On an inequality on Sobolev spaces involving Nečas inequality and other results

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February 4, 2024

Abstract

This paper establishes an inequality on Sobolev spaces which implies Nečas inequality and related results, by using a method due to Friedrichs.

1 Introduction

In the years 1946-1947, Friedrichs [9, 10] presented an interesting way to obtain inequalities in the Sobolev spaces. The application of its method is restricted to domains which have particular but not too restrictive properties as will be described in what follows. We found interesting to investigate the possibility to obtain by this way the well known Nečas' inequality (equivalent to Lions' lemma [1, 2]). We were led to another inequality, not so known but containing Nečas' inequality as a consequence. The present paper presents the establishment of this inequality and some of its consequences, including Nečas inequality, Korn's inequality, and a lot of related results. As exposed in the frame of Hilbert spaces, our presentation is as intuitive as possible.

2 Notations

In this text,

- V is a bounded and connected open set of $\mathbb{R}^n$.
- S is the boundary of V.
- Vector and tensor fields are generally noted by a bold character.
- Einstein's summation convention on repeated indices is freely used, except otherwise stated ("*no summation on* i").
- $D_i = \frac{\partial}{\partial x_i}$
- $D_{ij} = D_i D_j$
- $\mathbf{L}^2(V)$ stands for $\left(L^2(V)\right)^n$ and in general, product spaces are noted by a bold letter.
- $f \in L_0^2(V)$ if $f \in L^2(V)$ and $\int_V f dV = 0$

- $\mathbf{H^1(V)} = \left(H^1(V)\right)^n$
- $\mathbf{H_0^1(V)} = \left(H_0^1(V)\right)^n$
- $\|f\|^2 = \int_V f^2 dV$
- $\|\mathbf{f}\|^2 = \sum_i \int_V f_i^2 dV$
- $(f, g) = \int_V fg dV$
- $(\mathbf{f}, \mathbf{g}) = \int_V f_i g_i dV$
- $\mathcal{D}(V)$ is the set of indefinitely differentiable functions having a compact support on V
- $\|f\| =$ norm in $L^2(V)$
- $(f, g) =$ scalar product in $L^2(V)$
- $|f|_1 =$ semi-norm in $H^1(V)$ i.e. $\sqrt{\int_V D_i f D_i f dV}$
- $(f, g)_1 =$ scalar product associated to this semi-norm
- $\|f\|_1 =$ norm in $H^1(V)$, that is $\|f\|_1^2 = \|f\|^2 + |f|_1^2$
- $((f, g))_1 =$ scalar product associated to this norm
- $|r| = \sqrt{r_{ij} r_{ij}}$

3 The main result

Theorem 1 *Let $\theta \in L_0^2(V)$ be a scalar field and $r_{ij} \in L^2(V)$ be the components of an antisymmetrical tensor field verifying the condition*

$$D_i r_{jk} + D_j r_{ki} + D_k r_{ij} = 0 \tag{1}$$

whatever be the indices i, j, k from 1 to n. Then, if these two fields verifie the relation

$$D_i \theta = D_j r_{ij} \tag{2}$$

one has

$$\|\theta\| \leq C \|r\| \tag{3}$$

Our proof follows the ideas of Fiedrichs [9, 10]. We think that this method, although a bit technical, is very intuitive, due to its essentially agebraic nature. In counterpart, restrictions have to be made on the open set but as shown by Friedrichs in the above references, these conditions are verified in a large class of practical cases.

4 Conditions on the open set V

The domain V has to be bounded and connected and possesses a uniform Ω property, as explained hereafter. This property was initially introduced by Friedrichs who used the name "**Ω**-domains".

4.1 The $\Omega(\eta)$-property

We will say that a domain V possesses the $\Omega(\eta)$-property if, noting

$$V_\eta = \{\mathbf{x} \,|\, d(\mathbf{x}, \complement V) > \eta\} \tag{4}$$

there exists a vector field $\mathbf{\Omega}$ such that [1] [2]

$$\begin{aligned} \mathbf{\Omega} &\quad \text{is} \quad \text{Lipshitz - continuous} && (5)\\ |D_i\Omega_j| &\underset{a.e.}{\leq} L \quad \forall(i,j) && (6)\\ D_i\Omega_i &\underset{a.e.}{\geq} 1 \text{ in } V - V_\eta && (7)\\ \mathbf{\Omega} &= 0 \text{ on the boundary of } V && (8) \end{aligned}$$

4.2 The uniform Ω-property

We will say that a domain possess an uniform Ω-property if there exists a $\delta > 0$ such that, for all $\eta \in]0, \delta]$, the $\Omega(\eta)$-property is verified, with a uniformly bounded Lipschitz constant L.

5 Function H_ε

We will also need a function H_ε such that

$$\begin{aligned} H_\varepsilon &\in C^\infty && (9)\\ H_\varepsilon &\geq 0 && (10)\\ H_\varepsilon &= 1 \text{ in } V_\varepsilon && (11)\\ H_\varepsilon &= 0 \text{ outside of } V_{\varepsilon/3} && (12)\\ |D^p H_\varepsilon| &\leq \frac{K_p}{\varepsilon^p} && (13) \end{aligned}$$

As is well known, such a function may be obtained with a mollifier, i.e. a positive radial function $\rho_\varepsilon \in C^\infty(\mathbb{R}^n)$ verifying

$$\begin{aligned} \rho_\varepsilon &= 0 \quad \text{for} \quad |x| \geq \varepsilon\\ \int_{|x|\leq\varepsilon} \rho_\varepsilon dV &= 1 \end{aligned}$$

If $\delta_{V_{2\varepsilon/3}}$ is the characteristic function of $V_{2\varepsilon/3}$, it is easy to verify [11] that a suitable H_ε is given by

$$H_\varepsilon = \delta_{V_{2\varepsilon/3}} * \rho_{\varepsilon/3}$$

6 Proof of the theorem

6.1 Harmonicity of both fields

A direct result of the hypotheses of the theorem is that both fields θ and r_{ij} are harmonic on Ω. At first, $D_{ij}r_{ij} = 0$ from the fact that the differential operator

[1] This is a slightly modified version of original Friedrichs' conditions. In fact, Friedrichs required that $\mathbf{\Omega}$ be of class C^1

[2] In what follows, the precision *almost everywhere* will be omitted.

is symmetric while the tensor is antisymmetric. Using (2) leads then to

$$D_{ii}\theta = D_{ij}r_{ij} = 0 \tag{14}$$

Now, from (1),

$$D_{ii}r_{jk} + D_{ij}r_{ki} + D_{ik}r_{ij} = 0$$

and combining this result with (2) leads to

$$0 = D_{ii}r_{jk} + D_j(D_k\theta) + D_k(-D_j\theta) = D_{ii}r_{jk} \tag{15}$$

So, both fields are harmonic, which implies that they are of class $C^\infty(\Omega)$ [11, 12].

6.2 Identities

Our proof will be based on two identities.

6.2.1 First identity

The fundamental relation (2) directly implies

$$\theta D_i\theta = \theta D_j r_{ij} = D_j\left(\theta r_{ij}\right) - r_{ij}D_j\theta \tag{16}$$

By a second use of (2), the last term of the second member may be transformed as follows

$$-r_{ij}D_j\theta = -r_{ij}D_k r_{jk} = -D_k\left(r_{ij}r_{jk}\right) + r_{jk}D_k r_{ij} \tag{17}$$

In order to evaluate the last term, let us start from the fact that (1) implies

$$\begin{aligned} 0 &= r_{jk}\left(D_k r_{ij} + D_i r_{jk} + D_j r_{ki}\right) \\ &= r_{jk}D_k r_{ij} + r_{jk}D_i r_{jk} + r_{jk}D_j r_{ki} \end{aligned}$$

The last term of the last member may be transformed as follows

$$r_{jk}D_j r_{ki} = r_{kj}D_k r_{ji} = -r_{jk}D_k r_{ji} = r_{jk}D_k r_{ij}$$

that is to say, it is equal to the first term of the same member. One obtains thus

$$\begin{aligned} 0 &= 2r_{jk}D_k r_{ij} + r_{jk}D_i r_{jk} \\ &= 2r_{jk}D_k r_{ij} + \frac{1}{2}D_i\left(r_{jk}r_{jk}\right) \end{aligned}$$

that is

$$r_{jk}D_k r_{ij} = -\frac{1}{4}D_i\left(r_{jk}r_{jk}\right) \tag{18}$$

Assembling results (16), (17) and (18) leads to the

First identity :

$$\begin{aligned} D_i\left(\theta^2\right) &= 2\theta D_i\theta \\ &= 2D_j\left(\theta r_{ij}\right) - 2D_k\left(r_{ij}r_{jk}\right) - \frac{1}{2}D_i\left(r_{jk}r_{jk}\right) \end{aligned} \tag{19}$$

6.2.2 Second identity

Our purpose is to obtain an identity concerning $D_i\theta D_i\theta$. From (2),

$$D_i\theta D_i\theta = D_j r_{ij} D_k r_{ik} \tag{20}$$

First partial result - One has

$$D_{kk}\left(r_{ij}r_{ij}\right) = 2D_k\left(r_{ij}D_k r_{ij}\right) = 2\left(D_k r_{ij}\right)\left(D_k r_{ij}\right) + 2r_{ij}D_{kk}r_{ij}$$

and the last term vanishes from the harmonicity of the r_{ij}, so that

$$\left(D_k r_{ij}\right)\left(D_k r_{ij}\right) = \frac{1}{2}D_{kk}\left(r_{ij}r_{ij}\right) \tag{21}$$

Second partial result - Let us compute

$$\begin{aligned} D_{jk}\left(r_{ij}r_{ik}\right) &= D_j\left(r_{ij}D_k r_{ik} + r_{ik}D_k r_{ij}\right) \\ &= \left(D_j r_{ij}\right)\left(D_k r_{ik}\right) + r_{ij}D_{jk}r_{ik} + \left(D_j r_{ik}\right)\left(D_k r_{ij}\right) + r_{ik}D_{jk}r_{ij} \end{aligned}$$

Now from (2),

$$r_{ij}D_{jk}r_{ik} = r_{ij}D_{ji}\theta = 0$$

and

$$r_{ik}D_{jk}r_{ij} = r_{ik}D_{ki}\theta = 0$$

as $D_{ij}\theta$ is symmetrical and r_{ij}, antisymmetrical. One is thus led to

$$D_i\theta D_i\theta = \left(D_j r_{ij}\right)\left(D_k r_{ik}\right) = D_{jk}\left(r_{ij}r_{ik}\right) - \left(D_j r_{ik}\right)\left(D_k r_{ij}\right) \tag{22}$$

Third partial result - Relation (1) visibly implies

$$\begin{aligned} 0 &= \left(D_i r_{jk} + D_j r_{ki} + D_k r_{ij}\right)\left(D_i r_{jk} + D_j r_{ki} + D_k r_{ij}\right) \\ &= (D_j r_{jk})(D_i r_{jk}) + (D_j r_{ki})(D_j r_{ki}) + (D_k r_{ij})(D_k r_{ij}) \\ &\quad +2\left((D_i r_{jk})(D_j r_{ki}) + (D_j r_{ki})(D_k r_{ij}) + (D_k r_{ij})(D_i r_{jk})\right) \end{aligned}$$

By modifying the name of summation indices, one can see that all squared terms are equal to $(D_k r_{ij})(D_k r_{ij})$ and that all double products reduce to $2(D_j r_{ki})(D_k r_{ij})$. Then, using result (21) leads to

$$\begin{aligned} 0 &= 3(D_k r_{ij})(D_k r_{ij}) + 6(D_j r_{ki})(D_k r_{ij}) \\ &= \frac{3}{2}D_{kk}(r_{ij}r_{ij}) - 6(D_j r_{ik})(D_k r_{ij}) \end{aligned}$$

that is to say,

$$(D_j r_{ik})(D_k rij) = -\frac{1}{4}D_{kk}(r_{ij}r_{ij}) \tag{23}$$

Assembling results (20), (21) and (23) finally gives the

Second identity :

$$D_i\theta D_i\theta = D_{jk}(r_{ij}r_{ik}) - \frac{1}{4}D_{kk}(r_{ij}r_{ij}) \tag{24}$$

6.3 Step 1 : an integration by parts

Multiplying the *first identity* by Ω_i and integrating leads to

$$\int_V [\Omega_i D_i(\theta^2) - 2\Omega_i D_j(\theta r_{ij}) + 2\Omega_i D_k(r_{ij} r_{jk} + \frac{1}{2}\Omega_i D_i(r_{jk} r_{jk}]dV = 0 \quad (25)$$

An integration by parts formally leads to

$$\mathcal{I} = \int_V [\theta^2 D_i \Omega_i - 2\theta r_{ij} D_j \Omega_i + 2 r_{ij} r_{jk} D_k \Omega_i + \frac{1}{2} r_{jk} r_{jk} D_i \Omega_i]dV = 0 \quad (26)$$

But the validity of this integration by parts is not obvious, as no hypothesis has been made on the behaviour of the derivatives of θ and $\mathbf{r}$ near the boundary. In order to circumvent this difficulty, let us consider any $\eta \in]0, \delta[$ and replace in (25) Ω_i by $H_\eta \Omega_i$. The integration by parts is then valid, as H_η vanishes in the vicinity of the boundary. The result is $\mathcal{I}_\eta = 0$, where

$$\mathcal{I}_\eta = \int_V [\theta^2 D_i(H_\eta \Omega_i) - 2\theta r_{ij} D_j(H_\eta \Omega_i) + 2 r_{ij} r_{jk} D_k(H_\eta \Omega_i) + \frac{1}{2} r_{jk} r_{jk} D_i(H_\eta \Omega_i)]dV \quad (27)$$

This integral may be splitted in two terms, namely $\mathcal{I}_{\eta 1} = \int_{V_\eta} ...dV$ and $\mathcal{I}_{\eta 2} = \int_{V - V_\eta} ...dV$. In the first integral, $H_\eta = 1$ and $\mathcal{I}_{\eta 1}$ has the same form as $\mathcal{I}$, but integrated on V_η. In the second one, observe that

$$\begin{aligned} |\Omega_i| &\leq L\eta \\ |H_\eta| &\leq K_0 \\ |D_i H_\eta| &\leq \frac{K_1}{\eta} \end{aligned}$$

from which

$$\begin{aligned} |D_k(H\eta \Omega_i)| &\leq |D_k H_\eta||\Omega_i| + |H\eta||D_k \Omega_i| \\ &\leq \frac{K_1}{\eta} L\eta + K_0 L = L(K_0 + K_1) = C_1 \end{aligned}$$

Therefore, one obtains, with the appropriate constants,

$$\begin{aligned} \mathcal{I}_{\eta 2} &\leq \int_{V - V_\eta} [C_2 \theta^2 + C_3 |r||\theta| + C_4 |r|^2 + C_5 |r|^2]dV \\ &\leq C_6 \int_{V - V_\eta} (\theta^2 + |r|^2)dV \end{aligned}$$

a quantity which vanishes if η tends to zero. Simultaneously, $\mathcal{I}_{\eta 1}$ converges to $\mathcal{I}$, so that (26) is true.

6.4 Step 2

Let us now adopt a fixed value of $\varepsilon \in]0, \delta[$. From (26), it is easy to deduce

$$\int_V \theta^2 D_i \Omega_i = 2\int_V \theta r_{ij} D_j \Omega_i dV - 2\int_V r_{ij} r_{jk} D_k \Omega_i dV - \frac{1}{2}\int_V r_{jk} r_{jk} D_i \Omega_i dV \quad (28)$$

which may be rewritten as

$$\int_{V-V_\varepsilon} \theta^2 D_i\Omega_i = -\int_{V_\varepsilon} \theta^2 D_i\Omega_i + 2\int_V \theta r_{ij} D_j\Omega_i dV \\ - 2\int_V r_{ij} r_{jk} D_k\Omega_i dV - \frac{1}{2}\int_V r_{jk} r_{jk} D_i\Omega_i dV \quad (29)$$

Since $D_i\Omega_i \geq 1$ in $V - V_\varepsilon$, one has

$$\int_{V-V_\varepsilon} \theta^2 D_i\Omega_i dV \geq \int_{V-V_\varepsilon} \theta^2 dV$$

As the derivatves of the Ω_i are bounded, this leads to the following evaluation :

$$\int_{V-V_\varepsilon} \theta^2 dV \leq C_7^* \int_{V_\varepsilon} \theta^2 dV + C_8\|\theta\|\|r\| + C_9\|r\|^2 \quad (30)$$

which also implies, with $C_7 = C_7^* + 1$,

$$\int_V \theta^2 dV \leq C_7 \int_{V_\varepsilon} \theta^2 dV + C_8\|\theta\|\|r\| + C_9\|r\|^2 \quad (31)$$

6.5 Step 3

It remains to evaluate the integral of θ^2 on V_ε. First, from Poincaré's inequality,

$$\int_{V_\varepsilon} \theta^2 dV \leq \frac{1}{V_\varepsilon} \left| \int_{V_\varepsilon} \theta dV \right|^2 + C_{10} \int_{V_\varepsilon} D_i\theta D_i\theta dV \quad (32)$$

Now, it has to be noted that equation (2) remains valid if a constant is added to θ. For sake of simplicity, we will *momentarily* fix this constant by setting

$$\int_{V_\varepsilon} \theta dV = 0 \quad (33)$$

Starting from the *second inequality*

$$D_i\theta D_i\theta = D_{jk}(r_{ij} r_{ik}) - \frac{1}{4} D_{kk}(r_{ij} r_{ij})$$

let us multiply this result by H_ε and integrate on V. This leads to

$$\int_V H_\varepsilon D_i\theta D_i\theta dV = \int_V H_\varepsilon D_{jk}(r_{ij} r_{ik}) dV - \frac{1}{4}\int_V H_\varepsilon D_{kk}(r_{ij} r_{ij}) dV$$

After a double integration by parts, the second member of this equality reduces to

$$\int_V r_{ij} r_{ik} D_{jk} H_\varepsilon dV - \frac{1}{4}\int_V r_{ij} r_{ij} D_{kk} H_\varepsilon dV \quad (34)$$

and, since the derivatives of H_ε are bounded, the absolute value of this expression admits a bound of the form $C_{11}\|r\|^2$. Owing to the fact that H_ε is a positive function and equals 1 in V_ε, one has

$$\int_V H_\varepsilon D_i\theta D_i\theta dV \geq \int_{V_\varepsilon} D_i\theta D_i\theta dV$$

so that, finally,

$$\int_{V_\varepsilon} D_i\theta D_i\theta dV \leq C_{11}\|r\|^2 \tag{35}$$

Adding this result to (31) leads to

$$\|\theta\|^2 \leq C_8\|\theta\|\|r\| + C_{12}\|r\|^2$$

which, from the classical theory of second order inequations, leads to

$$\|\theta\| \leq \frac{C_8\|r\| + \sqrt{C_8^2\|r\|^2 + 4C_{12}\|r\|^2}}{2} = C_{13}\|r\| \tag{36}$$

6.6 Step 4 : Adjusting the condition on the mean of θ

In the preceeding step, use was made of the unnatural condition $\int_{V_\varepsilon} \theta dV = 0$. Let us consider a function θ whose integral on V is zero, and which verifies the fundamental relation (2). Any θ^* of the form $\theta^* = \theta - c$, where c is a constant, also verifies the fundamental relation. A possible choice is

$$0 = \int_{V_\varepsilon} \theta^* dV = \int_{V_\varepsilon} \theta dV - cV_\varepsilon$$

i.e.

$$c = \frac{1}{V_\varepsilon}\int_{V_\varepsilon} \theta dV$$

As just proved above, this θ^* verifies

$$\|\theta^*\|^2 \leq C_{13}^2\|r\|^2$$

Now

$$\begin{aligned} \int_V {\theta^*}^2 dV &= \int_V \theta^2 dV - 2c\int_V \theta dV + c^2 V \\ &= \int_V \theta^2 dV + c^2 V \\ &\geq \int_V \theta^2 dV \end{aligned}$$

so that it is also true that

$$\|\theta\| \leq C_{13}\|r\|$$

and theorem (1) is proved.

6.7 Case where θ is not of zero mean

When relation (2) is verified but $\bar{\theta} = \frac{1}{V}\int_V \theta dV \neq 0$, it is clear that $\hat{\theta} = \theta - \bar{\theta}$ verifies the conditions of theorem (1), so that

$$\|\hat{\theta}\| \leq C_{13}\|r\|$$

As

$$\|\theta\|^2 = \|\hat{\theta}\|^2 + \bar{\theta}^2 V$$

the general result is

$$\|\theta\|^2 \leq \bar{\theta}^2 V + C_{13}^2\|r\|^2 \tag{37}$$

7 Equation $\Delta\mathbf{u} = \mathbf{grad}p$

7.1 Preliminaries

Let us first recall the following expression of the laplacian of a vector field

$$\begin{aligned} \Delta u_i &= D_{jj}u_i \\ &= D_j(D_j u_i - D_i u_j) + D_{ij}u_j \\ &= D_i(\operatorname{div}\mathbf{u}) - D_j\omega_{ij}(\mathbf{u}) \end{aligned} \tag{38}$$

where

$$\omega_{ij}(\mathbf{u}) = D_i u_j - D_j u_i \tag{39}$$

For sake of brevity, we often will note ω_{ij} in place of $\omega_{ij}(\mathbf{u})$ when no confusion is possible. Visibly,

$$D_i\omega_{jk} + D_j\omega_{ki} + D_k\omega_{ij} = D_{ij}u_k - D_{ik}u_j + D_{jk}u_i - D_{ji}u_k + D_{ki}u_j - D_{kj}u_i = 0 \tag{40}$$

i.e. ω_{ij} verifies condition (1).
Now, it's a classical result [21] that in $\mathbf{H}_0^1(V)$,

$$|\mathbf{u}|_1^2 = \int_V D_j u_i D_j u_i dV$$

is equivalent to the natural norm of $\mathbf{H}^1(V)$

$$\|\mathbf{u}\|_1^2 = \|\mathbf{u}\|^2 + |\mathbf{u}|_1^2$$

Moreover, in $\mathbf{H}_0^1(V)$, the following identity holds :

$$\begin{aligned} |\mathbf{u}|_1^2 &= \int_V (|\operatorname{div}\mathbf{u}|^2 + \frac{1}{2}\omega_{ij}\omega_{ij})dV \\ &= \|\operatorname{div}\mathbf{u}\|^2 + \frac{1}{2}\|\omega\|^2 \end{aligned} \tag{41}$$

Indeed, for any vector field $\mathbf{e}$ whose components lie in $\mathcal{D}(V)$, one has

$$\begin{aligned} \int_V D_j e_i D_j e_i dV &= -\int_V e_i D_{jj} e_i dV \\ &= -\int_V e_i D_{ij} e_j dV + \int_V e_i D_j \omega_{ij}(\mathbf{e}) dV \\ &= \int_V D_i e_i D_j e_j dV - \int_V \omega_{ij}(\mathbf{e}) D_j e_i dV \end{aligned}$$

In the last term, renaming summation indexes gives

$$\begin{aligned} \omega_{ij}(\mathbf{e})D_j e_i &= \frac{1}{2}(\omega_{ij}(\mathbf{e})D_j e_i + \omega_{ji}(\mathbf{e})D_i e_j) \\ &= \frac{1}{2}\omega_{ij}(\mathbf{e})(D_j e_i - D_i e_j) \\ &= -\frac{1}{2}\omega_{ij}(\mathbf{e})\omega_{ij}(\mathbf{e}) \end{aligned}$$

so that (41) is true for $\mathbf{e}$. This result may be extended to $\mathbf{H}_0^1(V)$ from classical density arguments. The same is true for the scalar product, that is, when fields $\mathbf{u}$ and $\mathbf{v}$ are elements of $\mathbf{H}_0^1(V)$,

$$(\mathbf{u}, \mathbf{v})_1 = \int_V \left(\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v} + \frac{1}{2}\omega_{ij}(\mathbf{u})\omega_{ij}(\mathbf{v})\right) dV \tag{42}$$

7.2 Formulation of the problem

Let us consider the following problem :

Given a field $p \in L^2_0(V)$*, find a vector field* $\mathbf{u} \in \mathbf{H}^1_0(V)$ *such that, whatever be* $\mathbf{v} \in \mathbf{H}^1_0(V)$*,*

$$(\mathbf{u}, \mathbf{v})_1 = (p, \operatorname{div} \mathbf{v}) \tag{43}$$

The main result is :

Theorem 2 *The solution* $\mathbf{u}$ *of problem (43) verifies*

$$\|p\| \leq \frac{1}{\beta} |\mathbf{u}|_1 \tag{44}$$

Indeed, this problem is equivalent to

$$\Delta u_i = D_i p$$

that is

$$D_i \operatorname{div} \mathbf{u} - D_j \omega_{ij}(\mathbf{u}) = D_i p$$

or

$$D_j \omega_{ij}(\mathbf{u}) = D_i \theta$$

with

$$\theta = \operatorname{div} \mathbf{u} - p$$

We are thus in the frame of theorem (1) so that there exists a constant C depending on V such that

$$\|\theta\| \leq C \|\omega\|$$

or explicitely,

$$\| \operatorname{div} \mathbf{u} - p\| \leq C \|\omega\|$$

But this implies

$$\begin{aligned} \|p\| &\leq \| \operatorname{div} \mathbf{u}\| + \|p - \operatorname{div} \mathbf{u}\| \\ &\leq \| \operatorname{div} \mathbf{u}\| + C\sqrt{2} \frac{1}{\sqrt{2}} \|\omega\| \\ &\leq \sqrt{1 + 2C^2} \sqrt{\| \operatorname{div} \mathbf{u}\|^2 + \frac{1}{2} \|\omega\|^2} \end{aligned}$$

which is the announced result.

7.3 Corollaries

Corollary 1 *The solution of problem (43) verifies the inequality*

$$\|p\|^2 \leq \frac{1}{\beta^2} \| \operatorname{div} \mathbf{u}\| \tag{45}$$

Indeed,

$$\|p\|^2 \leq \frac{1}{\beta^2} |\mathbf{u}|_1^2 = \frac{1}{\beta^2} (p, \operatorname{div} \mathbf{u}) \leq \frac{1}{\beta^2} \|p\| \| \operatorname{div} \mathbf{u}\|$$

Corollary 2 *The solution of problem (43) verifies the inequality*

$$|\mathbf{u}|_1 \leq \frac{1}{\beta} \| \operatorname{div} \mathbf{u} \| \tag{46}$$

Indeed,

$$|\mathbf{u}|_1^2 = (p, \operatorname{div} \mathbf{u}) \leq \|p\| \| \operatorname{div} \mathbf{u} \|$$

and using corollary (1) leads to

$$|\mathbf{u}|_1^2 \leq \frac{1}{\beta^2} \| \operatorname{div} \mathbf{u} \|^2$$

8 Nečas inequality

8.1 Establishment of the inequality

The natural norm for the elements $\mathbf{f}$ of the dual $\mathbf{H}^{-1}(V)$ of $\mathbf{H}_0^1$ is

$$|f|_{-1} = \sup_{\mathbf{v} \in \mathbf{H}_0^1(V), \neq 0} \frac{| < f, \mathbf{v} > |}{|\mathbf{v}[_1}$$

In the case where $\mathbf{f} = \mathbf{grad} p$, with $p \in L_0^2(V)$, one has

$$< \mathbf{grad} p, \mathbf{v} >= -(p, \operatorname{div} \mathbf{v})$$

so that

$$|\mathbf{grad} p|_{-1} = \sup_{\mathbf{v} \in \mathbf{H}_0^1(V), \neq 0} \frac{|(p, \operatorname{div} \mathbf{v})|}{|\mathbf{v}|_1}$$

Let us first suppose that the mean of p is zero. Considering the vector field $\mathbf{u} \in \mathbf{H}_0^1(V)$ that verifies $\Delta \mathbf{u} = \mathbf{grad} p$, one has

$$\sup_{\mathbf{v} \in \mathbf{H}_0^1(V), \neq 0} \frac{|(p, \operatorname{div} \mathbf{v})|}{|\mathbf{v}|_1} = \sup_{\mathbf{v} \in \mathbf{H}_0^1(V), \neq 0} \frac{|(\mathbf{u}, \mathbf{v})_1|}{|\mathbf{v}|_1} = |\mathbf{u}|_1$$

that is to say

$$|\mathbf{u}|_1 = |\mathbf{grad} p|_{-1}$$

From result (44) then follows the first form of Nečas' inaquality

Theorem 3 (Nečas' inequality 1) *For any $p \in L_0^2(V)$, one has*

$$\|p\| \leq \frac{1}{\beta} |\mathbf{grad} p|_{-1} \tag{47}$$

In the case where the mean of p does not vanish, $\hat{p} = p - \bar{p}$ has the same gradient as p so that

$$\|\hat{p}\| \leq \frac{1}{\beta} |\mathbf{grad} p|_{-1}$$

As $\|p\|^2 = \|\hat{p}\|^2 + \bar{p}^2 V$, one obtains the second form of Nečas' inequality

Theorem 4 (Nečas' inequality 2) *For any $p \in L^2(V)$, one has*

$$\|p\|^2 \leq \bar{p}^2 V + \frac{1}{\beta^2} |\mathbf{grad} p|_{-1}^2 \tag{48}$$

8.2 An equivalent form

It is easy to see that inequality (47) is equivalent to

$$\forall p \in L_0^2(V) \qquad \sup_{\mathbf{v}\in\mathbf{H}_0^1(V),\neq 0} \frac{(p, \operatorname{div}\mathbf{v})}{|\mathbf{v}|_1} \geq \beta \|p\| \tag{49}$$

This may also be written

$$\inf_{p\in L_0^2(V),\neq 0} \sup_{\mathbf{v}\in\mathbf{H}_0^1(V),\neq 0} \frac{(p, \operatorname{div}\mathbf{v})}{|\mathbf{v}|_1} \geq \beta \tag{50}$$

This is precisely the so-called LBB condition, also referred as inf-sup condition, which ensures the existence of the solution of Stokes'problem in fluid mechanics [5, 6, 3].

8.3 H^{-1}-norms of the derivatives of a $p \in L_0^2(V)$

We have seen that the element $\mathbf{u} \in \mathbf{H}_0^1(V)$ which is the solution of the problem

$$\forall \mathbf{v} \in \mathbf{H}_0^1(V) \quad (\mathbf{u}, \mathbf{v})_1 = (p, \operatorname{div} v) \tag{51}$$

verifies the relation

$$|\mathbf{u}|_1^2 = |\mathbf{grad} p|_1^2 = (p, \operatorname{div}\mathbf{u})$$

Now, this problem may be split in the n following ones :

Find $u_i \in H_0^1(V)$ such that

$$\forall v_i \in H_0^1(V) \quad (u_i, v_i)_1 = (p, D_i u_i) \quad (\textit{no summation on } i) \tag{52}$$

As $(p, D_1 u_1) = - < D_1 p, u_1 >$ etc., the obtained u_i verifies

$$|u_i|_1^2 = |D_i p|_{-1}^2 = (p, D_i u_i) \quad (\textit{no summation on } i) \tag{53}$$

As $\|D_1 u_1\|^2 \leq |u_1|_1^2, \|D_2 u_2\|^2 \leq |u_2|_1^2$, etc., one has

$$|u_i|^2 \leq \|p\| |u_i|_1$$

so that

$$|D_i p|_{-1} = |u_i|_1 \leq \|p\| \tag{54}$$

which is an important property :

Theorem 5 *The H^{-1}-norms of the derivatives of any $p \in L_0^2(V)$ are bounded by $\|p\|$*

Now, as $|\mathbf{u}|_1^2 = \sum_i |u_i|_1^2$, one has

$$|\mathbf{grad} p|_{-1}^2 = \sum_i |D_i p|_{-1}^2 \tag{55}$$

and Nečas' inequality may be expressed in the following form :

Theorem 6 (Nečas' inequality 3) *For any $p \in L^2(V)$, one has*

$$\|p\|^2 \leq \overline{p}^2 V + \frac{1}{\beta^2} \sum_i |D_i p|_{-1}^2 \tag{56}$$

9 The set $\operatorname{div}\mathbf{H}_0^1(V)$

9.1 Introduction

Let q be an element of $L_0^2(V)$ that verifies

$$\forall \mathbf{v} \in \mathbf{H}_0^1(V) \quad (q, \operatorname{div}\mathbf{v}) = 0$$

As

$$(q, \operatorname{div}\mathbf{v}) = - < \mathbf{grad}q, \mathbf{v} >$$

this mean that q vanishes. This result means that *the set* $\operatorname{div}(\mathbf{H}_0^1(V)$ *is dense in* $L_0^2(V)$. The question is now : *would it be true that both sets coincide ?* In order to answer to this question, we need to develop some tools.

9.2 Subspace $\mathbf{B}_0(V)$

Problem (43) consists to find $\mathbf{u} \in \mathbf{H}_0^1(V)$ such that for any $\mathbf{v} \in \mathbf{H}_0^1(V)$

$$(\mathbf{u}, \mathbf{v})_1 = (p, div\mathbf{v}) \equiv < -\mathbf{grad}p, v > \tag{57}$$

In other words, the solution $\mathbf{u}$ is the Riesz representation of the functional $(-\mathbf{grad}p)$, what will be noted $\mathbf{u} = -\mathcal{R}\mathbf{grad}p$. It is clear that

$$|\mathcal{R}\mathbf{grad}p|_1^2 = |(p, \operatorname{div}\mathcal{R}\mathbf{grad}p)| \leq \|p\| \|\operatorname{div}\mathcal{R}\mathbf{grad}p\| \leq \|p\| |\mathcal{R}\mathbf{grad}p|_1$$

from which

$$|\mathcal{R}\mathbf{grad}p|_1 \leq \|p\|$$

that is, operator $\mathcal{R}\mathbf{grad}$ is bounded. Let $\mathbf{B}_0(V) = \mathcal{R}\mathbf{grad}L_0^2(V)$ be its image in $\mathbf{H}_0^1(V)$. From the above proven fact (47) that

$$\|p\| \leq \frac{1}{\beta}|\mathbf{grad}p|_1 \equiv \frac{1}{\beta}|\mathcal{R}\mathbf{grad}p|_1$$

this operator is also invertible, so that $\mathbf{B}_0(V)$ is a closed subspace of $\mathbf{H}_0^1(V)$. An important property of $\mathbf{B}_0(V)$ is a direct consequence of corollary (2) :

Theorem 7 *On subspace* $\mathbf{B}_0(V)$*, one has* $|\mathbf{u}|_1 \leq \frac{1}{\beta}\|\operatorname{div}\mathbf{u}\|$

9.3 Subspace $\mathbf{I}_0(V)$

Let us look at the elements $\mathbf{w} \in \mathbf{H}_0^1(V)$ which are orthogonal to $\mathbf{B}_0(V)$. These elements verify by definition

$$\forall \mathbf{u} \in \mathbf{B}_0(V) \quad (\mathbf{w}, \mathbf{u})_1 = 0$$

As the general form of the elements of $\mathbf{B}_0(V)$ is $\mathbf{u} = \mathcal{R}\mathbf{grad}p$ with $p \in L_0^2(V)$, this condition reduces to

$$\forall p \in L_0^2(V) \quad 0 = (\mathcal{R}\mathbf{grad}p, \mathbf{w})_1 = -(p, \operatorname{div}\mathbf{w})$$

which implies $\operatorname{div}\mathbf{w} = 0$. So, the set of elements that are orthogonal to $\mathbf{B}_0(V)$ is the incompressible subspace, as defined by

$$\mathbf{I}_0(V) = \{\mathbf{w} \in \mathbf{H}_0^1(V) \mid \operatorname{div}\mathbf{v} = 0\}$$

Being the kernel of the bounded operator div, it is a closed subspace. Thus,

$$\mathbf{H}_0^1(V) = \mathbf{B}_0(V) \oplus \mathbf{I}_0(V)$$

9.4 de Rham's theorem

Let us confider an element $\mathbf{f} \in \mathbf{H}^{-1}(V)$ such that for any incompressible field $\mathbf{v} \in \mathbf{H}_0^1(V)$, one has

$$< \mathbf{f}, \mathbf{v} >= 0 \tag{58}$$

It is equivalent to say that $(\mathcal{R}\mathbf{f}, \mathbf{v})_1 = 0$ for any incompressible field, so that $\mathcal{R}\mathbf{f} \in \mathbf{B}_0(V)$. Consequently, there exist a $p \in L_0^2(V)$ such that $\mathcal{R}\mathbf{f} = \mathcal{R}\mathbf{grad}p$ or, equivalently, $\mathbf{f} = \mathbf{grad}p$. One has thus proved the following theorem :

Theorem 8 (de Rham's theorem) *Any element $\mathbf{f} \in \mathbf{H^{-1}(V)}$ whose kernel is the incompressible subspace of $\mathbf{H}_0^1(V)$ is of the form $\mathbf{f} = \mathbf{grad}p$ with $p \in L_0^2$.*

9.5 Problem $\operatorname{div} \mathbf{v} = q$

An element $q \in L_0^2(V)$ being given, let us look for a vector field $\mathbf{v} \in \mathbf{B}_0(V)$ such that

$$\operatorname{div} \mathbf{v} = q$$

For this purpose, we will minimize in $\mathbf{B}_0(V)$ the squared distance

$$\| \operatorname{div} \mathbf{v} - q\|^2 = \| \operatorname{div} \mathbf{v}\|^2 - 2(q, \operatorname{div} \mathbf{v}) + \|q\|^2$$

which leads to the following variational problem

Find $\mathbf{v} \in \mathbf{B}_0(v)$ such that

$$\forall \mathbf{w} \in \mathbf{B}_0(V) \quad (\operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{w}) = (q, \operatorname{div} \mathbf{w})$$

Noting that in $\mathbf{B}_0(V)$,

$$\begin{aligned} |(\operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{w})| &\leq |\mathbf{v}|_1 |\mathbf{w}|_1 \quad \text{(boundedness)} \\ \| \operatorname{div} \mathbf{v}\|^2 &\geq \beta^2 |\mathbf{v}|_1^2 \quad \text{(ellipticity)} \end{aligned}$$

one can conclude that this problem admits an unique solution $\mathbf{v} \in \mathbf{B}_0(V)$. This solution verifies

$$(\operatorname{div} \mathbf{v} - q, \operatorname{div} \mathbf{w}) = 0$$

whatever be $\mathbf{w} \in \mathbf{B}_0(V)$. Moreover, as any element $\mathbf{r} \in \mathbf{H}_0^1(V)$ admits the decomposition

$$\mathbf{r} = \mathbf{r}_B + \mathbf{r}_I \quad \text{with} \quad \mathbf{r}_B \in \mathbf{B}_0(V),\, \mathbf{r}_I \in \mathbf{I}_0(V)$$

with, of course, $\operatorname{div} \mathbf{r}_I = 0$, one has also

$$\forall \mathbf{r} \in \mathbf{H}_0^1(V) \quad 0 = (\operatorname{div} \mathbf{v} - q, \operatorname{div} \mathbf{r}) \equiv < \mathbf{r}, -\mathbf{grad}(\operatorname{div} \mathbf{v} - q) >$$

whatever be $\mathbf{r} \in \mathbf{H}_0^1(V)$. This implies $(\operatorname{div} \mathbf{v} - q) =$ constant and as the mean of this expression vanishes,

$$\operatorname{div} \mathbf{v} = q$$

Furthermore, from theorem (7), this solution verifies

$$|\mathbf{v}|_1 \leq \frac{1}{\beta} \| \operatorname{div} \mathbf{v}\| = \frac{1}{\beta} \|q\|$$

We have thus obtained the following theorem:

Theorem 9 (Divergences of $\mathbf{H}_0^1(V)$) *Any element q of $L_0^2(V)$ is the divergence of an element $\mathbf{v} \in \mathbf{H}_0^1(V)$ verifying $|\mathbf{v}|_1 \leq \frac{1}{\beta}\|q\|$*

9.6 A remark

The above proposition implies Nečas' inequality. Indeed, considering the expression

$$\mathcal{Q}(p,\mathbf{v}) = \frac{(p, \operatorname{div}\mathbf{v})}{|\mathbf{v}|_1}$$

this theorem allows to choosing a particular $\mathbf{v}^*$ such that $\operatorname{div}\mathbf{v}^* = p$ and

$$|\mathbf{v}^*|_1 \leq \frac{1}{\beta}\|\operatorname{div}\mathbf{v}^*\| = \frac{1}{\beta}\|p\|$$

This choice leads to

$$\mathcal{Q}(p,\mathbf{v}^*) \geq \beta\frac{(p,p)}{\|p\|} \equiv \beta\|p\|$$

and the supremum of $\mathcal{Q}(p,\mathbf{v})$ is at least equal to this value. So, *Nečas' inequality and theorem (9) are equivalent.* This fact was pointed by Ciarlet et al. [1, 2].

Note that a direct proof of theorem (9) has been given by Bogovskii [4] by using the Calderon-Zygmund singular integral.

10 Korn's inequality

10.1 Strain and rotation tensors

In this section, a proof of Korn's inequality will be given, starting from Beltrami's relations. At any $\mathbf{u} \in \mathbf{H}^1(V)$ may be associated the two tensor fields

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}(D_i u_j + D_j u_i) \quad \text{(strain tensor)} \\ \omega_{ij} &= \frac{1}{2}(D_i u_j - D_j u_i) \quad \text{(rotation tensor)} \end{aligned}$$

Obviously,

$$D_i u_j = \varepsilon_{ij} + \omega_{ij}$$

It is clear that the strain tensor is symmetric, while the rotation tensor is anti-symmetric. As a result,

$$\varepsilon_{ij}\omega_{ij} = 0$$

so that

$$|\mathbf{u}|_1^2 = \int_V \varepsilon_{ij}\varepsilon_{ij}dV + \int_V \omega_{ij}\omega_{ij}dV = \|\varepsilon\|^2 + \|\omega\|^2 \tag{59}$$

Finally, one has

$$D_i\omega_{jk} + D_j\omega_{ki} + D_k\omega_{ij} = \frac{1}{2}(D_{ij}u_k - D_{ik}u_j + D_{jk}u_i - D_{ji}u_k + D_{ki}u_j - D_{kj}u_i) = 0$$

that is to say, the rotation tensor verifies the condition (1).

10.2 Beltrami's relations

Substracting the two relations

$$\begin{aligned} D_{ki} u_j &= D_k \varepsilon_{ij} + D_k \omega_{ij} \\ D_{ik} u_j &= D_i \varepsilon_{kj} + D_i \omega_{kj} \end{aligned}$$

leads to

$$\begin{aligned} 0 &= D_k \varepsilon_{ij} - D_i \varepsilon_{kj} + D_k \omega_{ij} - D_i \omega_{kj} \\ &= D_k \varepsilon_{ij} - D_i \varepsilon_{kj} + D_k \omega_{ij} + D_i \omega_{jk} \\ &= D_k \varepsilon_{ij} - D_i \varepsilon_{kj} - D_j \omega_{ki} \end{aligned}$$

where use has been made of condition (1). The result writes

$$D_j \omega_{ki} = D_k \varepsilon_{ij} - D_i \varepsilon_{kj} \tag{60}$$

and is generally attributed to Beltrami.

10.3 Bounding the rotation tensor

Beltrami's relations immediatly lead to

$$|D_j \omega_{ki}|_{-1} \le \|\varepsilon_{ij}\| + \|\varepsilon_{kj}\| \le \sqrt{2} \|\varepsilon_j\|$$

where use is made of the abusive notation

$$\|\varepsilon_j\| = \sqrt{\sum_l \|\varepsilon_{lj}\|^2}$$

Taking the sum of the squares leads to

$$\sum_j |D_j \omega_{ki}|_{-1}^2 \le 2 \sum_j \|\varepsilon_j\|^2 = 2\|\varepsilon\|^2$$

Now, using Nečas' inequality , one obtains

$$\|\omega_{ki}\|^2 \le \overline{\omega_{ki}}^2 V + \frac{2}{\beta^2} \|\varepsilon\|^2$$

Summing all components, one obtains

$$\|\omega\|^2 \le V \sum_{ij} \overline{\omega_{ij}}^2 + \frac{2n^2}{\beta^2} \|\varepsilon\|^2 \tag{61}$$

10.4 Korn's inequality

From the preceeding result, one may write

$$\begin{aligned} |\mathbf{u}|_1^2 &\le \|\varepsilon\|^2 + \|\omega\|^2 \\ &\le V \sum_{ij} \overline{\omega_{ij}}^2 + \left(1 + \frac{2n^2}{\beta^2}\right) \|\varepsilon\|^2 \end{aligned} \tag{62}$$

Concerning the norm in $\mathbf{H}^1(V)$, Poincaré's inequality

$$\|\mathbf{u}\|_1^2 \leq C_1 V \sum_i \overline{u_i}^2 + C_2 |\mathbf{u}|_1^2$$

leads to the general result

$$\|\mathbf{u}\|_1^2 \leq C_1 V \sum_i \overline{u_i}^2 + C_2 V \sum_{ij} \overline{\omega_{ij}}^2 + C_3 \|\varepsilon\|^2 \tag{63}$$

which is *Korn's inequality.*

Starting from Beltrami relations to prove Korn's inegality is not the most usual way. But the obtained result is optimal from a physical point of view, as it emphasizes the role of rigid body motions, namely, n translations and $n(n-1)/2$ rotations (as the rotation tensor is antisymmetric).

A completely different proof has been given by Gobert [13]. It seems to be the most general proof (open set having the cone property) but it makes use of the Calderon-Zygmund singular integral. An elegant proof has also been given by Oleinik [16, 17].

11 Traces of incompressible fields

11.1 Norm of the traces

Any element $\mathbf{u} \in \mathbf{H}^1(V)$ possess a trace $T\mathbf{u}$ on the boundary S of V. Reciprocally, at any trace is associated a class Class($T\mathbf{u}$) of elements of $\mathbf{H}^1(V)$ having this trace. Two different elements of a same class only differ by an element of $\mathbf{H}_0^1(V)$. A natural norm for $T\mathbf{u}$ is thus the norm of the quotient space $\mathbf{H}^1(V)/\mathbf{H}_0^1(V)$, which is the minimum value of the norm of an element of this class :

$$\|T\mathbf{u}\|_T = \inf_{\mathbf{v} \in \mathbf{H}_0^1(V)} \|\mathbf{u} - \mathbf{v}\|_1 \tag{64}$$

Minimizing this norm consists to find $\mathbf{v} \in \mathbf{H}_0^1(V)$ such that

$$\forall \mathbf{w} \in \mathbf{H}_0^1(V) \quad ((\mathbf{u} - \mathbf{v}, \mathbf{w}))_1 = 0$$

This defines a privileged element $RT\mathbf{u} = \mathbf{u} - \mathbf{v}$, the minimal one in Class($T\mathbf{u}$). Note that this element is orthogonal to $\mathbf{H}_0^1(V)$. By construction,

$$\|T\mathbf{u}\|_T = \|RT\mathbf{u}\|_1 \tag{65}$$

11.2 Traces of incompressible fields

An interesting question is to know if a given trace $T\mathbf{u}$ on the boundary may also be the trace of an incompressible field, in other words, if Class($T\mathbf{u}$) contains incompressible fields. Firstly, noting that any invompressible field $\mathbf{v}$ verifies

$$\int_S \mathbf{v} \cdot \mathbf{n} dS = \int_V \operatorname{div} \mathbf{v} dV = 0$$

a *necessary* condition on the trace $T\mathbf{u}$ is

$$\int_S T\mathbf{u} \cdot \mathbf{n} dS \equiv \int_V \operatorname{div}(RT\mathbf{u}) = 0 \tag{66}$$

We will now prove that if condition (66) is satisfied, Class($T\mathbf{u}$) contains an incompressible field. For this purpose, let us substract from $RT\mathbf{u}$ an element $\mathbf{b}$ of the above defined subspace $\mathbf{B}_0(V)$ such that

$$\operatorname{div}\mathbf{b} = \operatorname{div}(RT\mathbf{u}) \tag{67}$$

which is possible by virtue of (66). This substraction does not modify the trace, and the element $\mathbf{w} = RT\mathbf{u} - \mathbf{b}$ is incompressible by construction. Now, it remains to be proved that this $\mathbf{w}$ continuously depends on $RT\mathbf{u}$. From (67) follows

$$|b|_1 \leq \frac{1}{\beta}\|\operatorname{div}(RT\mathbf{u}\| \leq \frac{C_1}{\beta}\|RT\mathbf{u}\|_1$$

As $\mathbf{b} \in \mathbf{B}_0(V) \subset \mathbf{H}_0^1(V)$, Friedrichs' inequality applies and

$$\|\mathbf{b}\|_1 \leq C_2|\mathbf{b}|_1 \leq \frac{C_1C_2}{\beta}\|RT\mathbf{u}\|_1$$

Allowing to the fact that $\mathbf{b}$ and $RT\mathbf{u}$ are orthogonal, one obtains

$$\begin{aligned} \|\mathbf{w}\|_1 &= \sqrt{\|RT\mathbf{u}\|^2 + \|\mathbf{b}\|^2} \\ &\leq \sqrt{\left(1 + \frac{C_1^2C_2^2}{\beta^2}\right)}\|RT\mathbf{u}\|_1 \\ &\leq C_3\|T\mathbf{u}\|_T \end{aligned}$$

We have thus obtained the following theorem :

Theorem 10 *At any trace $T\mathbf{u}$ verifying $\int_S T\mathbf{u}\cdot\mathbf{n}dS = 0$ may be associated an uncompressible field $\mathbf{w}$ verfying $\|w\|_1 \leq C_3\|Tu\|_T$*

Corollary 3 *Let $\mathbf{u}$ be any element of $\mathbf{H}^1(V)$ whose trace on the boundary verifies condition (66). There exists an incompressible field $\mathbf{w}$ having the same trace on the boundary and verifying*

$$\|\mathbf{w}\|_1 \leq C_3\|\mathbf{u}\|_1 \tag{68}$$

Indeed, $\|T\mathbf{u}\|_T \leq \|\mathbf{u}\|_1$

12 A result concerning harmonic vector fields

Let us first consider a vector field which is harmonic in V and which verifies

$$\int_S \mathbf{u}\cdot\mathbf{n}dS \equiv \int_V \operatorname{div}\mathbf{u}dV = 0 \tag{69}$$

Fron (38), the harmonicity condition may be written

$$D_i \operatorname{div}\mathbf{u} - D_j\omega_{ij}(\mathbf{u}) = 0$$

and it is clear that te fields $\theta = \operatorname{div}\mathbf{u}$ and ω_{ij} comply with the exigencies of theorem (1). Therefore, one has

$$\|\operatorname{div}\mathbf{u}\| \leq C\|\omega(\mathbf{u})\| \tag{70}$$

If now condition (69) is not satisfied, it suffices to substract from $\mathbf{u}$ the particular vector field

$$\mathbf{v} = \frac{\overline{\theta}}{n}\mathbf{x} \quad \text{with} \quad \overline{\theta} = \frac{1}{V}\int_V \operatorname{div} \mathbf{u} dV$$

This leads to a harmonic field $\mathbf{u}^* = \mathbf{u} - \mathbf{v}$ which conforms to (69) and has the same rotation tensor as $\mathbf{u}$. Applying (70) to this field leads to

$$\| \operatorname{div} \mathbf{u}^* \| \leq C\|\omega(\mathbf{u}^*\| = C\|\omega(\mathbf{u})\|$$

and since

$$\| \operatorname{div} \mathbf{u}\|^2 = \| \operatorname{div} \mathbf{u}^* \|^2 + V\overline{\theta}^2$$

we have obtained the following theorem :

Theorem 11 *Any harmonic field* $\mathbf{u} \in \mathbf{H}^1(V)$ *verifies the following inequality :*

$$\| \operatorname{div} \mathbf{u}\|^2 \leq V\,\overline{\operatorname{div} \mathbf{u}}^2 + C_1\|\omega(\mathbf{u})\|^2 \tag{71}$$

13 Conclusions

We have thus established an inequality on Sobolev spaces by using a technique due to Friedrichs. This technique is essentially of algebraic nature and therefore, elementary. The obtained inequality has a lot of consequences, as Nečas' inequality, de Rham's theorem, the fact that the divergences are onto in L^2, Korn's inequality, results on the traces of incompressible fields and an inequality for harmonic vector fields. All these results have been obtained in the particularly intuitive Hilbertian frame.

It is well true that Friedrichs' technique makes assuptions on the domain. Friedrichs [9, 10] showed that his conditions are not too restrictives, as allowing corners and edges at the boundary. However, it could be interesting to further explore the characterization of Friedrichs' $\mathbf{\Omega}$-domains, as the existence of the $\mathbf{\Omega}$ field is a very useful property.

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