Different forms of the LBB condition for saddle point problems

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1 Introduction

The idea of the present paper is born from the reading of very interesting papers of Amrouche, Ciarlet and Mardane [1, 2] concerning the equivalence of a lot of properties related to Lions' lemma. All these properties are intimately related to Stokes' problem in fluid mechanics, one of them being the LBB condition of this problem. This led us to investigate the following question : are there similar results in the abstract case of a general saddle point problem ? We found that the answer is positive, and this is the object of the present paper. After obtention of the abstract results, a final section of this paper is devoted to Stokes' problem, in order to demonstrate the power of the abstract theory.

2 Frame of the present study

The LBB condition appears in saddle point problems of the following form. Let V and P be two Hilbert spaces, with scalar products $(u,v)_V$ and $(p,q)_P$. Let $a(u,v)$ be a symmetric bilinear form on $V \times V$ and $b(p,v)$ a bilinear form on $P \times V$, both continuous. Considering $f \in V'$ and $g \in P'$, the problem is as follows *Find $u \in V$ and $p \in P$ such that*

$$\forall v \in V \quad a(u,v) + b(p,v) = f(v) \tag{1}$$

$$\forall q \in P \quad b(q,v) = g(v) \tag{2}$$

Let Z be the subspace of V where $b(q,v) = 0$ whatever $q \in P$. It is well known [5, 6, 3] that this problem admits a unique solution if the following conditions are verified :

$$\forall u \in Z_\perp \qquad a(u,u) \geq \alpha \|u\|_V^2 \quad \text{(ellipticity)} \tag{3}$$

$$\forall p \in P \quad \sup_{v \in V - \{0\}} \frac{b(p,v)}{\|v\|_V} \geq \beta \|p\|_P \quad \text{(LBB)} \tag{4}$$

where α and β are strictly positive. Condition 4 is known as *LBB condition.* The purpose of the present paper is to highlight the equivalence of this condition with other ones.

3 Operators B and B'

It was said above that the bilinear form b is bounded. Precisely,

$$\forall p \in P \quad \text{and} \quad \forall v \in V \quad |b(p,v)| \leq N\|p\|_P\|v\|_V \tag{5}$$

For a given $p \in P$, $b(p,v)$ may be seen as a linear functional on V. From Riesz's representation theorem, this functional may be set on the form $(Bp,v)_V$. This defines an operator B which is bounded as

$$\|Bp\|_V = \sup_{v\in V-\{0\}} \frac{(Bp,v)_V}{\|v\|_V} = \sup_{v\in V-\{0\}} \frac{b(p,v)}{\|v\|_V} \leq N\|p\|_P$$

Conversely, for a given $v \in V$, $b(p,v)$ may be considered as a functional on P, and therefore written in the form $(p,B'v)_P$. The operator B' verifies

$$\|B'u\|_P = \sup_{q\in P-\{0\}} \frac{(q,B'u)_P}{\|q\|_P} = \sup_{q\in P-\{0\}} \frac{b(q,u)}{\|q\|_P} \leq N\|u\|_V$$

The above defined subset Z is then the kernel of B'. The LBB condition may be written

$$\sup_{v\in V-\{0\}} \frac{(Bp,v)_V}{\|v\|_V} \equiv \|Bp\|_V \geq \beta\|p|_P \quad \text{(LBB2)} \tag{6}$$

Here, a remark has to be done. *The LBB condition cannot be verified if* $\ker(B) \neq \{0\}$. This constitute a compatibility condition for space P :

Theorem 1 (Compatibility condition for P) *Space P has to comply to the following compatibility condition :*

$$\ker(B) = \{0\} \tag{7}$$

Practically, if this condition is not verified, P has to be restricted to $\ker(B)\perp$.

4 The DR condition

Theorem 2 *The LBB condition implies the DR condition which is as follows :*
DR condition : *To any element w of $Z_\perp$, it is possible to associate an element $p \in P$ such that $w = Bp$, and $\|w\|_V \geq \beta\|p\|_P$*

Proof - Trying to minimize the squared distance

$$\|Bp - w\|_V^2 = (Bp,Bp)_V - 2(Bp,w)_V + \|w\|_V^2$$

leads to the following variational problem :
Find $p \in P$ such that, whatever be $q \in P$, on has

$$(Bp,Bq)_V = (w,Bq)_V \tag{8}$$

The bilinear form $(Bp,Bq)_V$ is clearly bounded, and it is elliptic as $\|Bp\|_V^2 \geq \beta^2\|p\|_P$ from the LBB condition. So, our problem admits an unique solution $p \in P$ which verifies

$$\forall q \in P \quad 0 = (Bp - w, Bq)_V = (B'(Bp - w), q)_P$$

that is to say

$$B'(Bp - w) = 0$$

or $Bp - w \in Z$. Now, calculating the obtained distance, one obtains

$$d^2 = (Bp - w, Bp)_V - (Bp - w, w)_V = 0$$

as the first term vanishes by construction and the second one is the scalar product of $Bp - w \in Z$ and $w \in Z_\perp$. Thus, $Bp = w$. Finally, (8) implies

$$\|Bp\|_V^2 = (w, Bp)_V \leq \|w\|_V \|Bp\|_V$$

i.e. $\|Bp\|_V \leq \|w\|_V$ and from the LBB condition,

$$\|w\|_V \geq \beta \|p\|_P$$

as announced.

Remark - Let us consider a functional $f \in V'$ whose kernel is Z. Then, its Riez representation $\mathcal{R}f$ in V is necessarily in $Z_\perp$. This implies that there exists a $p \in P$ such that $\mathcal{R}f = Bp$. And reciprocally, a functional whose Riesz representation lies in $Z_\perp$ necessarily vanishes on Z. Moreover, one has

$$\|f\|'_V = \|\mathcal{R}f\|_V \geq \beta \|p\|_P$$

We have thus obained the

Theorem 3 *The DR condition is equivalent to the following one*
DR' condition : *Any functional $f \in V'$ whose kernel is Z verifies $\mathcal{R}f = Bp$ with $p \in P$, and $\|f\|_{V'} \geq \beta \|p\|_P$*

5 The NORB condition

Theorem 4 *The DR condition implies the NORB condition which is as follows :*
NORB condition : *In $Z_\perp$, the following inequality holds*

$$\|B'w\|_P \geq \beta \|w\|_V \tag{9}$$

Proof - Starting from

$$\|B'w\|_P = \sup_{p \in P - \{0\}} \frac{(B'w, q)_P}{\|q\|_P} = \sup_{p \in P - \{0\}} \frac{(w, Bq)_V}{\|q\|_P}$$

the DR condition implies that there exists an element $p \in P$ such that $w = Bp$ and $\|w\|_V \geq \beta \|p\|_P$. With this choice, the above supremum is not necessarily reached, so that

$$\|B'w\|_P \geq \frac{\|w\|_V^2}{\|p\|_P} \geq \beta \|w\|_V$$

as announced.

6 The ONTO condition

Theorem 5 *The NORB condition implies the ONTO condition which is as follows :*

ONTO condition : *Whatever be $p \in P$, there exists an element $w \in Z_\perp$ such that*

$$B'w = p \quad \textit{and} \quad \|w\|_V \leq \frac{1}{\beta}\|p\|_P \tag{10}$$

Proof - Let us minimize the squared distance

$$\|B'u - p\|_P^2 = \|B'u\|_P^2 - 2(p, B'u)_P + \|p\|_P^2$$

on $Z_\perp$. This leads to the following variational problem :

Find $w \in Z_\perp$ such that

$$\forall v \in Z_\perp \quad (B'w, B'v)_P - (p, B'v)_P = 0 \tag{11}$$

This problem is well posed as from the NORB condition, $\|B'u\|_P^2 \geq \beta^2\|u\|_V^2$ in $Z_\perp$. Its unique solution w verifies

$$\forall v \in Z_\perp \quad (B'w - p, B'v)_P = 0$$

Now, any element $z \in V$ may be decomposed as

$$z = z_1 + z_2 \quad \text{with} \quad z_1 \in Z_\perp,\, z_2 \in Z$$

From the above condition, $(B'w - p, B'z_1)_P = 0$. As for z_2, one has $B'z_2 = 0$. Thus one may write

$$\forall z \in V \quad (B'w - p, B'z)_P = 0$$

or equivalently

$$\forall z \in V \quad (B(B'w - p), z)_V = 0$$

which implies

$$B(B'w - p) = 0$$

So, $B'w - p \in \ker(B) = \{0\}$ from our hypotheses, and $B'w = p$. Finally, from the NORB condition,

$$\|w\|_V \leq \frac{1}{\beta}\|B'w\|_P \equiv \frac{1}{\beta}\|p\|_P$$

as announced.

7 Returning to the LBB condition

We will now prove the following theorem :

Theorem 6 *The ONTO condition implies the LBB condition.*

Proof - In the expression

$$Q(p,v) = \frac{(p, B'v)_P}{\|v\|_V}$$

one may, from the ONTO condition, choose a particular $w \in Z_\perp$ such that $B'w = p$ and $\|w\|_V \leq \frac{1}{\beta}\|p\|_P$. This choice leads to

$$Q(p,w) = \frac{\|p\|_P^2}{\|w\|_V} \geq \beta \frac{\|p\|_P^2}{\|p\|_P} \equiv \beta \|p\|_P$$

It is cleat that

$$\sup_{v \in V - \{0\}} Q(p,v) \geq Q(p,w) \geq \beta \|p\|_P$$

which is the LBB condition.

8 Equivalence theorem

We have thus proved the following implications

$$LBB \Rightarrow DR \Rightarrow NORB \Rightarrow ONTO \Rightarrow LBB \tag{12}$$

so that the following theorem may be written :

Theorem 7 *The four conditions LBB, DR, NORB and ONTO are equivalent.*

9 The case of Stokes' problem

As an illustration, let us consider Stoke's problem with zero speeds on the boundary. For this problem, in space $\mathbb{R}^n$,

$$V = (H_0^1(\Omega))^n \tag{13}$$
$$V' = (H^{-1}(\Omega))^n \tag{14}$$
$$P = L_0^2(\Omega) = \{p \in L^2(\Omega) \mid \int_\Omega p dx = 0\} \tag{15}$$
$$a(\mathbf{u},\mathbf{v}) = \mu \int_\Omega D_i u_j D_i v_j dx \tag{16}$$
$$b(p,\mathbf{v}) = \int_V p \operatorname{div} \mathbf{v} dx \tag{17}$$

Here and in what follows, use is made of Einstein's summation convention. On $L_0^2(\Omega)$, we will naturally use the classical scalar product

$$(p,q) = \int_\Omega pq dx \tag{18}$$

The simplest choice for a scalar product on $(H_0^1(\Omega))^n$ is

$$(\mathbf{u},\mathbf{v})_1 = (D_j u_i, D_j v_i) \tag{19}$$

to which corresponds the norm

$$|\mathbf{u}|_1^2 = (D_j u_i, D_j u_i) \tag{20}$$

With this choice, Riesz' representation theorem acquires a very simple meaning. Indeed,

$$(\mathcal{R}\mathbf{f}, \mathbf{v})_1 =< \mathbf{f}, \mathbf{v} >$$

is equivalent to

$$< f_j, v_j >= (D_i \mathcal{R} f_j, D_i v_j) = - < D_{ii} \mathcal{R} f_j, v_j >$$

and, this being true for any v_j,

$$-\Delta \mathcal{R}\mathbf{f} = \mathbf{f} \tag{21}$$

Considering the bilinear form b, one has

$$b(p, \mathbf{v}) = (p, \operatorname{div} \mathbf{v}) = - < D_i p, v_i >= -(\mathcal{R} D_i p, v_i)_1$$

that is to say

$$\begin{aligned} Bp &= -\mathcal{R}\mathbf{grad} p & (22)\\ B'\mathbf{u} &= \operatorname{div} \mathbf{u} & (23) \end{aligned}$$

Examining these results, one first see that $Z = \ker(B') = \ker(\operatorname{div})$ is the *incompressible subspace*. Secondly, $Bp = 0$ is equivalent to $\mathbf{grad} p = 0$ i.e. $p = constant$. This is why one has to work with $L_0^2(\Omega)$: one has to conform to the compatibility condition for space P. We are now able to see the different conditions of the problem

- LLB 1 condition :

$$\forall p \in L_0^2(\Omega) \qquad \sup_{\mathbf{v} \in (H_0^1(\Omega))^n - \{0\}} \frac{(p, \operatorname{div} \mathbf{v})}{|v|_1} \geq \beta \|p\|_{L_0^2(\Omega)} \tag{24}$$

- LLB2 condition :

$$\|\mathcal{R}\mathbf{grad} p\|_{(H_0^1(\Omega))^n} \geq \beta \|p\|_{L_0^2(\Omega)}$$

  or equivalently,

$$\|\mathbf{grad} p\|_{(H^{-1}(\Omega))^n} \geq \beta \|p\|_{L_0^2(\Omega)} \tag{25}$$

  This is *Nečas inequality* [17]

- DR' condition : Any $\mathbf{f} \in (H^{-1}(\Omega))^n$ whose kernel is Z verifies

$$\mathcal{R}\mathbf{f} = \mathcal{R}\mathbf{grad} p$$

  or equivalently

$$\mathbf{f} = \mathbf{grad} p \quad \text{with} \quad p \in L_0^2(\Omega) \tag{26}$$

  and one has

$$\|\mathbf{f}\|_{(H^{-1}(\Omega))^n} \geq \beta \|p\|_{L_0^2(\Omega)} \tag{27}$$

  This result is generally cited as *de Rham's theorem*

- NORB condition :

$$\forall \mathbf{w} \in Z_\perp \quad \|\operatorname{div} \mathbf{w}\|_{L^2_0(\Omega)} \geq \beta \|\mathbf{w}\|_{(H^1_0(\Omega))^n} \tag{28}$$

- ONTO condition :

$$\forall p \in L^2_0(\Omega) \quad \exists \mathbf{w} \in Z_\perp \mid \operatorname{div} \mathbf{w} = p \quad \text{and} \quad \|w\|_{(H^1_0(\Omega))^n} \leq \frac{1}{\beta}\|p\|_{L^2_0(\Omega)} \tag{29}$$

From theorem (7), all these conditions are *equivalent*. So, the general theory directly gives the equivalence results presented by Amrouche, Ciarlet and Mardane [1, 2].

10 Conclusion

We have thus found that the equivalence of different conditions for Stokes' problem, which has been pointed by Amrouche, Ciarlet and Mardane, is not an isolated fact related to this concrete problem. Such an equivalence remains true in the general abstract frame. The application of the abstract results to Stokes'problem has been exposed as an illustration. Incidentally, our developments are largely founded on Hilbertian techniques which, in the present case, proved to be intuitive and powerful.

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