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Energy-efficient production control of a make-to-stock system with buffer- and time-based policies

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ABSTRACT

Increasing energy efficiency in manufacturing has significant environmental and cost benefits. Turning on or off a machine dynamically while considering the production rate requirements can offer substantial energy savings. In this work, we examine the optimal policies to control production and turn on and off a machine that operates in working, idle, off, and warmup modes for the case where demand inter-arrival, production, and warmup times have phase-type distributions. The optimal control problem that minimises the expected costs associated with the energy usage in different energy modes and the inventory and backlog costs is solved using a linear program associated with the underlying Markov Decision Process. We also present a matrix-geometric method to evaluate the steady-state performance of the system under a given threshold control policy. We show that when the inter-arrival time distribution is not exponential, the optimal control policy depends on both the current phase of the inter-arrival time and inventory position. The phase-dependent policy implemented by estimating the current phase based on the time elapsed since the last arrival yields a buffer- and time-based policy to control the energy mode and production. We show that policies that only use the inventory position information can be effective if the control parameters are chosen appropriately. However, the control policies that use both the inventory and time information further improve the performance.

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Optimal production and energy control; energy efficiency; markovian models; SDG 12: responsible consumption and production; switch-on and -off policy; linear programming

1. Introduction

Manufacturing is a major contributor to global energy consumption and greenhouse gas emissions (Frigerio, Cornaggia, and Matta 2021; Masmoudi, Delorme, and Gianessi 2019; Tan, Karabağ, and Khayyati 2023). Given the looming threat of climate change, there is a heightened importance to address energy consumption and improve energy efficiency in manufacturing (Zavanella et al. 2015). Data-driven energy control is emerging as a powerful tool for achieving this goal. This benefits the environment and improves manufacturers' bottom line as energy costs increase steadily. This article will explore data-driven optimal energy control policies for manufacturing systems.

Turning on and off machines based on the data collected from a manufacturing system effectively implements data-driven energy control in manufacturing. On/off type of controlling approaches have become increasingly important in recent years as new technologies have made it easier to collect, analyse, and act on data in real time. The Industrial Internet of Things (IIoT) technologies introduced by Industry 4.0 allow one to collect real-time data and easily control machines based on this data (Wang et al. 2019; Zhang et al. 2018). In addition, many machines are built to operate in various energy modes, including idle, working, and standby; each mode has its own distinct energy consumption levels (Loffredo et al. 2023; Squeo, Frigerio, and Matta 2019; Wang, Li, and Liu 2013). By collecting data such as the current load, energy mode of the machines, and the number of waiting parts to be produced, manufacturers can determine when machines should be turned on or off or operated on standby to optimise energy consumption.

Our study has been motivated by a current project related to the on/off scheduling of a painting oven at a plant of a major automotive manufacturer. While a painting oven for regular colors is utilised heavily at this plant, another oven for special color vehicles has sufficient idle time to switch it on and off, according to the arrivals. A control policy that dynamically switches the oven on

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and off based on the system's state must account for the warmup delay when a switched-off oven is switched on and the production requirements. A buffer-based policy switches the oven on and off depending on the number of special color vehicles in a designated upstream area. Since the vehicles' arrivals at the oven are generated by the upstream line, the distribution of the inter-arrival times has memory. The proposed buffer- and time-based policy in this paper improves the control policy by incorporating the time since the last arrival of a special vehicle and the number of waiting vehicles. Various implementations of similar buffer-based on/off policies in different industries have been reported in the literature. Examples include implementations in an engine block production line (Chang et al. 2012), an automobile assembly line (Sun and Li 2012), a CNC system processing aluminum parts (Frigerio and Matta 2014), a machining centre for power-train applications (Marzano, Frigerio, and Matta 2019), and a motor cylinder head production system (Loffredo et al. 2023).

In this work, we determine the optimal policy to control production and turn on/off a make-to-stock machine to minimise the expected energy usage costs due to operating in different energy modes and the inventory and backlog costs. We examine a machine that operates in various energy modes, such as working, off, idle, and warmup, with random demand inter-arrival, production, and warmup times. Among these modes, the off mode incurs the lowest energy cost; the energy cost associated with staying in the idle mode falls between the energy costs of staying in the off and working modes. When the machine is turned off, it goes through a warmup delay before it is turned on. The warmup mode usually incurs a higher energy cost than the working mode. Our objective is to devise a control policy that determines when to turn off or make idle a working machine, when to turn on a turned-off machine, and when to start production on an idle machine in a dynamic manner according to the system state.

The optimal policy to control energy mode and production for systems in which the demand inter-arrival, production, and warmup times are exponential has been shown to have a threshold-type policy (Özkan and Tan 2023; Tan, Karabağ, and Khayyati 2023). According to the optimal policy, the machine operates either between working/off or working/idle modes, and it does not use working, off, and idle modes together. When it operates with working/off modes, it is turned off when the inventory position reaches an upper threshold and turned on when it drops to a lower threshold. Then, it starts production as soon as the warmup period is completed. When it operates with working/idle modes, it stops production and becomes idle when the inventory position reaches an upper threshold. Then, it resumes production when the inventory position reduces to a lower threshold. These policies are commonly referred to as *buffer-based* policies in the literature because they are characterised by the current inventory position and the machine's energy mode (Frigerio and Matta 2015).

In this study, we extend the machine turn-on/off control problem to the case where the inter-event times have phase-type distributions. We model this problem as a Markov Decision Process (MDP) and obtain the optimal solution using linear programming. Furthermore, we provide a matrix-geometric approach to assess the system's performance when operating under the optimal control policy. When inter-event times are modelled as phase-type distributions instead of exponential distributions, we reveal that the optimal on/off policy depends not only on the inventory position but also on the time since the occurrence of the last event. Correspondingly, the optimal control policy is shown to have a *time-and buffer-based* structure. In the literature, *time-and buffer-based* on/off policies have been proposed without determining the optimal policy and shown to be effective in minimising energy costs (Frigerio and Matta 2015).

Our results extend the work on the optimal energy mode and production control policies for manufacturing systems and offer an analytical basis to analyse and implement *time-and buffer-based* policies. Specifically, this study provides two notable contributions to the existing literature. First, by analysing the MDP underlying our control problem, we show that when the inter-arrival time distribution is not exponential, the optimal policy is contingent upon both the inventory position and the current phase of the inter-arrival time. We further show that the current phase can be predicted using the time elapsed since the last arrival. Therefore, based on the optimal solution of the control problem, we show that the optimal policy to control production and turn on/off the machine when the inter-arrival times are not exponential is a type of *time-and buffer-based* policy. Additionally, we introduce a computationally efficient approach to assess the system's performance under the optimal policy.

Furthermore, our extensive numerical analysis shows that policies that use only the inventory position information can be effective if the control parameters are chosen appropriately. Specifically, our numerical results yield that when the production and warmup times become more variable, a *buffer-based* policy performs reasonably well compared to a phase-dependent policy that integrates the elapsed time since the last arrival with the inventory position. For systems with low utilisation, buffer- and time-based policies significantly outperform production control policies that do not change the energy mode of the machine dynamically.

The subsequent sections of this paper are structured as follows. Section 2 provides a review of the relevant literature. Section 3 presents both the model and the optimal control problem. In Section 4, we present the solution to the MDP associated with the optimal control problem using linear programming. In Section 5, the optimal policy to control production and turn on/off the machine is discussed. Section 6 provides a detailed description of the matrix-geometric approach used to assess the system's performance under a specified control policy. Section 7 presents the numerical results that compare the performance of *time-and buffer-based* policies. Lastly, the conclusions and limitations of this study are presented in Section 8.

2. Literature review

The growing apprehension surrounding rising energy costs, unpredictable energy supply, and carbon emission regulations has impacted professionals and prompted a surge in research efforts to address energy-saving strategies and technologies. Extensive systematic overviews related to this literature stream have been proposed by Gunasekaran and Spalanzani (2012), Ngai et al. (2013), Biel and Glock (2016), and Bänsch et al. (2021). Here, we only review the most relevant and recent studies, i.e. the ones focussing on implementing on/off types of energy-controlling policies on production systems and server vacation models in queueing systems.

Under on/off control policies, a switch-off command is utilised to deactivate specific auxiliary units, resulting in the machine entering a low power consumption state and temporarily interrupting its operation (Frigerio, Cornaggia, and Matta 2021). Operational readiness is resumed with a switch-on command. These control policies generally rely on two types of real-time information: (i) *Buffer information*, and (ii) *Time information*.

The first group studies systems in which the production is interrupted and restarted again based on the number of parts accumulated in buffers such that machines can be controlled during starvation periods (Gahm et al. 2016). As one of the early studies within the stream of *buffer-based* control policies, Wang et al. (2016) utilise a Markovian modelling approach to examine the performances of on/off switching policies based on real-time data of buffers and machines in a serial manufacturing system. Wang et al. (2019, 2017) consider a very similar manufacturing system to that of Wang et al. (2016); but, different than their previous work, the authors use a fuzzy logic modelling approach. Jia et al. (2016) develop an approximate analytical method to assess the performance of a serial line with Bernoulli machines, in which a subset of the machines is controlled with an on/off switching policy depending on buffer levels. Su et al. (2016) and Pei et al. (2022) focus on a production line that includes only two unreliable machines. They develop an integrated model that minimises energy consumption and aims to maintain a given productivity rate. Su et al. (2017) revise the analytical model given in Su et al. (2016) to deal with large production lines. Renna (2018) uses a simulation-based approach to study dynamic and adaptive switching on/off policies in a pull-controlled production line. The author reveals that energy-saving control policies perform better for pull-type production lines when the production line is balanced, or the bottleneck is the last machine of the production line. Cui et al. (2021) utilise a Markovian modelling approach to characterise the interdependent relationships between production rate and energy consumption for serial production lines. In Frigerio, Tan, and Matta (2023), a simulation-based modelling approach is used to analyse *buffer-based* threshold policies that simultaneously control several serially connected machines for energy-saving purposes. Additionally, Loffredo et al. (2021, 2023) use similar types of *buffer-based* control policies in parallel-line production systems.

More recently, Tan, Karabağ, and Khayyati (2023) characterised the optimal policy to control production and turn on/off the machine for a make-to-stock system with exponentially distributed production, warmup, and demand inter-arrival times. They also use the optimal policy for the exponential case as an approximate policy for systems with inter-event times modelled as Markov Arrival Processes (MAPs). Özkan and Tan (2023) prove the asymptotic optimality of working/idle and working/off policies for the same system having exponentially distributed production, demand inter-arrival, and warmup times under heavy traffic. Our work is closely related to these two studies and extends them to analysing the optimal policy with phase-type inter-event times. Our findings reveal that the optimal policy for this case is characterised not only by the inventory position as the case for the exponential case but also by the inter-arrival time phase.

The second group of studies analyses systems using *time-based* control policies, which operate with thresholds on machine idle time to determine when production should be interrupted and restarted again. Mouzon, Yıldırım, and Twomey (2007) and Mouzon and Yıldırım (2008) focus on a single machine operating in a job shop environment and present various on/off types of control policies that depend on the machine's idle times. These two works highlight the potential of effectively utilising a *time-based* on/off control policy to reduce energy consumption, particularly in non-bottleneck machines. Sun and Li (2012) introduce an approximate analytical method that determines the optimal duration of on/off times to decrease energy consumption in a serial production line. By using a simulation-based approach, Wang, Li, and Liu (2013), Frigerio and Matta (2015), and Squeo, Frigerio, and Matta (2019) evaluate how *time-based* control policies perform in reducing the energy consumption of systems composed of a single machine with multiple energy consumption modes. Frigerio, Cornaggia, and Matta (2021); Frigerio, Marzano, and Matta (2020) employ machine learning techniques to establish a *time-based* control policy that can adapt its parameters over time based on a real data set received from the system.

In the queuing literature, models that include *servers with vacations* use vacations for modelling planned off/idle times. Some works in this literature consider threshold-type policies and fluid queues, e.g. Barron (2016, 2018) and Xu and Wang (2019). However, the settings in these articles do not allow the policy parameter to depend on the environment state directly. Furthermore, these parameters do not change with the information from the arrival process or with the anticipation of arrivals. In addition, although a setup cost is incorporated when a switched-off machine is switched on, the warmup delay is not modelled explicitly.

Based on the review given above, we can summarise the contribution of this study as follows: First, in contrast to the aforementioned existing studies, we determine the optimal policy to control production and turn on/off the machine for a make-to-stock system where production, warmup, and demand inter-arrival times have phase-type distributions by solving the LP associated with the Markovian Decision Process formulation. Second, different from existing studies that rely on simulation-based approaches to analyse systems with heuristic *time and buffer-based* control policies, we use analytical methods to derive optimal control policies and present an analytical method to evaluate the system performance under the *time and buffer-based* policies. By comparing the optimal policies for the phase-type inter-event times with *time- and buffer-based* policies, we show that policies based only on inventory position information can be effective if the control parameters are chosen appropriately.

3. Model

We consider a machine that can operate in Working (W), Idle (I), Warmup (R), and Off (O) energy modes. The control problem we examine is determining in which energy mode the machine should be operated to minimise expected energy usage, inventory, and backlog costs given the observed inventory/backlog level, the machine operation mode, and the phase of the inter-arrival time process. The decisions are taken at the instances when production is completed, warmup is completed, or a change in the inter-arrival process occurs. Upon completion of a part, the control policy determines whether the machine will be put in idle mode, switched off, or continue production. Similarly, when a change in the inter-arrival time process occurs, a machine in idle mode can resume production, and a switched-off machine can be switched on that requires a warmup delay before the machine becomes fully operational again. When the warmup is completed, the machine can resume production or stay idle. The energy cost the manufacturer incurs depends on the energy mode of the machine. The problem is making the dynamic energy mode decisions to minimise the expected energy consumption, inventory, and backlog costs.

At time t, the state of the system is denoted by $\mathbf{S(t)} = (P(t), N(t), A(t))$ where $P(t)$ is the machine's energy mode state, $N(t)$ is the inventory/backlog position, and $A(t)$ is the phase of the inter-arrival time process at time t.

We consider inter-arrival, production, and warmup times with phase-type distributions. A phase-type distribution can be represented by a random variable describing the time until absorption of a Markov process with one absorbing state. Each state of the Markov process is referred to as a phase. With this representation, a phase-type distribution can approximate a given general distribution (Neuts 1994).

The demand inter-arrival time follows a phase-type distribution with K_A phases, and the expected inter-arrival time is $\frac{1}{\lambda}$. At time t, the demand inter-arrival process is in one of K_A phases, i.e. $A(t) \in \{A_1, \ldots, A_{K_A}\}$. The process starts in phase A_1 and ends in phase A_{K_A}. The transition rates among different arrival phases are given in the matrix Q_A.

The production time follows a phase-type distribution with K_W phases where the process starts in phase W_1 and ends in phase W_{K_W}. The transition rates among different production phases are given in the matrix Q_W, and the expected production time is $\frac{1}{\mu}$. Similarly, the warmup time has a phase-type distribution with K_R phases. The process starts in phase R_1 and ends in phase R_{K_R}. The transition rates among different warmup phases are given in the matrix Q_R. The expected warmup time is $\frac{1}{\tau}$.

Since the production time and warmup time have phase-type distributions with K_W and K_R phases, together with the Idle and Off states, the model has $K_W + K_R + 2$ machine-related states. Therefore, at time t, the machine state is $P(t) \in \{\mathrm{W}_1, \ldots, \mathrm{W}_{K_W}, \mathrm{I}, \mathrm{O}, \mathrm{R}_1, \ldots, \mathrm{R}_{K_R}\}$.

When production starts in phase W_1, the machine mode cannot be changed. That is, it cannot be turned off or made idle before the production is completed in phase W_{K_W}. Similarly, while the warmup time process goes through phases 1 to K_R, the machine cannot be turned off again before the warmup is completed in phase K_R. As a result, the decision to be taken at any given time t does not change with the changes in the phases of production time and warmup time processes. To designate the machine modes that are used for the decisions, the production time states $\{W_1, \ldots, W_{K_W}\}$ are lumped into state W and the warmup states $\{R_1, \ldots, R_{K_R}\}$ are lumped into state R. Correspondingly, the set of energy modes used in the decisions is denoted by $\mathcal{M} = \{W, I, O, R\}$.

The proposed method to derive the optimal control policy and analyse the system applies to the case where Markovian Arrival Processes (MAP) are used to model the inter-event times. This enables analysis of the production control problems with correlated inter-event times (Manafzadeh Dizbin and Tan 2019). This extension is not given in the paper for brevity.

The energy mode control policy is denoted by $\mathbf{u}(t) = \{u_{a,m,n}(t)\}$ where $u_{a,m,n}(t) \in U_m$ is the decision to be taken depending on the inter-arrival process phase a, energy mode m, and inventory position n at time t in the long-run. U_m includes the admissible discrete actions that can be taken when the energy mode is m. Figure 1 depicts the transitions among the different states and the corresponding decision variables. The transitions from Working to Idle and Off when the production is completed, from Idle to Working, from Off to Warmup, and from Warmup completion to Idle or Working are instantaneous. The time in the Working state corresponds to the random processing time. Whenever a warmup is initiated, it is completed after a random warmup time. When the machine starts working on a part and when it is going through a warmup time, its state cannot be changed until the completion of the processing or warmup.

The per-unit time energy cost consumed by the machine when it is in mode m is c_m, $m \in \mathcal{M} = \{W, I, O, R\}$. $I_m(t)$ is a binary variable that is 1 if the mode

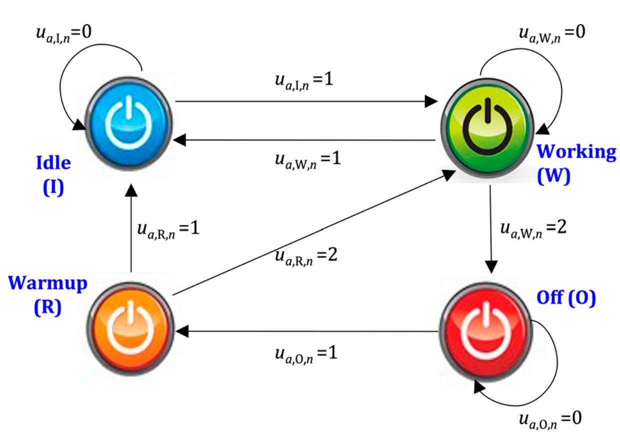

Figure 1. State transitions among the energy states and the corresponding decisions.

is m at time t and 0 otherwise. The energy costs at each production and warmup phases are the same.

The expected energy consumption, inventory, and backlog cost rate in the long run, depending on the control policy $\mathbf{u}(t)$, $\chi^{\mathbf{u}(t)}$ is given as:

$$\chi^{\mathbf{u}(t)} = \mathbb{E}^{\mathbf{u}(t)}\left[\lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\mathcal{C}(\mathbf{S}(t)) + \sum_{m\in\mathbf{Q}} c_m I_m(t)\,\mathrm{d}t \mid \mathbf{S}(0)\right], \quad (1)$$

where $\mathcal{C}(\mathbf{S}(t)) = c^- N-(t) + c^+ N^+(t)$ is the total inventory carrying and backorder cost, c^+ is the inventory carrying cost, $N^+(t) = \max\{N(t), 0\}$ is the inventory level, c^- is the backlog cost and $N^-(t) = \max\{-N(t), 0\}$ is the backlog level.

The energy mode control problem is stated as minimising the expected energy consumption, inventory, and backlog cost rate, $\chi^{\mathbf{u}(t)}$ by determining the state-dependent decision $\mathbf{u}(t)$:

$$\mathbf{u}^*(t) = \inf_{\mathbf{u}(t)} \chi^{\mathbf{u}(t)}. \quad (2)$$

4. Determining the optimal energy mode control policy

Since the inter-event times are modelled as phase-type distributions, the evolution of $\mathbf{S}(\mathbf{t}) = (P(t), N(t), A(t))$ can be described as a continuous time discrete state space Markov Chain (CTMC). This enables us to formulate our problem as an MDP and use well-known efficient solution methods to compute its optimal policy. In this study, we utilise a solution approach based on linear programming to find the optimal policy to control production and turn on/off the machine. The LP-based solution approach has been previously used to find the optimal control policies in production and/or energy mode control settings (see, e.g. Karabağ and Tan 2019; Loffredo et al. 2023; Tan, Karabağ, and Khayyati 2023). Using the LP approach allows us to leverage advanced optimisation software packages, empowering us to solve the problem more effectively to determine the relevant optimal policies in a shorter time than classical methods like value/policy iterations.

Let us define $c(s, d)$ as a function that represents the cost of taking action d in state s and $p(j \mid s, d)$ as a transition probability from state s to state j under action d. The decision variable $y(s, d)$ in the LP formulation represents the proportion of time, in the long run, that the system occupies state s when action d is executed. Considering these parameters and decision variables, the general form of the linear programming formulation to determine the optimal policy can be given as:

$$\min \quad \sum_{s\in\mathbf{S}} \sum_{d\in\mathbf{D}_s} c(s,d)y(s,d), \tag{3}$$

s.t.

$$\sum_{d\in\mathbf{D}_j} y(j,d) - \sum_{s\in\mathbf{S}} \sum_{d\in\mathbf{D}_s} p(j\,|\,s,d)y(s,d) = 0, \quad \forall\, j \in \mathbf{S}, \tag{4}$$

$$\sum_{s\in\mathbf{S}} \sum_{d\in\mathbf{D}_s} y(s,d) = 1, \tag{5}$$

$$y(s,d) \geq 0, \quad \forall\, s \in \mathbf{S},\ \forall d \in \mathbf{D}_s. \tag{6}$$

Equation (3) represents the objective function minimising the long-run average cost. Equation (4) denotes the system balance equation. That is, the balance equations ensure that, in the long run, the flow-in and the flow-out rates for each system state are equal. Given that the decision variables form a probability mass function, it is necessary for each decision variable to be non-negative and their sum to equal exactly 1. These two conditions are integrated into the formulation via Equations (5) and (6). Note that our problem has three important characteristics: (i) it has a discrete state space; (ii) both state and action costs are strictly positive and bounded; and (iii) there exists a finite number of allowable actions for each state. These problem characteristics assure an optimal deterministic and stationary policy obtained via the LP formulation presented above (Bertsekas 2015; Puterman 2014). The explicit LP formulation for a specific system where the demand inter-arrival times have Erlang and the production and warmup times have exponential distributions is given in Appendix A.

In our problem definition, there is no restriction on the inventory level. However, truncating the inventory level's lower and upper sides is necessary to use the linear programming approach with a finite number of variables and constraints (see, Appendix A). The upper truncation level bounds the inventory levels, whereas the lower truncation level bounds the backlog levels. Both truncation levels are chosen to ensure that the long-run probabilities at those levels remain lower than a prespecified tolerance level. Since the LP approach allows for different decisions in each state, the considered solution space is large. As the number of states that represent the inter-arrival, production, warmup times, and truncation levels increase, the state space gets larger. In this case, finding the solution with the LP approach gets computationally inefficient and may be prone to computational issues. As such, the LP approach can be used to show what type of policies are better. However, in order to evaluate the performance of a given system under a given threshold policy, we present a computationally efficient method that uses a matrix geometric approach in Section 6.

5. Structure of the control policy

In what follows, we consider a particular system in which demand inter-arrival times have an Erlang distribution and production and warmup times have exponential distributions. Based on our extensive numerical analyses of this specific system, we discuss the general structure of the integrated optimal production and energy control policy.

When the demand inter-arrival time distribution is Erlang, a monotonic phase-dependent threshold policy is the optimal policy to control production and turn on/off the machine. Figure 2 shows the optimal policy's structure for a specific system with Erlang(2) distributed demand inter-arrival times and exponentially distributed production and warmup times. In this case, the switching decision from one mode to another is contingent upon whether the inventory/backlog level is higher or lower than specific thresholds that depend on the inter-arrival time phase. Given the inventory/backlog level n and the inter-arrival time phase a, the transition from state i to state j is triggered depending on the inventory/backlog level n being above or below the phase-dependent threshold $S_{i,j}(a)$ where $i,j \in \{\mathrm{W, I, O, R}\}$, $a \in \{1,\dots,K_A\}$ at the decision epochs that are the times when production is completed, warmup is completed, an inter-arrival phase change occurs that leads to a part arrival at the last phase.

Figure 3 shows the phase-dependent threshold policy that governs the transitions among the Working, Idle, Warmup, and Off states. A working machine that completes production of a part when the inter-arrival time process is in phase a either continues production with the next part if the current inventory level n is lower than $S_{\mathrm{WI}}(a)$ or switches to idle mode if a sufficient number of parts have been produced, i.e. $S_{\mathrm{WO}}(a) > n \geq S_{\mathrm{WI}}(a)$. If the inventory level is quite high, $n \geq S_{\mathrm{WO}}(a)$ then the machine is turned off.

When the machine is either off or idle, the inventory/backlog level decreases. When the machine is idle, it stays idle as long as the inventory/backlog level is still above $S_{\mathrm{IW}}(a)$ and production resumes as soon as the inventory level falls below $S_{\mathrm{IW}}(a)$. An idle machine is not switched off because keeping the machine off rather than idle would decrease the cost.

When the machine is off and the inventory level reaches below $S_{\mathrm{OW}}(a)$, the machine is switched back on, and the warmup period starts. Upon the completion of warmup, if the current inventory position is higher

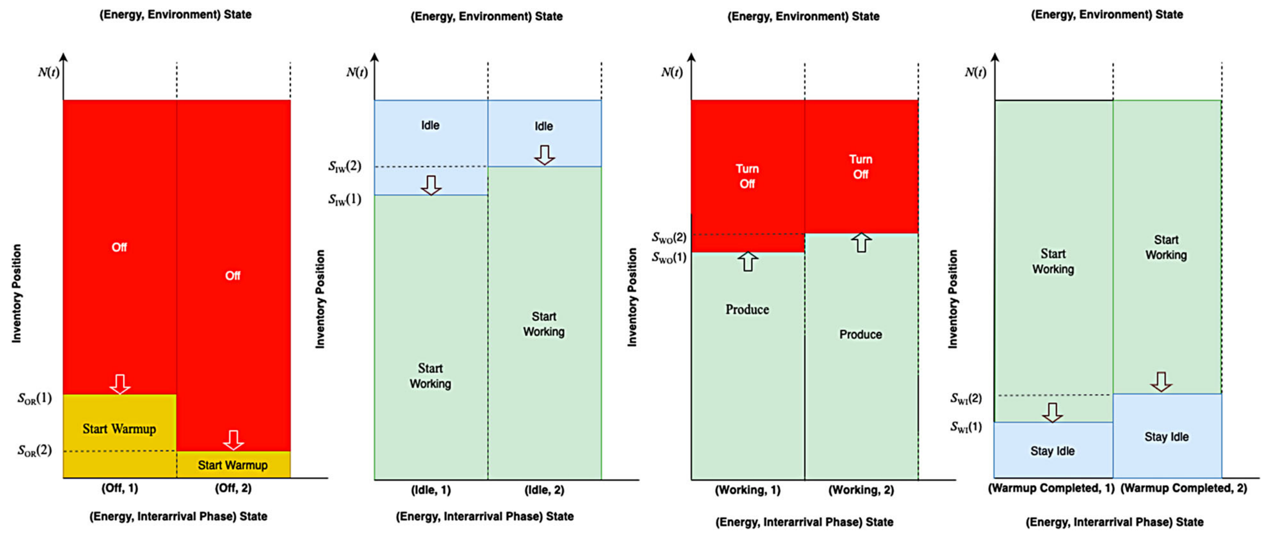

Figure 2. The optimal policy structure for a specific system where inter-arrival times have Erlang(2) and production and warmup times have exponential distributions.

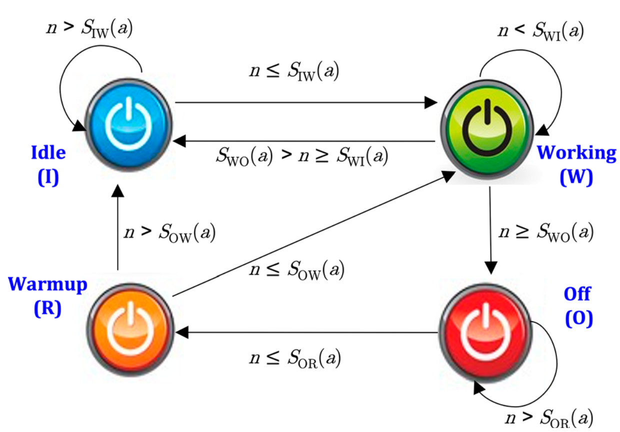

Figure 3. Phase-dependent threshold policy.

than $S_{OW}(a)$, the machine is kept idle (ready for production, but production does not start). Otherwise, i.e. $n \leq S_{OW}(a)$, production starts immediately following the completion of the warmup period.

Accordingly, the state-dependent decisions $u_{a,m,n}(t)$ depicted in Figure 1 are expressed depending on the thresholds $S_{ij}(a)$ as:

$$u_{a,\mathrm{W},n} = \begin{cases} 2, & \text{if } n \geq S_{\mathrm{WO}}(a) \\ 1, & \text{if } S_{\mathrm{WO}}(a) > n \geq S_{\mathrm{WI}}(a) \\ 0, & \text{if } n < S_{\mathrm{WI}}(a) \end{cases},$$

$$u_{a,\mathrm{O},n} = \begin{cases} 1, & \text{if } n \leq S_{\mathrm{OR}}(a) \\ 0, & \text{if } n > S_{\mathrm{OR}}(a) \end{cases}, \tag{7}$$

$$u_{a,\mathrm{R},n} = \begin{cases} 1, & \text{if } n > S_{\mathrm{OW}}(a) \\ 2, & \text{if } n \leq S_{\mathrm{OW}}(a) \end{cases},$$

$$u_{a,\mathrm{I},n} = \begin{cases} 1, & \text{if } n \leq S_{\mathrm{IW}}(a) \\ 0, & \text{if } n > S_{\mathrm{IW}}(a) \end{cases}. \tag{8}$$

5.1. Relationship between phase-dependent and time-based energy control policies

To implement the phase-dependent policies described in the preceding section, the phases of the inter-arrival time process need to be known when taking an action. However, the phase-type distribution used to model a given inter-arrival time distribution and the phase of the fitted distribution is typically not observable in practice. Accordingly, the current phase must be estimated from the elapsed times to implement this control policy for real-life cases. This section proposes a policy for dynamically controlling production and energy modes for make-to-stock systems with Erlang-distributed inter-event times. We show that the current phase of an inter-event time distribution modelled as a phase-type distribution can be predicted using the monitored inter-arrival times and predetermined time thresholds. As a result, a phase-dependent policy can be implemented by using these time thresholds and the inventory position. This relationship supports the *time- and buffer-based* policies that have been proposed as effective control policies in the literature (see, e.g. Frigerio and Matta 2015).

5.1.1. Prediction of the current phase based on the elapsed time for Erlang-type inter-event times

Let the transition time for an event, denoted with T, be a random variable having an Erlang(K) distribution with mean $E[T]$ and $P(t) \in \{1, \ldots, K\}$ be the phase of the Erlang distribution at time t. Let us assume that we can observe when the event is initiated, denoted with t_0, and ends, denoted with t_e; but we cannot observe the intermediary phase where the process is in at time $t = t_0 + \tau$, $t_0 \le t \le t_e$. Theorem 5.1 states that the current phase of the process at time t can be predicted based on the elapsed time τ.

Theorem 5.1: *Based on the elapsed time τ since an event that has an inter-event time distribution of Erlang(K) and mean $E[T]$ is initiated at time t_0, the phase of the Erlang(K) distribution that has the highest transient probability at $t_0 + \tau$ is k^*, i.e.* $\text{Prob}[P(t_0 + \tau) = k^*] > \text{Prob}[P(t_0 + \tau) = k], \forall k \neq k^*$ *where*

$$k^* = \begin{cases} \arg\min_k \left\{ k \mid \tau < \dfrac{k}{K} E[T] \right\}, \\ \quad \text{if } \tau < E[T],\ k = 1 \ldots, K, \\ K, \quad \text{if } \tau \ge E[T]. \end{cases} \tag{9}$$

Equivalently,

$$k^* = \begin{cases} k, \quad \text{if } \dfrac{k-1}{K} E[T] \le \tau < \dfrac{k}{K} E[T], \\ \qquad k = 1 \ldots, K, \\ K, \quad \text{if } \tau \ge E[T]. \end{cases} \tag{10}$$

Proof: For an Erlang process, the transient probability of being in state k at time t is given as:

$$\text{Prob}[P(t_0 + \tau) = k \mid P(t_0) = 1] = \frac{(\mu\tau)^{k-1}\, \mathrm{e}^{-\mu\tau}}{(k-1)!}, \quad k = 1, \ldots, K, \tag{11}$$

where $\mu = K/E[T]$. The condition for the phase with the highest probability at time $t_0 + \tau$, $\text{Prob}[P(t_0 + \tau) = k^*] > \text{Prob}[P(t_0 + \tau) = k], \forall k \neq k^*$ with the closed-form expression for the probability of being in a given phase given in Equation (11) gives the result in Equation (10). ■

The result given in Equation (10) shows that the current phase of the distribution can be predicted by using the time since the event is initiated and a set of critical times, $\frac{kE[T]}{K}$ where $k = 1, \ldots, K$ that depend on the mean $E[T]$ and the coefficient of variation (cv) of the inter-event time distribution where $K = 1/cv^2$. Therefore, a phase-dependent policy can be implemented by first predicting the current phase based on the elapsed time since the initiation of the event and the critical times that are determined in advance based on the mean and cv of the inter-event time distribution. Consequently, when the inter-arrival times follow an Erlang distribution, a phase-dependent policy is equivalent to a *time-based* policy.

This approach can be extended to predict the current phase for other phase-type distributions based on the observed elapsed times since the last arrivals. However, predicting the current phase for a general phase-type distribution requires using not only the most recent elapsed time since the last arrival, as in the case of Erlang distribution but a sequence of the elapsed times. Then, the sequence of the elapsed times can be analysed using an appropriate statistical method to predict the current phase. The predicted phase can then be used to implement the phase-dependent threshold policy.

6. Matrix geometric solution for the phase dependent control policy

The LP approach given in Section 4 yields the optimal policy and the performance under the optimal policy that uses phase-dependent thresholds. In this section, we present a matrix geometric method that evaluates the system's performance under optimal and approximate threshold policies that use fewer parameters that depend on a collection of phases.

The matrix geometric approach analyses the continuous-time Markov Chain (CTMC) that describes the dynamics of the system under a given control policy. We develop an algorithm to generate the state space and determine the transitions among the states automatically for a machine with the given inter-event time processes and the given control policy.

Figure 4 depicts the state-transition diagram for a system that operates in the working, idle, warmup, and off modes and is controlled by the optimal phase-dependent threshold policy. The system parameters determine the order of the optimal phase-dependent thresholds. Based on this order, certain thresholds are reached before the others. As a result, certain transitions from a given state may not be possible. Correspondingly, the system's dynamics can be determined by a subset of the thresholds. In this example, the dynamics of the system are determined by five thresholds.

After determining the state space and the transition rates, the system performance for the given control parameters can be evaluated by finding the long-run probabilities of the underlying CTMC. Then, the optimal control parameters can be determined by using an optimisation approach that evaluates the performance for different values of the control parameters at each step.

Since the backlog level can be infinite, determining the long-run probabilities by directly utilising the transition rate matrix is impossible. Truncating the backlog

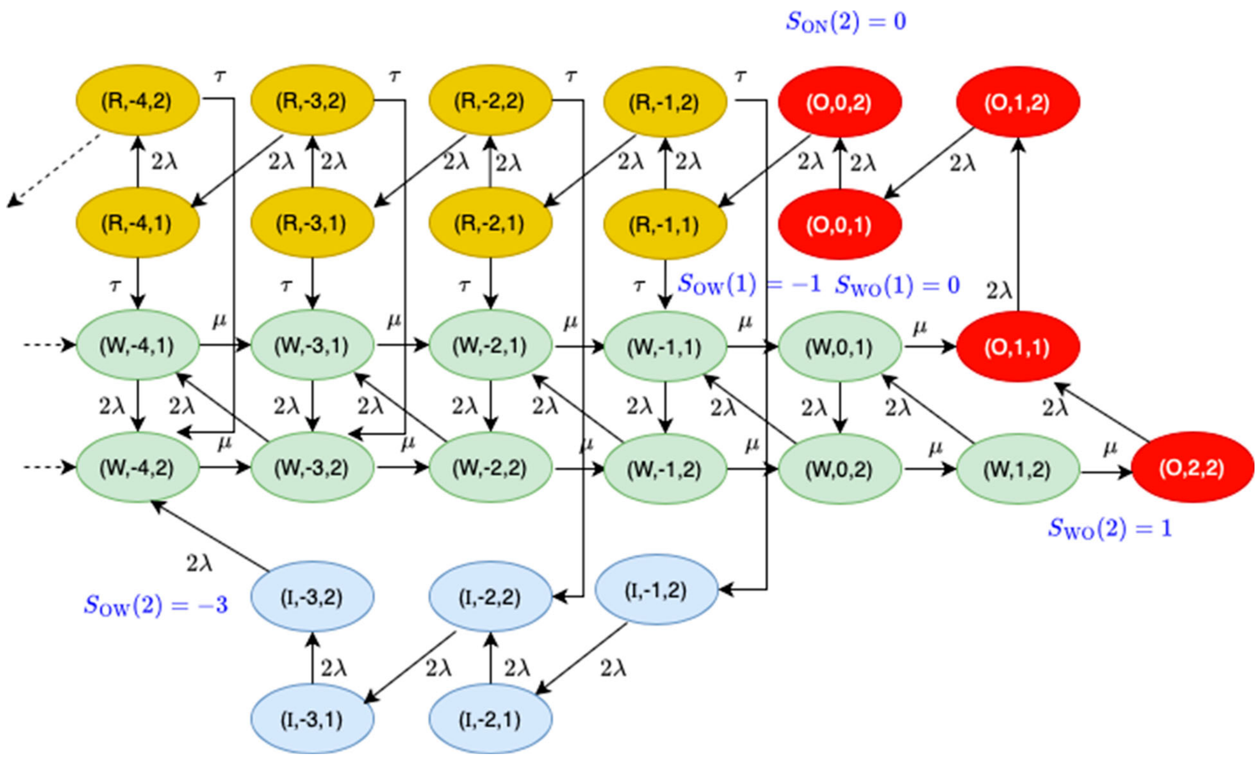

Figure 4. Continuous Time Markov Chain state transition diagram for a specific system that operates with the Working, Off and Idle modes.

level at a given level introduces evaluation errors. Furthermore, even after truncation, the size of the state space can be very large for efficient analysis (Tan, Karabağ, and Khayyati 2023). We present a matrix-geometric method to assess the system's performance controlled with the optimal policy efficiently and accurately without the need to truncate the backlog levels.

In the following part, we introduce an efficient method to determine the long-run probabilities of a machine where the arrival process is modelled as Marked Markovian Arrival Process (MMAP) and the warmup times and the production time process are modelled as Markovian Arrival Processes. Using MMAPs and MAPs for modelling inter-event times allows modelling correlated inter-event times with general inter-event time distributions that can be represented as phase-type distributions. Tan, Karabağ, and Khayyati (2023) use a similar approach to analyse a system with a *buffer-based threshold-type* energy mode and control policy. Our work extends this approach to a system controlled with a phase-dependent threshold policy.

6.1. MMAP model

We consider a setting where the arrival process, the warmup times, and the production time process are modelled with an MMAP, MAP, and MAP, respectively. Markov Arrival Processes (MAP) are used to model correlated arrivals with general distributions described as phase-type distribution using a continuous time Markov chain (Neuts 1979). As an extension of MAPs, MMAPs are used to model different types of arrivals referred to as marked arrivals together with the arrival process (He and Neuts 1998; Khayyati and Tan 2020). In our setting, arrivals are considered marked arrivals when the arrival process is in a given phase or one of the bundled phases. When a particular marking is observed, the corresponding threshold for this marking is used. Each time an event occurs, a marking is generated depending on the system state following this event and the corresponding state-dependent threshold to be used. As shown in Figure 3, the phase-dependent threshold policy is determined by five thresholds for each phase of the inter-arrival process a: $S_{WI}(a)$, $S_{WO}(a)$. $S_{OR}(a)$, $S_{OW}(a)$ and $S_{IW}(a)$. Using Marked Markov Arrival Processes facilitates modelling the system dynamics for such systems as a continuous time Markov process.

A MMAP X is described by the matrices $(\mathbf{X0}, \mathbf{X1}_1, \ldots, \mathbf{X1}_C)$. In this representation, $\mathbf{X0}$ has the transition rates that do not result in an arrival. In addition, $\mathbf{X1}_c$ has the transition rates that result in an arrival marked c, $c = 1, \ldots, C$ where C is the number of markings. The infinitesimal generator matrix of the process X is given as $\mathbf{X0} + \sum_{c=1}^{C} \mathbf{X1}_c$.

In our case, the state space is constructed to record the inventory position $N(t)$, the marking that indicates which phase-dependent threshold to use, the inter-arrival time process phase $A(t)$, and the machine mode $P(t)$ (either Off, Warmup including Warmup phase, Idle or Working including Processing time phase). Furthermore, for the optimal phase-dependent policy, each marking corresponds to a different phase of the inter-arrival process, i.e. $C = K_A$. For the phase-independent policy, $C = 1$. For an approximate policy that uses bundles of phases, the number of markings will be less than the number of arrival phases, i.e. $C < K_A$. For example, an approximate policy may bundle the arrival phases into two categories, such as high and low, and use different thresholds for the high and low phases. In this case, C will be 2. In this example, this approach decreases the number of policy parameters from $5K_A$ to 10 and will be computationally more efficient when the inter-arrival cv is low.

Since the state-dependent energy-mode control policy uses the current phase of the inter-arrival time, the energy mode, and the inventory/backlog level, the event arrivals coinciding with arrival-process phase changes and the production or warmup phase changes need to be differentiated. Accordingly, let $(\mathbf{A0}, \mathbf{A1}_1, \ldots, \mathbf{A1}_C, \mathbf{A2}_1, \ldots, \mathbf{xA2}_C)$ denote the inter-arrival time process for the markings, where $\mathbf{A0}$ has the transition rates that do not result in an arrival of a marking or a demand, $\mathbf{A1}_1, \ldots, \mathbf{A1}_C$ are for the arrivals of markings that coincide with the arrivals of demands, and $\mathbf{A2}_1, \ldots, \mathbf{A2}_C$ are for transitions that result in arrival of a marking without a demand, e.g. transitions for the phase changes that coincide with marking arrivals, before reaching the last phase that generates the demand arrival. In this case, $Q_A = \mathbf{A0} + \sum_{c=1}^{C}(\mathbf{A1}_c + \mathbf{A2}_c)$.

Let $\mathbf{Y}$ denote the CTMC model of the system. Let π denote the long-run probability vector for the transition matrix $\mathbf{Y}$. If the state-space is truncated at an appropriate backlog level, the long-run probability vector π can be obtained by finding the solution of the equations $\pi\mathbf{Y} = 0$ and $\pi u = 1$ where u is a column vector of 1s. In the next section, we present a matrix geometric method that exploits the special structure of the transition matrix to obtain the long-run probabilities without the need to construct the transition matrix fully and then truncate it.

6.2. Obtaining the long-run probabilities and the performance measures

The transition matrix $\mathbf{Y}$ for the system controlled with a phase-dependent control policy has a block-triangular structure as shown in Equation (12).

$$\mathbf{Y} = \begin{bmatrix} \mathbf{B}_{0,0} & \mathbf{B}_{0,1} & & & & \\ \mathbf{B}_{1,0} & \mathbf{B}_{1,1} & \mathbf{G}_0 & & & \\ & \mathbf{B}_{2,1} & \mathbf{G}_1 & \mathbf{G}_0 & & \\ & & \mathbf{G}_2 & \mathbf{G}_1 & \mathbf{G}_0 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \tag{12}$$

The blocks of the transition matrix $\mathbf{Y}$ are generated according to the events related to arrivals, markings, and energy mode changes. The formulas for obtaining $\mathbf{Y}$ directly from the matrices are associated with the MMAP representation of demand inter-arrival, warmup, and production times and then forming the submatrices defining the block tridiagonal structure of $\mathbf{Y}$ are given in A.

The long-run probabilities can be determined without truncating the infinite-size state space by utilising the matrix geometric method (Neuts 1994). Once the blocks of the transition matrix $\mathbf{Y}$ are determined based on the MMAP representation of the inter-event times, the balance equations corresponding to the equations $\pi\mathbf{Y} = 0$ and $\pi u = 1$ are written using these blocks. To rewrite the balance equations using the blocks given above (G_0, G_1, G_2, $B_{1,1}$, $B_{2,1}$, $B_{0,0}$, $B_{0,1}$, $B_{1,0}$), we represent the long-run probability vector as a concatenation of vectors for the boundary and repeating states using $\pi = [\mathbf{b}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots]$. Then, the balance equations can be expressed as:

$$\mathbf{b}\mathbf{B}_{0,0} + \mathbf{v}_0\mathbf{B}_{1,0} = \mathbf{0}, \tag{13}$$

$$\mathbf{b}\mathbf{B}_{0,1} + \mathbf{v}_0\mathbf{B}_{1,1} + \mathbf{v}_1\mathbf{B}_{2,1} = \mathbf{0}, \tag{14}$$

$$\mathbf{v}_j\mathbf{G}_0 + \mathbf{v}_{j+1}\mathbf{G}_1 + \mathbf{v}_{j+2}\mathbf{G}_2 = \mathbf{0}, \quad j = 0, 1, 2, \ldots, \tag{15}$$

$$\mathbf{b1} + \sum_{j=0}^{\infty} \mathbf{v}_j\mathbf{1} = 1. \tag{16}$$

A general solution for the long-run probability vectors $\mathbf{v}_j$, $j = 0, 1, \ldots$ is $\mathbf{v}_j = \mathbf{v}_0\mathbf{R}^j$ where $\mathbf{R} \in \mathbb{R}^{\beta\times\beta}$ is calculated by iteratively applying

$$\mathbf{R} = -\left(\mathbf{G}_0 + \mathbf{R}^2\mathbf{G}_2\right)\mathbf{G}_1^{-1} \tag{17}$$

until convergence. Substituting $\mathbf{v}_j = \mathbf{v}_0\mathbf{R}^j$ in Equations (13)–(16) yields a set of linear equations that can be solved to obtain the long-run distribution $\pi = [\mathbf{b}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots]$.

6.3. Obtaining the performance measures

Once the long-run probability vector π is obtained, the probability of the inventory level being s is determined as $(\mathbf{e}_{i=\bar{S}-s+1,X} \otimes \mathbf{1}_{1,C} \otimes \mathbf{1}_{1,K_A} \otimes \mathbf{1}_{1,4} \otimes \mathbf{1}_{1,K_R} \otimes \mathbf{1}_{1,K_W})\pi^{\mathrm{T}}$ and the probability of being in energy mode m is $(\mathbf{1}_{1,X} \otimes \mathbf{1}_{1,C} \otimes \mathbf{1}_{1,K_A} \otimes \mathbf{e}_{m,4} \otimes \mathbf{1}_{1,K_R} \otimes \mathbf{1}_{1,K_W})\pi^{\mathrm{T}}$, where $\mathbf{e}_{i,x}$ is a row vector of size x with its i^{th} entry as 1 and 0 elsewhere, $\mathbf{1}_{i,j}$ denotes a $i \times j$ matrix of ones and π^{T} is the transpose of π.

The long-run probabilities of being in particular energy modes and the probabilities of the inventory level being at different levels allow determining the average total cost rate that includes the energy cost and also the inventory holding and backlog costs for given threshold levels.

6.4. Determination of control parameters

The policy considered here has five thresholds for each marking $S_{\mathrm{WI}}(a)$, $S_{\mathrm{WO}}(a)$, $S_{\mathrm{OR}}(a)$, $S_{\mathrm{OW}}(a)$ and $S_{\mathrm{IW}}(a)$ where a is the phase of the demand inter-arrival process. Hence, optimising the system requires optimising $5C$ parameters. The optimisation can be performed using heuristic methods, e.g. the genetic algorithm. The computational effort for performance evaluation of a set of thresholds depends on two main steps. The first step is calculating the matrix $\mathbf{R}$ for given system parameters by using Equation (17), and the second step is solving the balance equations given in Equations (13)–(16) by using $\mathbf{R}$. This step is computationally very efficient. For example, for an approximate policy that bundles the phases in high and low categories, the performance of the system under this policy with its 10 thresholds can be calculated under 0.2 s. The time to determine the optimal parameters depends on the algorithm used for optimisation and its parameters. The process of optimising the system is improved by calculating $\mathbf{R}$ once and then repeating the second step while evaluating the performance of a set of thresholds.

7. Numerical results

This section presents our numerical results for the system with Erlang demand inter-arrival times and exponential processing and warmup times. Our main objective in these numerical experiments is to understand how the system parameters affect the control policy and how incorporating the phase of the inter-arrival time that indicates the elapsed time since the last arrival improves the performance.

To examine how the system parameters affect the performance of the control policy, the optimal average cost is obtained for different arrival rates, inter-arrival time cvs, and warmup costs according to the values given in Table 1. The phase-dependent solution is obtained by solving the corresponding LP formulation given in Appendix A. The solution time for the LP solution for each case is under 30 s using Gurobi optimiser (version 5.5.0) in GAMS on a computer with Intel(R) Core(TM) i7-1165G7 CPU (2.80 GHz, 16 GB RAM). The phase-independent solution is obtained by using the Matrix Geometric approach given in Section 6 in 2 s on the same computer. The number of parameters defining the optimal phase-dependent policies in these experiments varies between 5 for the cases with $cv_2^a = 1$ and 50 for the cases with $cv_2^a = 0.1$. The number of parameters of the phase-independent policy is 5 for all values of cv_2^a.

Table 1. Parameter of the numerical experiments with the Erlang inter-event times.

Parameter	Set	Parameter	Set
μ	{1}	c^+	{1}
λ	{0.5, 0.6, 0.7, 0.8, 0.9}	c_I	{50}
τ	{0.2}	c_W	{100}
cv_a^2	{0.1, 0.25, 0.5, 1}	c_O	{0}
cv_s^2	{1}	c_R	{150, 200, 250, 300}
cv_R^2	{1}	c^-	{3}

Figure 5 illustrates how the optimal cost changes with the arrival rate for different values of the inter-arrival time cv and the warmup cost. The figure indicates that the average cost increases as the demand inter-arrival rate and the warmup cost increase. A higher cv of the demand inter-arrival rate and the warmup cost leads to a higher cost for the same demand inter-arrival rate and the warmup cost.

We analyse the effect of using the phase of the inter-arrival time that indicates the elapsed time since the last arrival in the control policy by comparing the average cost obtained by utilising the phase-dependent, phase-independent thresholds policies, and always-on policy.

The energy-saving level that will be obtained by using the proposed policy can be evaluated by comparing the results for the proposed policy with the always-on policy. The always-on policy controls only the machine for inventory and backlog costs by keeping it working or idle without turning it off. The phase-dependent policies use the current phase of the inter-arrival time together with the current inventory position. In contrast, the phase-independent policies only use the current inventory position to decide machine mode changes. The optimal parameters of the policies are determined by using the approach described in Section 6.

Table 2 gives the comparison of the total costs obtained by the phase-dependent and the always-on policies for a production system where the demand inter-arrival times

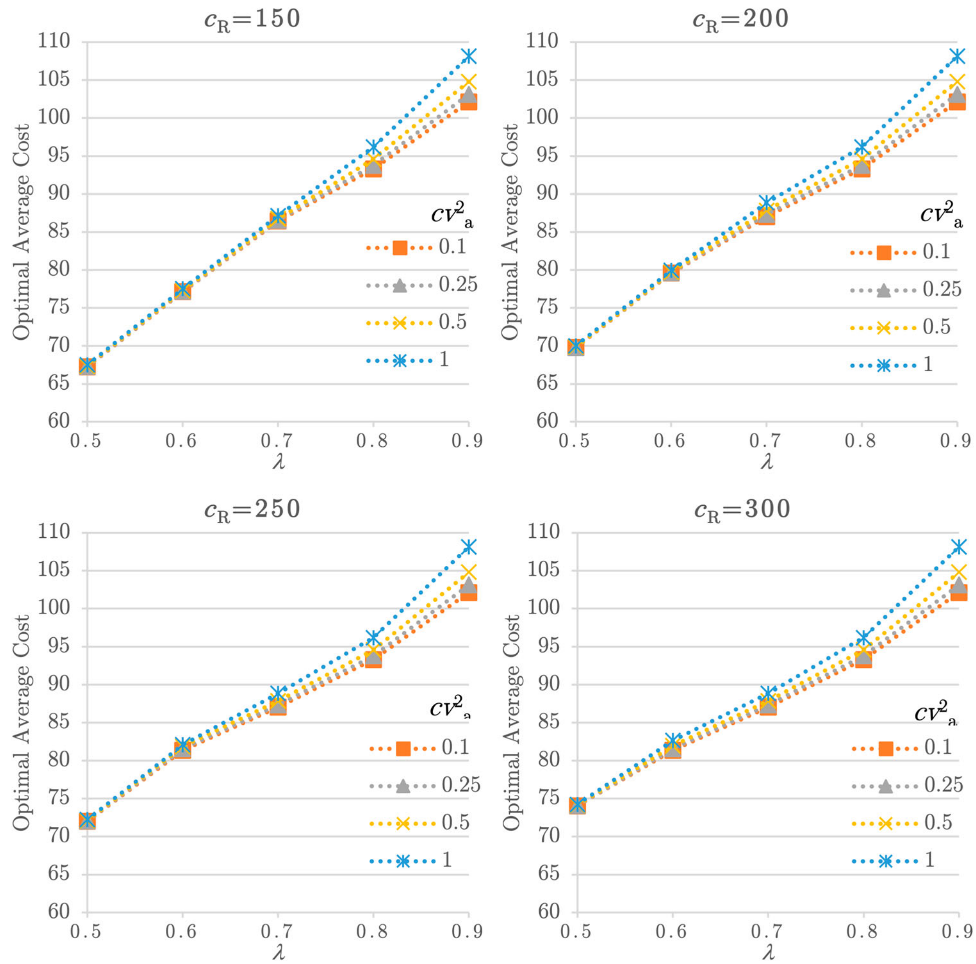

Figure 5. The effect of the arrival rate on the optimal cost for different values of the inter-arrival time *cv* and the warmup cost ($c_W = 100$, $c_I = 50$, $c_O = 0$, $c^+ = 1$, $c^- = 3$, $\mu = 1$, $\tau = 0.2$).

have Erlang(2), the production and warmup times have exponential distributions for different utilisation levels and warmup costs. The always-on policy uses a single threshold parameter to decide when to stop production and stay idle, depending on the inventory level.

Our numerical results indicate that the control policy that utilises the current phase yields significant advantages compared to the always-on policy that does not control energy modes together with the production especially when the system utilisation is low. For example, when the utilisation is 50% ($\lambda = 0.5$), the proposed policy yields an energy saving of 12% compared to the always-on policy and when the utilisation is 60% ($\lambda = 0.6$), the proposed policy yields an energy saving of 6% compared to the always-on policy. However, as the utilisation increases, the energy-saving potential diminishes. This is because when the arrival rate is high, the production resource cannot be turned off due to the warmup delay. In this case, the production resource switches between the idle and working modes and, therefore, operates according to the always-on policy. In this case, there will be no energy saving for a heavily utilised resource. However, when the arrival rate is low, the unutilised time can be managed effectively by turning off the machine and then making it operational at the right time, considering the warmup delay to meet the demand.

Table 2 also compares the total costs obtained by the same system's phase-dependent and phase-independent threshold policies. The phase-independent policy uses 5 thresholds S_{WI}, S_{WO}. S_{OR}, S_{OW} and S_{IW} that determine the rules of switching among the energy modes independent of the inter-arrival process. The results show that the phase-independent policy with the right parameters yields an average cost very close to the cost obtained with the optimal phase-dependent policy for the case with Erlang inter-arrival and exponential production and warmup times. The average and maximum percentage differences for all the cases are 0.06% and 0.23%, respectively. For this case, having exponential production and exponential warmup times diminishes the effect of incorporating the elapsed time information in the policy. That is, the effect of changing the decision based on the elapsed

Table 2. Comparison of the total costs obtained by the phase-dependent, phase-independent, and always-on policies for a system where the inter-arrival times have Erlang, the production and warmup times have exponential distribution ($c_W = 100$, $c_I = 50$, $c_O = 0$, $c^+ = 1$, $c^- = 3$, $\mu = 1$, $\tau = 0.2$).

		Optimal Cost with Phase-Dependent Policy				Optimal Cost with Phase-Independent Policy				Optimal Cost with Always On Policy			
		Coef. of Var. of Inter-arrival times (cv^2_a)				Coef. of Var. of Inter-arrival times (cv^2_a)				Coef. of Var. of Inter-arrival times (cv^2_a)			
c_R	λ	0.10	0.25	0.50	1.00	0.10	0.25	0.50	1.00	0.10	0.25	0.50	1.00
150	0.5	67.241	67.281	67.349	67.500	67.242	67.281	67.363	67.500	76.141	76.278	76.580	77.167
150	0.6	77.139	77.198	77.304	77.542	77.145	77.198	77.311	77.542	81.464	81.737	82.104	82.852
150	0.7	86.458	86.557	86.739	87.163	86.460	86.559	86.758	87.163	87.109	87.399	87.974	89.004
150	0.8	93.310	93.793	94.595	96.194	93.337	93.796	94.602	96.194	93.343	93.870	94.650	96.264
150	0.9	102.131	103.135	104.806	108.151	102.134	103.139	104.810	108.151	102.152	103.158	104.842	108.185
200	0.5	69.774	69.808	69.869	70.009	69.776	69.815	69.893	70.009	76.141	76.278	76.580	77.167
200	0.6	79.590	79.641	79.732	79.944	79.595	79.645	79.758	79.944	81.464	81.737	82.104	82.852
200	0.7	87.028	87.335	87.844	88.868	87.030	87.395	87.854	88.868	87.109	87.399	87.974	89.004
200	0.8	93.310	93.793	94.595	96.194	93.337	93.796	94.602	96.194	93.343	93.870	94.650	96.264
200	0.9	102.131	103.135	104.806	108.151	102.134	103.139	104.810	108.151	102.152	103.158	104.842	108.185
250	0.5	72.017	72.048	72.103	72.229	72.024	72.061	72.110	72.229	76.141	76.278	76.580	77.167
250	0.6	81.380	81.599	81.896	82.086	81.464	81.635	81.896	82.086	81.464	81.737	82.104	82.852
250	0.7	87.028	87.335	87.844	88.868	87.030	87.395	87.854	88.868	87.109	87.399	87.974	89.004
250	0.8	93.310	93.793	94.595	96.194	93.337	93.796	94.602	96.194	93.343	93.870	94.650	96.264
250	0.9	102.131	103.135	104.806	108.151	102.134	103.139	104.810	108.151	102.152	103.158	104.842	108.185
300	0.5	74.052	74.081	74.132	74.238	74.060	74.083	74.188	74.238	76.141	76.278	76.580	77.167
300	0.6	81.380	81.599	81.972	82.660	81.464	81.635	81.974	82.660	81.464	81.737	82.104	82.852
300	0.7	87.028	87.335	87.844	88.868	87.030	87.395	87.854	88.868	87.109	87.399	87.974	89.004
300	0.8	93.310	93.793	94.595	96.194	93.337	93.796	94.602	96.194	93.343	93.870	94.650	96.264
300	0.9	102.131	103.135	104.806	108.151	102.134	103.139	104.810	108.151	102.152	103.158	104.842	108.185

time is limited since the system dynamics following the change are affected more by the variability in the production and warmup times. Our additional numerical experiments with Erlang inter-arrival and Erlang production and warmup times show that as the production and warmup times become less variable, the benefit of using a phase-dependent policy that incorporates the elapsed time since the last arrival increases.

8. Conclusion

In this paper, we explore the optimal control policies to control production and the energy mode for a machine operating in various energy modes, including working, off, idle, and warmup, with inter-arrival, production, and warmup times that have phase-type distributions. The optimal control solution that minimises the expected costs associated with energy usage in different modes and inventory and backlog costs is determined by solving a linear program associated with the MDP. A matrix-geometric method is presented to analyse the system performance under the optimal control policy.

This study extends the results for the dynamic on/off control problem given in the literature for systems with exponential inter-arrival times and presents the optimal policy when the inter-arrival times have phase-type distribution. More specifically, based on the LP solution of the underlying MDP of the problem for a wide range of parameters, we reveal that when the inter-arrival time distribution is not exponential, the optimal policy is defined by the current phase of the inter-arrival time together with the inventory position. In addition, we present a method to predict the current phase of the inter-arrival process by using the time elapsed since the last arrival. Therefore, by analysing the structure of the optimal policies, we offer an analytical basis to analyse and implement *time- and buffer-based* policies. Our numerical analysis shows that buffer-based policies can yield near-optimal results for systems with high production and warmup time variability if their parameters are chosen appropriately. In addition, when the system utilisation is low, using the current phase to control energy modes outperforms the *always-on* policy, which does not coordinate energy modes with production.

This work can be extended in several ways. First, our work focuses on a system with a single machine. Extending our work to determine the optimal energy and production control strategies for a system with multiple machines is a future research direction. Second, in this study, the optimal policy to control energy mode and production in a system with phase-type inter-arrival times is obtained numerically through the solution of the LP associated with the MDP. Exploring the analytical characterisation of the optimal policy structure is a challenging future research direction. Third, developing a method that estimates the current state to implement time- and buffer-based policies in systems with MAP inter-event times is also left for future research. Lastly, in our work, we assume that the machine never breaks down, i.e. is reliable. Extending this work to a setting with an unreliable machine enables us to examine the impact of maintenance decisions on the system and policy performance metrics. This is also left for future research.

In conclusion, we show that policies that only use the inventory position information can be effective if the control parameters are chosen appropriately. However, as the inter-arrival time variability decreases, the benefit of using the time- and buffer-based policies increases.

Disclosure statement

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Data availability statement

Data sharing is not applicable to this article as no new data were created or analysed in this study.

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Appendices

Appendix 1. LP formulation for the Case with Erlang inter-arrival, Exponential Production, and Warmup Times

In this section, we give the linear programming formulation for the case in which the time between two successive demand arrivals is considered to follow an Erlang process. The inventory level is limited to a range between $\underline{B}$ and $\overline{B}$, with any excess or shortage above or below this range being truncated with these two levels. We determine these truncation levels using the procedure described in the main text.

The decision variables of the LP formulation are denoted by $Y_{a,m,n,u_{a,m,n}}$ where $a \in A$, $m \in \mathbf{M}$, $n \in \mathcal{N}$, $u_{a,m,n} \in U_m$. To simplify the notation, in the formulation, Y_{a,m,n,u_m} is used instead of $Y_{a,m,n,u_{a,m,n}}$. These decision variables are interpreted as the long-run fraction of the time spent in state (a, m, n) with action u_m. With the given sets and decision variables, the formulation is:

$$\begin{aligned}\min \quad \pi = &\sum_{a\in\mathbf{A}}\sum_{m\in\mathbf{M}}\sum_{n\in\mathcal{N}_1}\sum_{u_m\in U_m} c^+ n Y_{a,m,n,u_m} \\ &+\sum_{a\in\mathbf{A}}\sum_{m\in\mathbf{M}}\sum_{n\in\mathcal{N}_2}\sum_{u_m\in U_m} c^- |n| Y_{a,m,n,u_m} \\ &+\sum_{a\in\mathbf{A}}\sum_{n\in\mathcal{N}}\sum_{u_I\in U_I} c_I Y_{a,I,n,u_I} \\ &+\sum_{a\in\mathbf{A}}\sum_{n\in\mathcal{N}}\sum_{u_R\in U_R} c_R Y_{a,R,n,u_R} \\ &+\sum_{a\in\mathbf{A}}\sum_{n\in\mathcal{N}}\sum_{u_W\in U_W} c_W Y_{a,W,n,u_W} \\ &+\sum_{a\in\mathbf{A}}\sum_{n\in\mathcal{N}}\sum_{u_O\in U_0} c_O Y_{a,0,n,u_0} \end{aligned} \tag{A1}$$

subject to

$$\begin{aligned}&\sum_{u_W\in U_W} Y_{1,W,\underline{B},u_W} - \sum_{u_W\in U_W} \lambda Y_{K_A,W,\underline{B},u_W} \\ &- \sum_{u_W\in U_W} \lambda Y_{K_A,W,\underline{B}+1,u_W} - \lambda Y_{K_A,I,\underline{B},1} \\ &- \lambda Y_{K_A,I,\underline{B}+1,1} - \delta Y_{1,R,\underline{B},1} \\ &- \sum_{u_W\in U_W} (1-\lambda-\mu) Y_{1,W,\underline{B},u_W} \\ &= 0, \quad \forall\, a \in \{1\}, m \in \{W\}, n \in \{\underline{B}\},\end{aligned} \tag{A2}$$

$$\begin{aligned}&\sum_{u_W\in U_W} Y_{1,W,n,u_W} - \mu Y_{1,W,n-1,1} \\ &- \sum_{u_W\in U_W} \lambda Y_{K_A,W,n+1,u_W} - \lambda Y_{K_A,I,n+1,1} \\ &- \delta Y_{1,R,n,1} \\ &- \sum_{u_W\in U_W} (1-\lambda-\mu) Y_{1,W,n,u_W} \\ &= 0, \quad \forall\, a \in \{1\}, m \in \{W\}, n \in \mathcal{N}_4,\end{aligned} \tag{A3}$$

$$\begin{aligned}&\sum_{u_W\in U_W\setminus\{1\}} Y_{1,W,\bar{B}-1,u_W} - \mu Y_{1,W,\bar{B}-2,1} \\ &- \lambda Y_{K_A,I,\bar{B},1} - \delta Y_{1,R,\bar{B}-1,1} \\ &- \sum_{u_W\in U_W\setminus\{1\}} (1-\lambda-\mu) \\ &\times Y_{1,W,\bar{B}-1,u_W} = 0, \\ &\forall\, a \in \{1\}, m \in \{W\}, n \in \{\bar{B}-1\},\end{aligned} \tag{A4}$$

$$\begin{aligned}&\sum_{u_W\in U_W} Y_{a,W,\underline{B},u_W} - \sum_{u_W\in U_W} \lambda Y_{a-1,W,\underline{B},u_W} \\ &- \lambda Y_{a-1,I,\underline{B},1} - \delta Y_{a,R,\underline{B},1} \\ &- \sum_{u_W\in U_W} (1-\lambda-\mu) Y_{a,W,\underline{B},u_W} \\ &= 0, \quad \forall\, a \in A\setminus\{1\}, m \in \{W\}, n \in \{\underline{B}\},\end{aligned} \tag{A5}$$

$$\begin{aligned}&\sum_{u_W\in U_W} Y_{a,W,n,u_W} - \mu Y_{a,W,n-1,1} \\ &- \sum_{u_W\in U_W} \lambda Y_{a-1,W,n,u_W} \\ &- \lambda Y_{a-1,I,n,1} - \delta Y_{a,R,n,1} \\ &- \sum_{u_W\in U_W} (1-\lambda-\mu) Y_{a,W,n,u_W}\end{aligned}$$

$$= 0, \quad \forall\, a \in A \setminus \{1\}, m \in \{\mathrm{W}\}, n \in \mathcal{N}_4, \tag{A6}$$

$$\sum_{u_\mathrm{W} \in U_\mathrm{W} \setminus \{1\}} Y_{a,\mathrm{W},\bar{B}-1,u_\mathrm{W}} - \mu Y_{a,\mathrm{W},\bar{B}-2,1} - \lambda Y_{a-1,\mathrm{I},\bar{B}-1,1} - \delta Y_{a,\mathrm{R},\bar{B}-1,1} - \sum_{u_\mathrm{W} \in U_\mathrm{W} \setminus \{1\}} \lambda Y_{a-1,\mathrm{W},\bar{B}-1,u_\mathrm{W}} - \sum_{u_\mathrm{W} \in U_\mathrm{W} \setminus \{1\}} (1-\lambda-\mu) Y_{a,\mathrm{W},\bar{B}-1,u_\mathrm{W}} = 0,$$
$$\forall\, d \in A \setminus \{1\}, m \in \{\mathrm{W}\}, n \in \left\{\bar{B}-1\right\}, \tag{A7}$$

$$\sum_{u_\mathrm{I} \in U_\mathrm{I}} Y_{1,\mathrm{I},\underline{B},u_\mathrm{I}} - \lambda Y_{K_A,\mathrm{I},\underline{B},0} - \lambda Y_{K_A,\mathrm{I},\underline{B}+1,0} - \delta Y_{1,\mathrm{R},\underline{B},0} - \sum_{u_\mathrm{I} \in U_\mathrm{I}} (1-\lambda) Y_{1,\mathrm{I},\underline{B},u_\mathrm{I}}$$
$$= 0, \quad \forall\, a \in \{1\}, m \in \{\mathrm{I}\}, n \in \left\{\underline{B}\right\}, \tag{A8}$$

$$\sum_{u_\mathrm{I} \in U_\mathrm{I}} Y_{1,\mathrm{I},n,u_\mathrm{I}} - \lambda Y_{K_A,\mathrm{I},n+1,0} - \mu Y_{1,\mathrm{W},n-1,0} - \delta Y_{1,\mathrm{R},n,0} - \sum_{u_\mathrm{I} \in U_\mathrm{I}} (1-\lambda) Y_{1,\mathrm{I},n,u_\mathrm{I}} = 0,$$
$$\forall\, a \in \{1\}, m \in \{\mathrm{I}\}, n \in \mathcal{N}_3, \tag{A9}$$

$$\sum_{u_\mathrm{I} \in U_\mathrm{I} \setminus \{1\}} Y_{1,\mathrm{I},\bar{B},u_\mathrm{I}} - \mu Y_{1,\mathrm{W},\bar{B}-1,0} - \delta Y_{1,\mathrm{R},\bar{B},0} - \sum_{u_\mathrm{I} \in U_\mathrm{I} \setminus \{1\}} (1-\lambda) Y_{1,\mathrm{I},\bar{B},u_\mathrm{I}} = 0,$$
$$\forall\, a \in \{1\}, m \in \{\mathrm{I}\}, n \in \left\{\bar{B}\right\}, \tag{A10}$$

$$\sum_{u_\mathrm{I} \in U_\mathrm{I}} Y_{a,\mathrm{I},\underline{B},u_\mathrm{I}} - \lambda Y_{a-1,\mathrm{I},\underline{B},0} - \delta Y_{a,\mathrm{R},\underline{B},0} - \sum_{u_\mathrm{I} \in U_\mathrm{I}} (1-\lambda) Y_{a,\mathrm{I},\underline{B},u_\mathrm{I}} = 0,$$
$$\forall\, a \in A \setminus \{1\}, m \in \{\mathrm{I}\}, n \in \left\{\underline{B}\right\}, \tag{A11}$$

$$\sum_{u_\mathrm{I} \in U_\mathrm{I}} Y_{a,\mathrm{I},n,u_\mathrm{I}} - \lambda Y_{a-1,\mathrm{I},n,0} - \mu Y_{a,\mathrm{W},n-1,0} - \delta Y_{a,\mathrm{R},n,0} - \sum_{u_\mathrm{I} \in U_\mathrm{I}} (1-\lambda) Y_{a,\mathrm{I},n,u_\mathrm{I}} = 0,$$
$$\forall\, a \in A \setminus \{1\}, m \in \{\mathrm{I}\}, n \in \mathcal{N}_3, \tag{A12}$$

$$\sum_{u_\mathrm{I} \in U_\mathrm{I} \setminus \{1\}} Y_{a,\mathrm{I},\bar{B},u_\mathrm{I}} - \lambda Y_{a-1,\mathrm{I},\bar{B},0} - \mu Y_{a,\mathrm{W},\bar{B}-1,0} - \delta Y_{a,\mathrm{R},\bar{B},0} - \sum_{u_\mathrm{I} \in U_\mathrm{I}} (1-\lambda) Y_{a,\mathrm{I},\bar{B},u_\mathrm{I}} = 0,$$
$$\forall\, a \in A \setminus \{1\}, m \in \{\mathrm{I}\}, n \in \left\{\bar{B}\right\}, \tag{A13}$$

$$\sum_{u_\mathrm{O} \in U_\mathrm{O}} Y_{1,\mathrm{O},\underline{B},u_\mathrm{O}} - \lambda Y_{K_A,\mathrm{O},\underline{B},0} - \lambda Y_{K_A,\mathrm{O},\underline{B}+1,0} - \lambda Y_{K_A,\mathrm{I},\underline{B},2} - \lambda Y_{K_A,\mathrm{I},\underline{B}+1,2} - \sum_{u_\mathrm{O} \in U_\mathrm{O}} (1-\lambda) Y_{1,\mathrm{O},\underline{B},u_\mathrm{O}} = 0,$$
$$\forall\, a \in \{1\}, m \in \{\mathrm{O}\}, n \in \left\{\underline{B}\right\}, \tag{A14}$$

$$\sum_{u_\mathrm{O} \in U_\mathrm{O}} Y_{1,\mathrm{O},n,u_\mathrm{O}} - \lambda Y_{K_A,\mathrm{O},n+1,0} - \lambda Y_{K_A,\mathrm{I},n+1,2} - \mu Y_{1,\mathrm{W},n-1,2} - \sum_{u_\mathrm{O} \in U_\mathrm{O}} (1-\lambda) Y_{1,\mathrm{O},n,u_\mathrm{O}} = 0,$$
$$\forall\, a \in \{1\}, m \in \{\mathrm{O}\}, n \in \mathcal{N}_3, \tag{A15}$$

$$\sum_{u_\mathrm{O} \in U_\mathrm{O}} Y_{1,\mathrm{O},\bar{B},u_\mathrm{O}} - \mu Y_{1,\mathrm{W},\bar{B}-1,2} - \sum_{u_\mathrm{O} \in U_\mathrm{O}} (1-\lambda) Y_{1,\mathrm{O},\bar{B},u_\mathrm{O}} = 0,$$
$$\forall\, a \in \{1\}, m \in \{\mathrm{O}\}, n \in \left\{\bar{B}\right\}, \tag{A16}$$

$$\sum_{u_\mathrm{O} \in U_\mathrm{O}} Y_{a,\mathrm{O},\underline{B},u_\mathrm{O}} - \lambda Y_{a-1,\mathrm{O},\underline{B},0} - \lambda Y_{a-1,\mathrm{I},\underline{B},2} - \sum_{u_\mathrm{O} \in U_\mathrm{O}} (1-\lambda) Y_{a,\mathrm{O},\underline{B},u_\mathrm{O}} = 0,$$
$$\forall\, a \in A \setminus \{1\}, m \in \{\mathrm{O}\}, n \in \left\{\underline{B}\right\}, \tag{A17}$$

$$\sum_{u_\mathrm{O} \in U_\mathrm{O}} Y_{a,\mathrm{O},n,u_\mathrm{O}} - \lambda Y_{a-1,\mathrm{O},n,0} - \lambda Y_{a-1,\mathrm{I},n,2} - \mu Y_{a,\mathrm{W},n-1,2} - \sum_{u_\mathrm{O} \in U_\mathrm{O}} (1-\lambda) Y_{a,\mathrm{O},n,u_\mathrm{O}} = 0,$$
$$\forall\, a \in A \setminus \{1\}, m \in \{\mathrm{O}\}, n \in \mathcal{N} \setminus \left\{\underline{B}\right\}, \tag{A18}$$

$$\sum_{u_\mathrm{R} \in U_\mathrm{R}} Y_{1,\mathrm{R},\underline{B},u_\mathrm{R}} - \sum_{u_\mathrm{R} \in U_\mathrm{R}} \lambda Y_{K_A,\mathrm{R},\underline{B},u_\mathrm{R}} - \sum_{u_\mathrm{R} \in U_\mathrm{R}} \lambda Y_{K_A,\mathrm{R},\underline{B}+1,u_\mathrm{R}} - \lambda Y_{K_A,\mathrm{O},\underline{B},1} - \lambda Y_{K_A,\mathrm{O},\underline{B}+1,1} - \sum_{u_\mathrm{R} \in U_\mathrm{R}} (1-\lambda-\delta) Y_{1,R,\underline{B},u_\mathrm{R}} = 0,$$
$$\forall\, a \in \{1\}, m \in \{\mathrm{R}\}, n \in \left\{\underline{B}\right\}, \tag{A19}$$

$$\sum_{u_\mathrm{R} \in U_\mathrm{R}} Y_{1,\mathrm{R},n,u_\mathrm{R}} - \sum_{u_\mathrm{R} \in U_\mathrm{R}} \lambda Y_{K_A,\mathrm{R},n+1,u_\mathrm{R}} - \lambda Y_{K_A,\mathrm{O},n+1,1} - \sum_{u_\mathrm{R} \in U_\mathrm{R}} (1-\lambda-\delta) \times Y_{1,R,n,u_\mathrm{R}} = 0,$$
$$\forall\, a \in \{1\}, m \in \{\mathrm{R}\}, n \in \mathcal{N}_4, \tag{A20}$$

$$\sum_{u_R\in U_R} Y_{1,R,\bar{B}-1,u_R} - \sum_{u_R\in U_R\setminus\{1\}} \lambda Y_{K_A,R,\bar{B},u_R} - \lambda Y_{K_A,O,\bar{B},1} - \sum_{u_R\in U_R} (1-\lambda-\delta) \times Y_{1,R,\bar{B}-1,u_R} = 0,$$
$$\forall a \in \{1\}, m \in \{R\}, n \in \{\bar{B}-1\}, \quad \text{(A21)}$$

$$\sum_{u_R\in U_R} Y_{a,R,n,u_R} - \sum_{u_R\in U_R} \lambda Y_{a-1,R,n,u_R} - \lambda Y_{a-1,O,n,1} - \sum_{u_R\in U_R} (1-\lambda-\delta) \times Y_{a,R,n,u_R} = 0,$$
$$\forall a \in A\setminus\{1\}, m \in \{R\}, n \in N\setminus\{\bar{B}\}, \quad \text{(A22)}$$

$$\sum_{u_R\in U_R\setminus\{1\}} Y_{a,R,\bar{B},u_R} - \lambda Y_{a-1,O,\bar{B},1} - \sum_{u_R\in U_R\setminus\{1\}} (1-\lambda-\delta) Y_{a,R,\bar{B},u_R} = 0,$$
$$\forall a \in A\setminus\{1\}, m \in \{R\}, n \in \{\bar{B}\}, \quad \text{(A23)}$$

$$\sum_{a\in A}\sum_{m\in \mathbf{M}}\sum_{n\in\mathcal{N}}\sum_{u_m\in U_m} Y_{a,m,n,u_m} = 1, \quad \text{(A24)}$$

$$Y_{a,W,\bar{B},u_W} = 0 \quad \forall a \in A, m \in \{W\}, n \in \{\bar{B}\}, \quad \text{(A25)}$$
$$Y_{a,W,\bar{B}-1,u_W} = 0 \quad \forall a \in A, m \in \{W\}, n \in \{\bar{B}-1\}, \quad \text{(A26)}$$
$$Y_{1,R,\bar{B},u_R} = 0 \quad \forall a \in \{1\}, m \in \{R\}, n \in \{\bar{B}\}, \quad \text{(A27)}$$
$$Y_{a,R,\bar{B},1} = 0 \quad \forall a \in A\setminus\{1\}, m \in \{R\}, n \in \{\bar{B}\}, \quad \text{(A28)}$$
$$Y_{a,m,n,u_m} \geq 0 \quad \forall a \in A, m \in M, n \in \mathcal{N}, u_m \in U_m. \quad \text{(A29)}$$

In Equation (A1), we give the objective function aiming at minimising the system's long-run expected cost rate. Equations (A2) to (A23) stand for the balance equations of the system. They ensure that the probabilities of leaving from and entering a particular state are in equilibrium in the long run. The unreachable states in the formulation are eliminated by Equations (A25) to (A28). Lastly, the sum of decision variables must be one and each of them must be greater than or equal to zero as they constitute a probability mass function. With Equations (A24) and (A29), we ensure that these two conditions are satisfied.

Appendix 2. Forming the CTMC model of the system

We use the following equations for obtaining the CTMC model of the system from the matrices that describe demand inter-arrival, production, and warmup time processes. The transitions related to the arrival of a part and inter-arrival time process phase change are given as:

$$\mathbf{Y}_1 = \sum_{i=1}^{X}\sum_{c_1=1}^{C}\sum_{c_2=1}^{C}\Big(\mathbf{J}_{i,i+\delta_{i<X},X} \otimes \mathbf{J}_{c_1,c_2,C} \otimes \mathbf{A1}_{c_2}, \otimes \Big(\Omega_{1,\bar{S}-(i+\delta_{i<X})+1,c_2} + \mathbf{J}_{2,2,4} + \Omega_{3,\bar{S}-(i+\delta_{i<X})+1,c_2} + \mathbf{J}_{4,4,4}\Big) \otimes \mathbf{I}_{K_R} \otimes \mathbf{I}_{K_W}\Big),$$
$$+ \sum_{i=1}^{X}\sum_{c_1=1}^{C}\sum_{c_2=1}^{C}\big(\mathbf{J}_{i,i,X} \otimes \mathbf{J}_{c_1,c_2,C} \otimes \mathbf{A2}_{c_2}, \otimes \big(\Omega_{1,\bar{S}-i+1,c_2} + \mathbf{J}_{2,2,4} + \Omega_{3,\bar{S}-i+1,c_2} + \mathbf{J}_{4,4,4}\big) \otimes \mathbf{I}_{K_R} \otimes \mathbf{I}_{K_W}\big),$$

and $\mathbf{Y}_2 = \sum_{i=1}^{X} (\mathbf{J}_{i,i,X} \otimes \mathbf{I}_C \otimes \mathbf{A0} \otimes \mathbf{I}_4 \otimes \mathbf{I}_{K_R} \otimes \mathbf{I}_{K_W})$, where $\mathbf{J}_{i,j,x}$ denotes a $x \times x$ square matrix with 1 in row i and column j and 0 elsewhere, $\mathbf{I}_x$ denotes the identity matrix of size x, $s = \bar{S} - i + 1$ denotes the inventory/stock level, and

$$\Gamma_{s,c} = \begin{bmatrix} \delta_{s>S_{ON}(c)} & \delta_{s\leq S_{ON}(c)} & 0 & 0 \\ 0 & 0 & \delta_{s>S_{OW}(c)} & \delta_{s\leq S_{OW}(c)} \\ 0 & 0 & \delta_{s>S_{IW}(c)} & \delta_{s\leq S_{IW}(c)} \\ \delta_{s\geq S_{WO}(c)} & 0 & \delta_{s\geq S_{WI}(c)\wedge s<S_{WO}(c)} & \delta_{s<S_{WO}(c)\wedge s<S_{WI}(c)} \end{bmatrix},$$

and

$$\Omega_{m,s,c} = \Gamma_{s,c} \cdot \left((\mathbf{e}_{m,4})^T \otimes \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\right)$$

that controls the evolution of the energy modes of the machine.

The transitions related to the completion of a warmup process and warmup process phase change are given as:

$$\mathbf{Y}_3 = \sum_{i=1}^{X}\sum_{c=1}^{C} (\mathbf{J}_{i,i,X} \otimes \mathbf{J}_{c,c,C} \otimes \mathbf{I}_{K_A} \otimes \Omega_{2,\bar{S}-i+1,c} \otimes \mathbf{R1} \otimes \mathbf{I}_{K_W}),$$

and

$$\mathbf{Y}_4 = \sum_{i=1}^{X} (\mathbf{J}_{i,i,X} \otimes \mathbf{I}_C \otimes \mathbf{I}_{K_A} \otimes \mathbf{J}_{2,2,4} \otimes \mathbf{R0} \otimes \mathbf{I}_{K_W}).$$

Similarly, the transitions related to the completion of a production process and production process phase change are:

$$\mathbf{Y}_5 = \sum_{i=2}^{X}\sum_{c=1}^{C} \Big(\mathbf{J}_{i,i-1,X} \otimes \mathbf{J}_{c,c,C} \otimes \mathbf{I}_{K_A} \otimes \Omega_{4,\bar{S}-(i-1)+1,c} \otimes \mathbf{I}_{K_R} \otimes \mathbf{W1}\Big),$$

and

$$\mathbf{Y}_6 = \sum_{i=2}^{X} (\mathbf{J}_{i,i,X} \otimes \mathbf{I}_C \otimes \mathbf{I}_{K_A} \otimes \mathbf{J}_{4,4,4} \otimes \mathbf{I}_{K_R} \otimes \mathbf{W0}).$$

Finally, $\mathbf{Y} = \sum_{i=1}^{6} \mathbf{Y}_i$ yields the transition matrix for the CTMC model of the system.

Let $f_1(i,j)$ denote the block of $\mathbf{Y}$ between rows $(i-1)\beta+1$ and $i\beta$, and columns $(j-1)\beta+1$ and $j\beta$, and let $f_2(i_1,i_2,j_1,j_2)$ denote the block of $\mathbf{Y}$ between rows $(i_1-1)\beta+1$ and $i_2\beta$, and columns $(j_1-1)\beta+1$ and $j_2\beta$, where $\beta = 4CK_AK_RK_W$. Then, the repeating and the boundary levels to be used in the matrix geometric method can be defined as:

$$G_0 = f_1(X-1, X), \quad \text{(A30)}$$

$$G_1 = f_1(X-1, X-1), \tag{A31}$$

$$G_2 = f_1(X-1, X-2), \tag{A32}$$

$$B_{1,1} = f_1(X-3, X-3), \tag{A33}$$

$$B_{2,1} = f_1(X-2, X-3), \tag{A34}$$

$$B_{0,0} = f_2(1, X-4, 1, X-4), \tag{A35}$$

$$B_{0,1} = f_2(1, X-4, X-3, X-3), \tag{A36}$$

$$B_{1,0} = f_2(X-3, X-3, 1, X-4). \tag{A37}$$