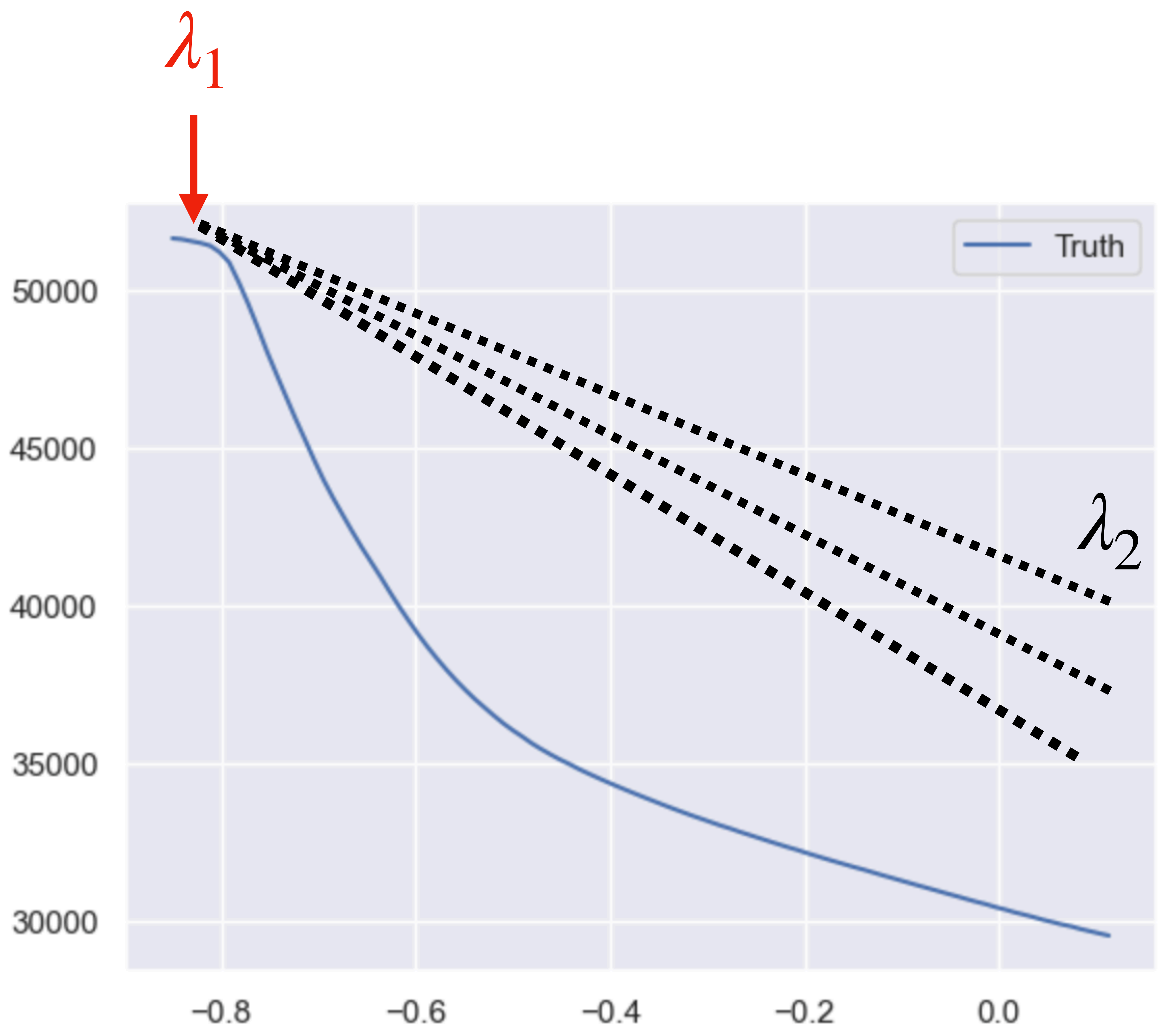
$$s.t. A_1 y + \lambda_1 A_1 z \leq b_1$$

$$A_1 y + \lambda_2 A_1 z \leq b_1$$

$$(A_2 + \lambda_1 D)(y + \lambda_1 z) \leq b_2$$

$$(A_2 + \lambda_2 D)(y + \lambda_2 z) \leq b_2$$

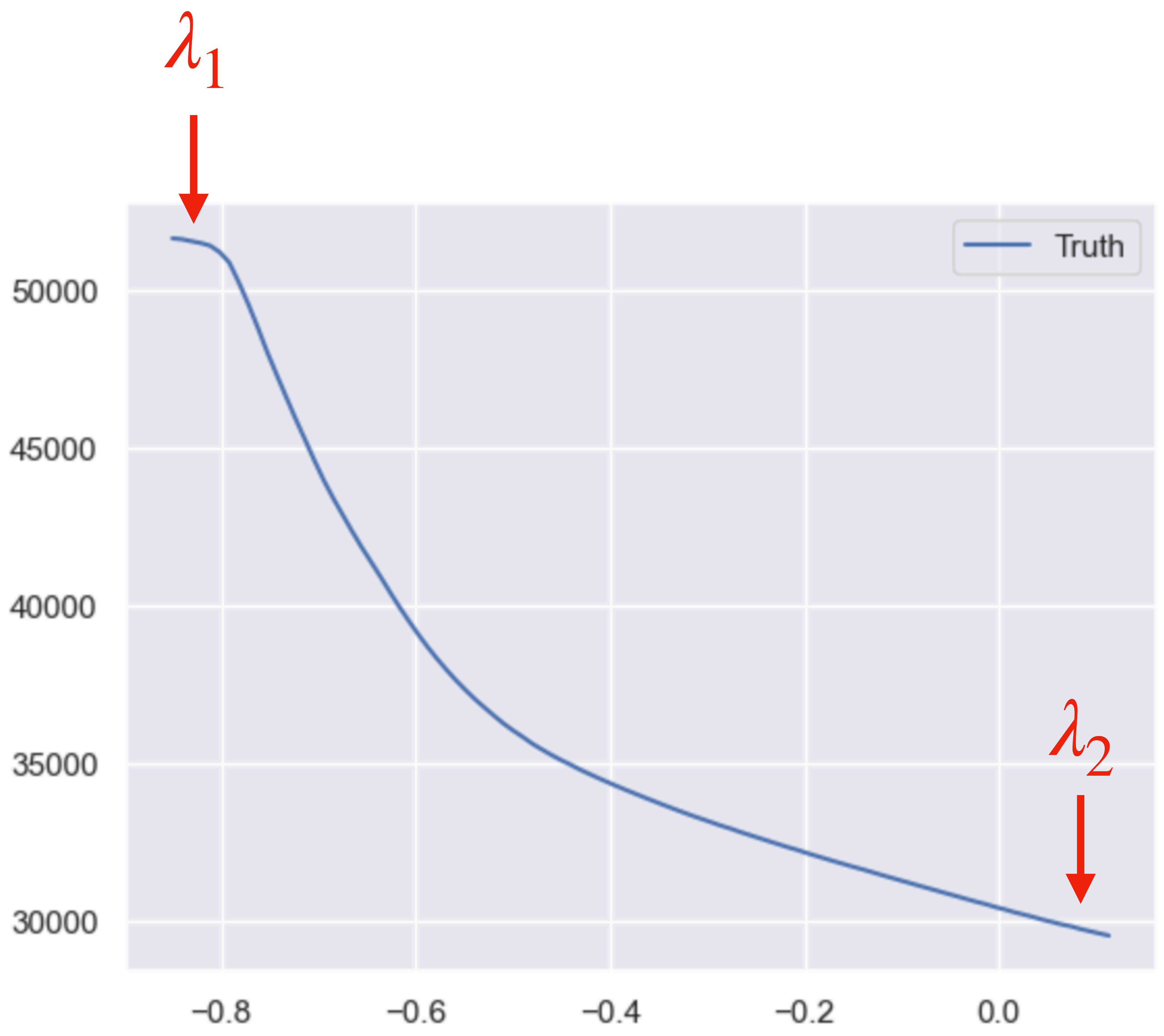
$$A_2 y + Dz\lambda_1\lambda_2 + (Dy + A_2 z)\frac{\lambda_1 + \lambda_2}{2} \leq b_2$$



Fixed-slope variable robust

Robust bounds

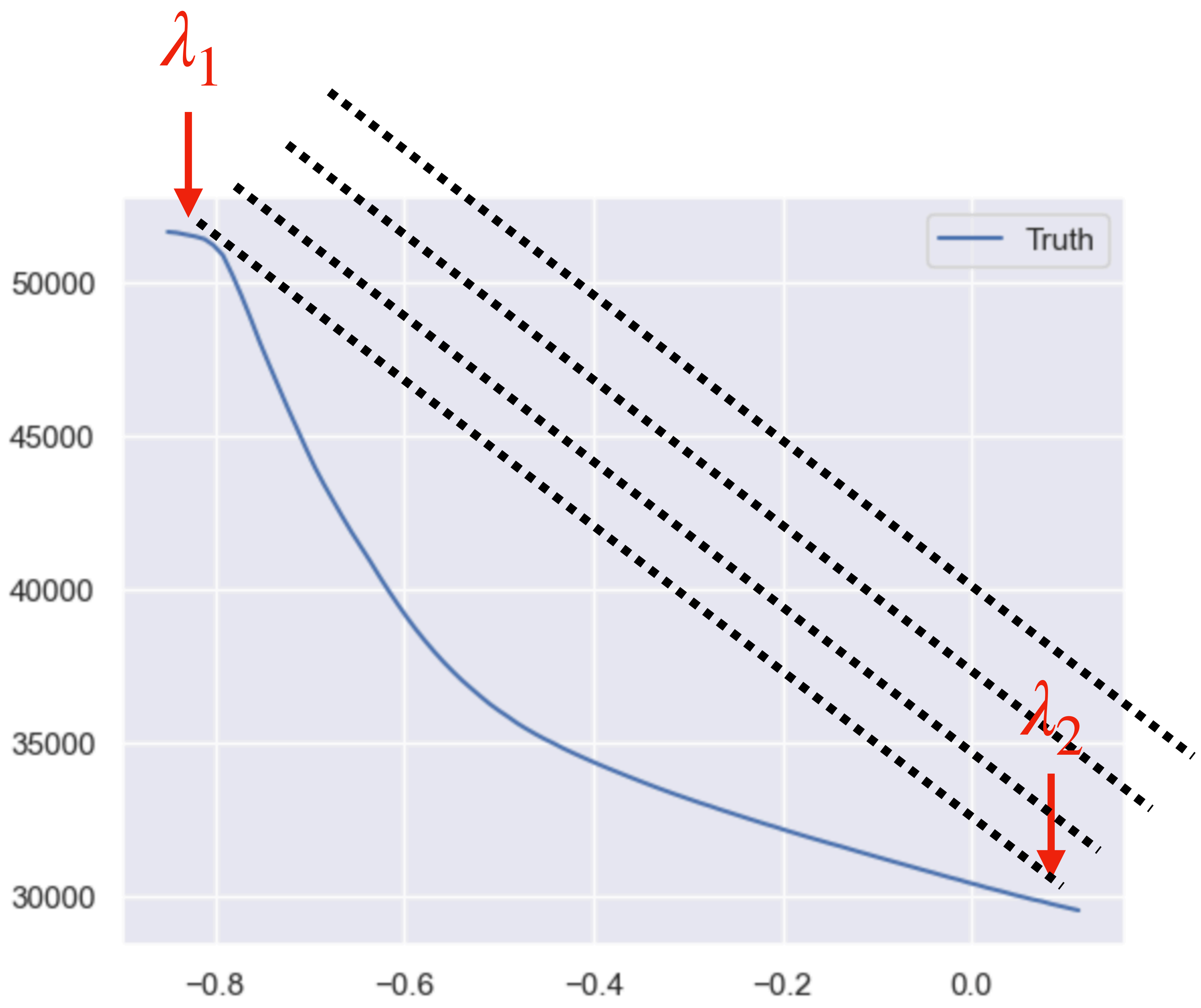
$$
\begin{aligned}
\min \quad & \underline{c^t(y+\lambda z)} \\
s.t.\, & A_1 y + \lambda_1 A_1 z \le b_1 \\
& A_1 y + \lambda_2 A_1 z \le b_1 \\
& (A_2 + \lambda_1 D)(y + \lambda_1 z) \le b_2 \\
& (A_2 + \lambda_2 D)(y + \lambda_2 z) \le b_2 \\
& A_2 y + D z \lambda_1 \lambda_2 + (Dy + A_2 z)\frac{\lambda_1 + \lambda_2}{2} \le b_2
\end{aligned}
$$



Fixed-slope variable robust

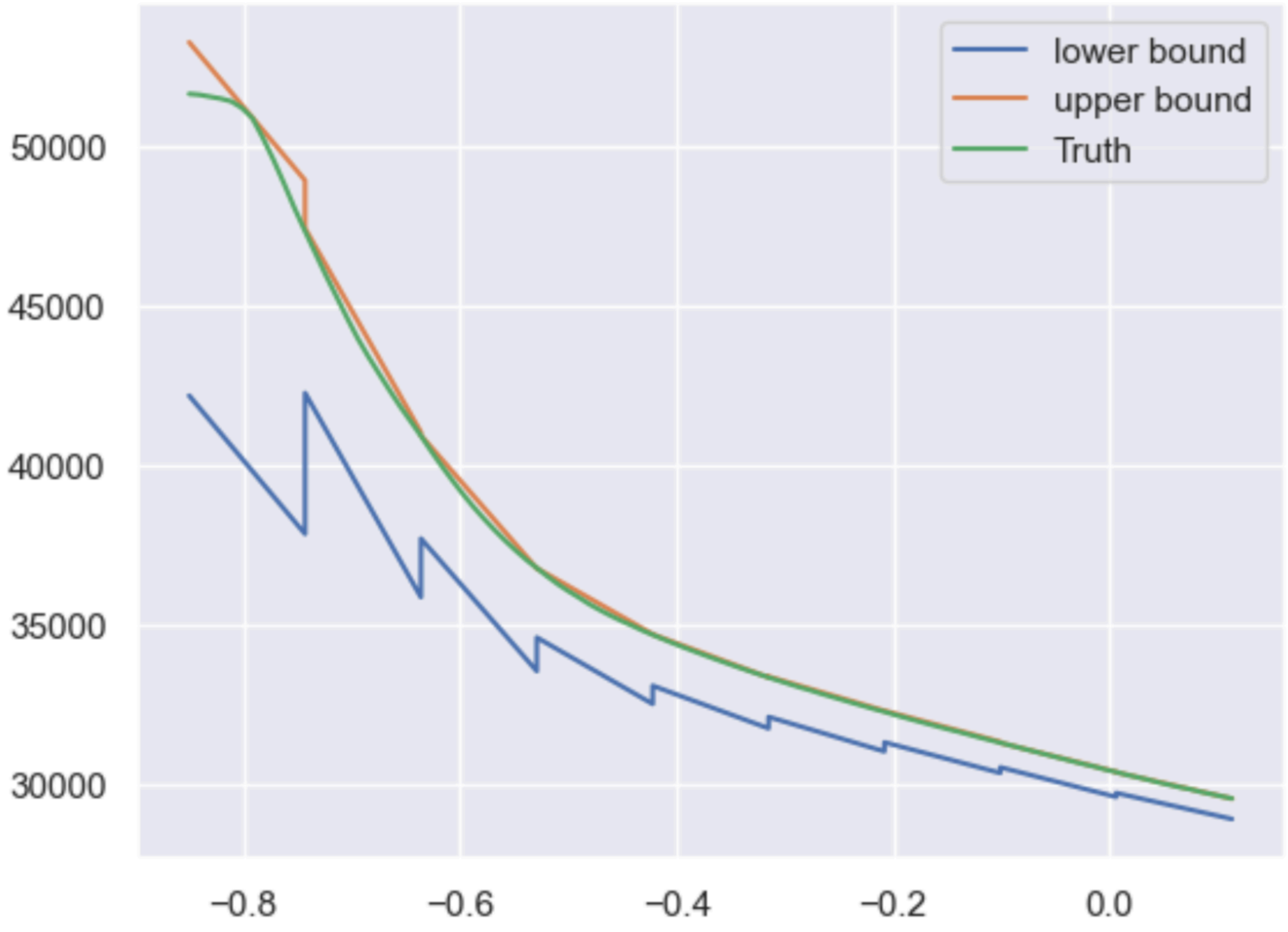
Robust bounds

$$
\begin{aligned}
\min \quad & \underline{c^t(y + \lambda z)} \\
s.t. \; & A_1 y + \lambda_1 A_1 z \leq b_1 \\
& A_1 y + \lambda_2 A_1 z \leq b_1 \\
& (A_2 + \lambda_1 D)(y + \lambda_1 z) \leq b_2 \\
& (A_2 + \lambda_2 D)(y + \lambda_2 z) \leq b_2 \\
& A_2 y + D z \lambda_1 \lambda_2 + (Dy + A_2 z)\frac{\lambda_1 + \lambda_2}{2} \leq b_2
\end{aligned}
$$



Variable robust solution

Robust bounds



Variable robust solution

Robust bounds

- If we note by n_1 and n_2 the number of constraints in A_1 and A_2 respectively and m the number of variables in $f(\lambda)$
- The variable robust bound reformulates the problem using $2n_1 + 3 * n_2$ constraints and $2m$ variables
- (+) Significantly tighter
- (-) Computationally more costly

Lagrangian bound

Lagrangian relaxation

- Let us consider $\lambda \in [\lambda_1, \lambda_2]$

$$h(\lambda, \alpha) = \min \quad c^t x + \alpha(A_2 x + \lambda D x - b_2) \quad s.t. \quad A_1 x \leq b_1$$

$$\leq$$

$$f(\lambda) = \min \quad c^t x \quad s.t. \quad A_1 x \leq b_1, \quad A_2 x + \lambda D x \leq b_2$$

- If we consider α_1 the optimal dual variables of the constraints $A_2 x + \lambda D x \leq b_2$ for $\lambda = \lambda_1$ and α_2 for $\lambda = \lambda_2$, we have

$$\max(\min(f(\lambda_1), h(\alpha_1, \lambda_2)), \min(f(\lambda_2), h(\alpha_2, \lambda_1))) \leq \min f(\lambda)$$

Lagrangian bound

Lagrangian relaxation

- If we note by n_1 and n_2 the number of constraints in A_1 and A_2 respectively and m the number of variables in $f(\lambda)$
- The reformulation has n_1 constraints and m the number of variables
- BUT requires 4 optimizations
- (-) Rarely exists

Benchmarking

Protocol

- Benchmark on an energy system
    - Uncertainty on electrolysis efficiency
- Problem is made-up of
    - 1.98M variables
    - 4.04M constraints
    - 35040 constraints depending on lambda

Benchmarking

Per point/bound

Time in seconds	Primal	Dual
Computing one point	200s	200s
Robust constant	210s	110s
Robust variable	1565s	1542s
Lagrangian	Na	Na

Summary

Bounds for parametric linear problems

- These bounds offer an overview of the evolution of the objective with respect to the modification
- There is a tradeoff between tightness and time
- Can help practitioners identify problematic behaviors that need further assessment
- More to come