TOWARDS A UNIFIED FRAMEWORK FOR SCATTERING MODELS AT MICROWAVE AND OPTICAL WAVELENGTHS: THE BISPINORIAL DESCRIPTION OF POLARIZATION STATES

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ABSTRACT

In literature, a variety of models, representations and formalisms exist that describe electromagnetic waves and their interaction with media. The choice of model often depends on the required properties of the electromagnetic wave, e.g. the *degree of polarization* or whether the modeled electromagnetic wave is described coherently (amplitude and phase) or incoherently (only amplitude). In this study, we propose a more unified theoretical framework that can represent all cases of coherent, non-coherent, fully polarized, partially polarized, and non-polarized electromagnetic waves. The novel framework also represents them in such a way that the principles of energy conservation and conservation of polarization states are already manifested in the equations.

Index Terms— complexification, Jones, Stokes, Mueller, Fresnel, reflection, transmission, scattering, spinor helicity formalism, bispinor

1. INTRODUCTION

A unifying parameterization and spectrum-overarching modeling of object properties for the prediction and analysis of wave-media interaction over a broad electromagnetic spectrum is becoming increasingly important, especially in the light of the growing fleet of earth observation sensors. The objective of this research is to find a formalism that can not just represent all cases of coherent, non-coherent, fully polarized and partially polarized electromagnetic waves, but also describe them in such a way that the principles of energy conservation and conservation of an incoming polarization state are already manifested in the equations. This would result in a more time efficient, numerically stable and physically robust computation of the modelled scattering matrices. The unified theoretical approach would also be the basis for physical models spanning from optical to radar spectra.

2. THE BISPINORIAL REPRESENTATION OF POLARIZATION STATES (*BRS*)

2.1. Notation and signature of the *Stokes* vector

Remarkably, a number of authors have recognized that the *Stokes* vector $S^\mu = \left(s^0\ s^1\ s^2\ s^3\right)$ with $\mu = [0,3]$ behaves like *Minkowskian four-vectors* [1]. This fact implies that S^μ has a *Lorentzian metric signature* [2, p. 1] [3, p. 73]. The *metric* of the *Stokes* vector is embedded in the determinant of the *coherency* matrix

$$P^{\alpha\dot{\alpha}} = \frac{1}{2}\begin{bmatrix} s^0+s^3 & s^1-is^2 \\ s^1+is^2 & s^0-s^3 \end{bmatrix}, \tag{1}$$

with $\alpha, \dot{\alpha} = [0,1]$, which reads:

$$\det\left(P^{\alpha\dot{\alpha}}\right) = \frac{1}{4}\left[\left(s^0\right)^2 - \left(s^1\right)^2 - \left(s^2\right)^2 - \left(s^3\right)^2\right]. \tag{2}$$

The determinant has one positive and three negative entries, which is a $(1,3)$ (reads as *one comma three*) *signature* or *Lorentzian signature* [2, p. 3][3, p. 73]. The intention is to describe the *coherency* matrix with a pair of spinors (*bispinors*). The signature of the *coherency* matrix plays a crucial role. E.g., in order to obtain a $(1,3)$ signature the *coherency* matrix $P^{\alpha\dot{\alpha}}$ must be a *Hermitian* matrix. It forces the *bispinors*, which will be examined in the next section, to be the complex conjugate of each other. Thus we lose two degrees of freedom. A possible solution to this problem at hand is to complexify the *metric* tensor [4, p. 6]. This will force $S^\mu \in \mathbb{C}$. The resulting complexified space will be denoted as $\mathbb{M}^{\mathbb{S}}_{\mathbb{C}}$. This complexification will achieve that the two dependent *bispinors* will become two independent *two vectors*. It follows from here that we can study the physics of polarization in a complex space with almost no constraint, like the constrain $\left(s^0\right)^2 \geq \left(s^1\right)^2 + \left(s^2\right)^2 + \left(s^3\right)^2$ and impose the reality condition (*Lorentzian metric signature*) at the end of the calculations.

With the intention to provide a better description of the different *DOP* (*Degree of Polarization*) within the *BRS* (bispinorial representation of polarization states), the *Stokes* vectors describing a fully polarized electromagnetic wave will be referred as *null Stokes* vectors, unpolarized electromagnetic waves as *single Stokes* vectors and partially polarized electromagnetic waves as *general Stokes* vectors.

2.2. The description of *BRS* in $\mathbb{M}_{\mathbb{C}}^{\mathrm{S}}$ for all degrees of polarization and polarization states

A *null* Stokes vector transforms in a *coherency* matrix with a *null* determinant. Such a matrix can always be written as an outer-product of a pair of two dimensional vectors

$$P^{\alpha\dot{\alpha}} = \boldsymbol{\pi}^{\alpha} \otimes \tilde{\boldsymbol{\pi}}^{\dot{\alpha}} = \boldsymbol{\pi}^{\alpha}\tilde{\boldsymbol{\pi}}^{\dot{\alpha}} = \begin{bmatrix} \boldsymbol{\pi}^1\tilde{\boldsymbol{\pi}}^{\dot{1}} & \boldsymbol{\pi}^1\tilde{\boldsymbol{\pi}}^{\dot{2}} \\ \boldsymbol{\pi}^2\tilde{\boldsymbol{\pi}}^{\dot{1}} & \boldsymbol{\pi}^2\tilde{\boldsymbol{\pi}}^{\dot{2}} \end{bmatrix}, \quad (3)$$

where $\boldsymbol{\pi}^{\alpha}$ and $\tilde{\boldsymbol{\pi}}^{\dot{\alpha}}$ are known as the *spinor variables*, with $\alpha, \dot{\alpha} = [1, 2]$ [4, p. 6]. Consequently, every matrix characterized by these variables has manifestly vanishing determinants, since the corresponding *Stokes* vector must be *null*. This fact reflects one of the key advantage of the *bispinors*, since they are able to describe a fully polarized electromagnetic wave without the quadratic constraint $\left(s^0\right)^2 = \left(s^1\right)^2 + \left(s^2\right)^2 + \left(s^3\right)^2$.

It is important to note that equation (3) does not uniquely fix $\boldsymbol{\pi}^{\alpha}$ and $\tilde{\boldsymbol{\pi}}^{\dot{\alpha}}$. In other words if

$$\boldsymbol{\pi}^{\alpha} \mapsto z\boldsymbol{\pi}^{\alpha}, \quad (4)$$

where $z \in \mathbb{C}$ then $\tilde{\boldsymbol{\pi}}^{\dot{\alpha}} \mapsto z^{-1}$, such that $P^{\alpha\dot{\alpha}}$ remains invariant under this transformation [4, p. 6, 7].

The total intensity of a *Stokes* vector corresponds to the trace of the *coherency* matrix. With *bispinors* this can be expressed as

$$\sum_{a=1}^{n} \boldsymbol{\pi}^{\alpha}_{(a)} \cdot \tilde{\boldsymbol{\pi}}^{\dot{\alpha}}_{(a)} = \vec{\boldsymbol{\pi}}^{a\alpha} \cdot \vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}} = 0, \quad (5)$$

where $\boldsymbol{\pi}^{\alpha}_{(a)} \cdot \tilde{\boldsymbol{\pi}}^{\dot{\alpha}}_{(a)}$ is the dot product of the *a´th bispinor*. The corresponding transformation group of $\mathbb{M}_{\mathbb{C}}^{\mathrm{S}}$ is $\mathrm{SL}\left(2,\mathbb{C}\right) \otimes \mathrm{SL}\left(2,\mathbb{C}\right)$. This means, that there are two linear conditions on $\boldsymbol{\pi}^{\alpha}$ and $\tilde{\boldsymbol{\pi}}^{\dot{\alpha}}$. In other words, the statement of energy conservation means that each of the n dimensional *two vectors* $\vec{\boldsymbol{\pi}}^{a\alpha}$ and $\vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}$ define a *two plane* that are orthogonal to each other, since their dot product vanishes [5, p. 15].

In contrast to the *null Stokes* vector, the *general Stokes* vector is transformed into a *coherency* matrix whose determinant does not vanish. That means it is a rank two matrix. A rank two matrix can be written as an outer-product of two pairs of two dimensional vectors [4, p. 7]

$$P^{\alpha\dot{\alpha}}_{(a)} = \vec{\boldsymbol{\pi}}^{a\alpha}_{I} \otimes \vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}_{I} = \vec{\boldsymbol{\pi}}^{a\alpha}_{I}\vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}_{I}, \quad (6)$$

where $I = [1, 2]$. The connection of the *bispinors* with the determinant of the *coherency* matrix is expressed by the dependence $\det\left(P^{\alpha\dot{\alpha}}_{(a)}\right) = \vec{\boldsymbol{\pi}}^{a\alpha}_{1}\vec{\boldsymbol{\pi}}^{2}_{a\alpha} \cdot \vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}_{1}\vec{\tilde{\boldsymbol{\pi}}}^{2}_{a\dot{\alpha}}$, where $(\cdot)$ indicates an ordinary multiplication.

Nevertheless, equation (6) represents a fundamentally difficult problem, since it cannot be solved explicitly in a simple way. The reason for this is that the *bispinors* do not have just one exact solution. The problem can be resolved by the constraint that the determinants of the *bispinors* are equal to each other [4, p. 7]

$$\vec{\boldsymbol{\pi}}^{a\alpha}_{1}\vec{\boldsymbol{\pi}}^{2}_{a\alpha} = \vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}_{1}\vec{\tilde{\boldsymbol{\pi}}}^{2}_{a\dot{\alpha}}. \quad (7)$$

In contrast to the behaviour seen in the *null Stokes* case, the transformation that leaves the *coherency* matrix invariant is in the *general Stokes* case a $\mathrm{SL}\left(2,\mathbb{C}\right)$ transformation: if $\vec{\boldsymbol{\pi}}^{I}_{a\alpha} \mapsto W^{I}{}_{J}\vec{\boldsymbol{\pi}}^{J}_{a\alpha}$ then $\vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}_{I} \mapsto \left(W^{-1}\right)^{J}{}_{I}\vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}_{J}$ where $W^{I}{}_{J} \in \mathrm{SL}\left(2,\mathbb{C}\right)$ [4, p. 7].

Regarding the results of this transformation, some interesting implications may be made. One particularly useful implication is that the two indices I and J can be expanded in terms of a pair of *null Stokes* vectors $\vec{\boldsymbol{\pi}}^{a\alpha}, \vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}$ and $\vec{\boldsymbol{\eta}}^{a\alpha}, \vec{\tilde{\boldsymbol{\eta}}}^{a\dot{\alpha}}$ [4, p. 9]

$$\begin{aligned} \vec{\boldsymbol{\pi}}^{a\alpha}_{I} &= \vec{\boldsymbol{\pi}}^{a\alpha} \otimes \xi^{+}_{I} + \vec{\boldsymbol{\eta}}^{a\alpha} \otimes \xi^{-}_{I} \\ \vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}_{J} &= \vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}} \otimes \xi^{+}_{J} + \vec{\tilde{\boldsymbol{\eta}}}^{a\dot{\alpha}} \otimes \xi^{-}_{J} \end{aligned}, \quad (8)$$

where $\vec{\boldsymbol{\pi}}^{a\alpha}, \vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}$ are the *null Stokes bispinors* as discussed in the previous section. The variables ξ^{+}_{I} and ξ^{-}_{I} are two basis vectors representing the $\mathrm{SL}\left(2,\mathbb{C}\right)$ space, which will be discussed in the following paragraphs.

It is interesting, that this approach provides enough flexibility to handle complex issues like the geometrization of energy conservation and the conservation of an incoming polarization state, even if the individual components of the bispinors have not yet been determined at all. Moreover, the description of an arbitrarily polarized state is symmetrized by describing it as a pair of fully polarized elements.

As already stated in the previous paragraphs, the two components of the *bispinors* are independent entities in $\mathbb{M}_{\mathbb{C}}^{\mathrm{S}}$. This means we can freely define $\vec{\boldsymbol{\pi}}^{a\alpha}_{I}$ or $\vec{\tilde{\boldsymbol{\pi}}}^{a\dot{\alpha}}_{I}$. However, we have to take care that the conditions shown in equations (5) and (7) are obeyed. For a *null Stokes* vector we can define the *bispinors* $\boldsymbol{\pi}^{\alpha}$ and $\tilde{\boldsymbol{\pi}}^{\dot{\alpha}}$ as

$$\boldsymbol{\pi}^{\alpha} = \frac{1}{\sqrt{2k^{+}}}\begin{pmatrix} k^{+} \\ u^{+} \end{pmatrix}, \ \tilde{\boldsymbol{\pi}}^{\dot{\alpha}} = \frac{1}{\sqrt{2k^{+}}}\begin{pmatrix} k^{+} \\ u^{-} \end{pmatrix}, \quad (9)$$

where $k^{\pm} = D \pm s^3,\ u^{\pm} = s^1 \pm is^2$. The function D scales the polarized part of the electromagnetic wave according to whether it has an *incoming* or *outgoing* energy contribution $D = \mathrm{sgn}\left(s^0\right)\sqrt{\left(s^1\right)^2 + \left(s^2\right)^2 + \left(s^3\right)^2}$. This particular solution offers a significant advantage with respect to the determination of the parameter z, emerging from the equation (4).

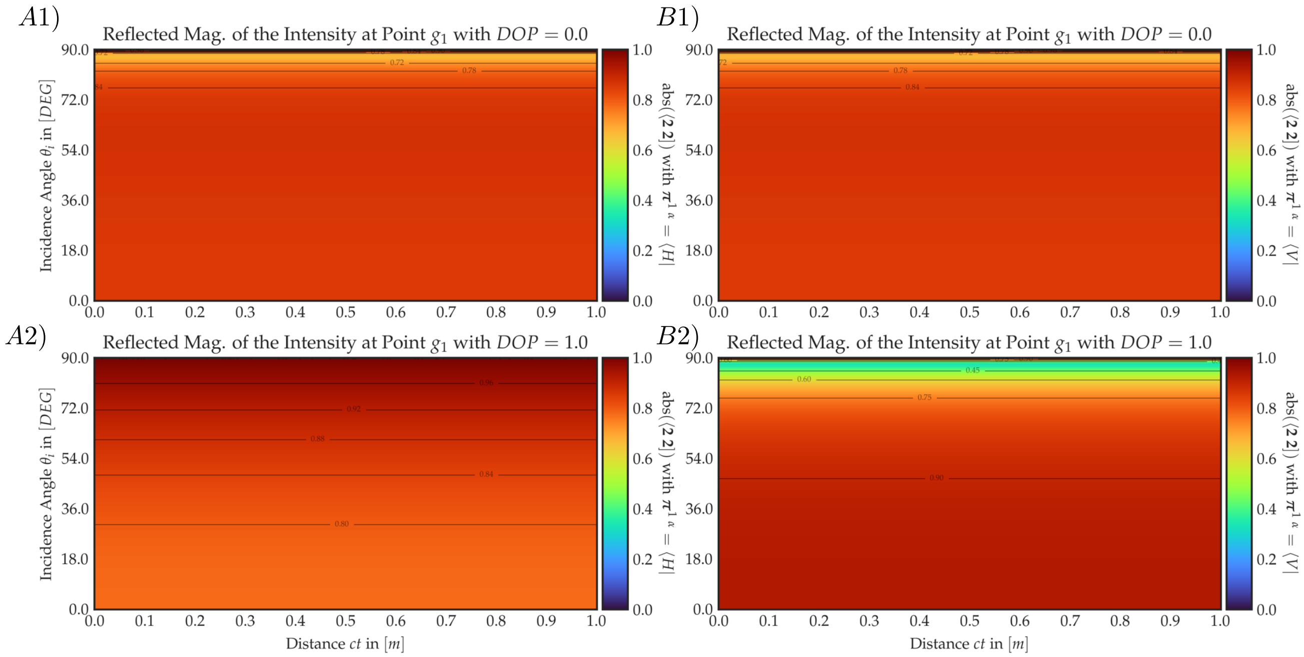

Fig. 1. The simulated reflected intensity of saline water at 100 [GHz] at a temperature of 20° [C] and a salinity of 35 [ppt] at different *DOP*. Pictures $A1 - A2$: Horizontal part of the reflected intensity. Pictures $B1 - B2$: Vertical part of the reflected intensity.

The way we have specified the spinor, however, allows us to interpret the parameter z as the phase of the electromagnetic wave, which in turn is defined as an action of the group $\mathrm{U}(1)$.

It has already been seen above that, in the *general Stokes* case, the *bispinors* can be expanded in terms of a pair of *null Stokes* vectors $\vec{\boldsymbol{\pi}}^{a\alpha}$ and $\vec{\boldsymbol{\eta}}^{a\alpha}$ with respect to the basis ξ_I^+ and ξ_I^- (8). To simplify the formulation of *BRS*, we are free to define ξ_I^+ and ξ_I^- as the standard basis $\xi_I^+ = (1\,0)$ and $\xi_I^- = (0\,1)$. Under this definition of basis vectors and respecting the constraints defined in equations (5) and (7), the bispinors can be derived as:

$$\begin{aligned}\boldsymbol{\pi}_I^{\alpha} &= h^+ \begin{pmatrix} k^+ \\ u^+ \end{pmatrix} \otimes \xi_I^+ + h^- \begin{pmatrix} -u^- \\ k^+ \end{pmatrix} \otimes \xi_I^- \\ \tilde{\boldsymbol{\pi}}_I^{\dot{\alpha}} &= h^+ \begin{pmatrix} k^+ \\ u^- \end{pmatrix} \otimes \xi_I^+ + h^- \begin{pmatrix} -u^+ \\ k^+ \end{pmatrix} \otimes \xi_I^- \end{aligned}. \tag{10}$$

The new parameter $h^{\pm}$ extends the term $\frac{1}{\sqrt{2k^+}}$ in (9) with respect to the sign function D and is defined as $h^{\pm} = \sqrt{\frac{s^0 \pm D}{2D}} \frac{1}{\sqrt{2k^+}}$.

3. RESULTS AND DISCUSSION

In this section, the main results of the *BRS* formalism are summarized by discussing a straightforward example. It involves salt water at a refractive index at 100 [GHz], a temperature of 20° [C], and a salinity of 35 [ppt] (parts per thousand). We place the simulated transmitter and receiver one meter above the medium. This means, that the reflected wave travels from zero meter (surface) to one meter (sensor). The angles of incidence are varied from 0° to 90°. The simulation includes only the *Fresnel* reflection, described by the process g_1 at the surface of the medium. The *DOP* of the incident wave is varied between zero (unpolarized) and one (fully polarized).

Figure 1 ($A1 - B2$) visualizes the simulated reflected intensity at different *DOP* from zero (unpolarized) to one (totally polarized). Figure 1 ($A1 - A2$) illustrates the horizontal component and Figure 1 ($B1 - B2$) the vertical component of the reflected intensity. It is very interesting to see that at low *DOP* even the horizontally polarized part of the wave shows a *Brewster's* angle effect, that will disappear at $DOP = 1$. The reason for this is that at low DOP, the horizontal component of the received electromagnetic wave includes vertically polarized parts of the unpolarized incoming wave, where the Brewster's angle is predominant. A visual representation of this can be seen in Figure 2 ($A1 - A2$). This figure clearly shows that the *Brewster's* angle is induced by the unpolarized part of the wave (which contains also the vertical polarization) and its effect decreases with increasing *DOP* (Figure 2, $A2$).

In the vertically polarized component of the wave the opposite is observed (Figure 1 ($B1 - B2$)). In this case, the *Brewster's* angle is not predominantly present at low *DOP* and it reaches its maximum at $DOP = 1$. This is largely

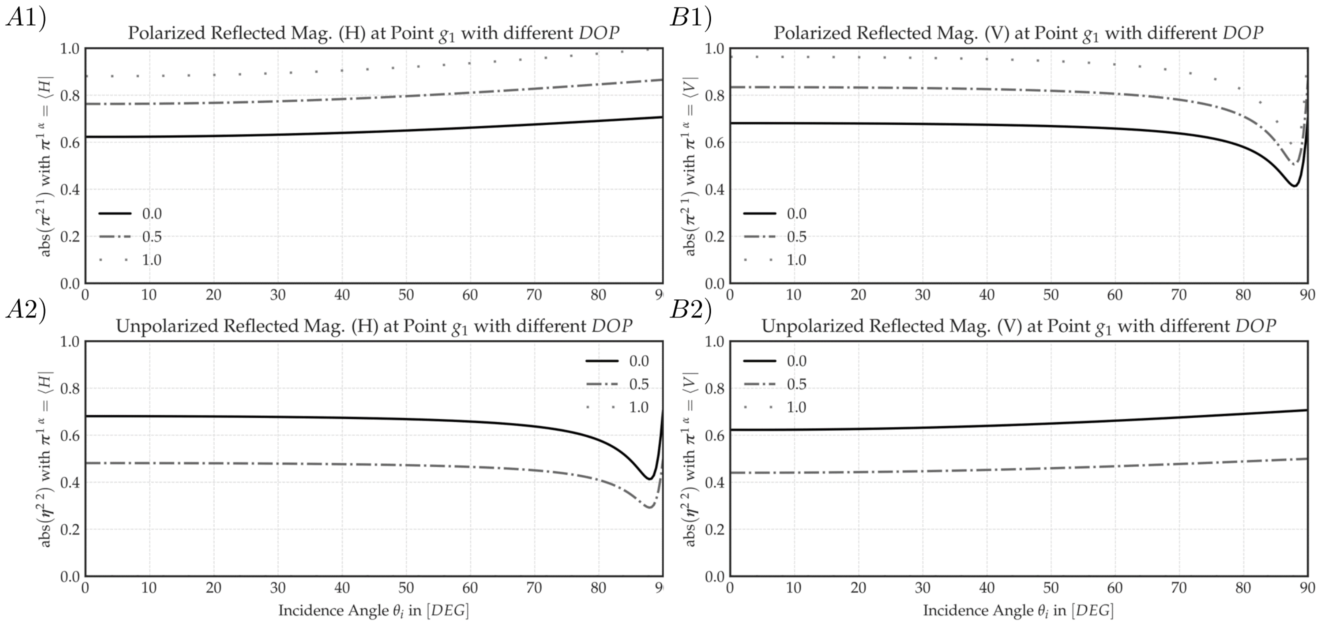

Fig. 2. The polarized (*null Stokes*) and unpolarized part (*single Stokes*) of the simulated reflected intensity at the surface for different *DOP*. The *x-axis* represents the varying incident angle and the *y-axis* represents the intensity. Pictures $A1 - A2$: Horizontal part of the polarized (top) and unpolarized (bottom) intensity. Pictures $B1 - B2$: Vertical part of the polarized (top) and unpolarized (bottom) reflected intensity.

attributed to the presence of a horizontal part in the vertical component of the received wave. This phenomenon is graphically depicted in Figure 2 ($B1 - B2$). From this figure it can be seen that the decrease of the *Brewster's* angle is caused by the unpolarized part of the wave, which contains also the horizontal polarization. Its effect weakens with increasing *DOP*, where the *Brewster's* angle gains more and more dominant (Figure 2, $B1$).

4. CONCLUSION AND OUTLOOK

The most significant observation of this study is related to the appearance of the *Brewster's* angle in horizontally polarized waves for *DOP* less than one. In addition, the intensity in the horizontal or vertical part of the wave is lower as the wave becomes more depolarized. An important note must be made here regarding the *DOP* and the phase of the electromagnetic wave. It is a well-known problem that the commonly used formalisms, as reffered to *Stokes* or *Jones* vector are not able to calculate the *Fresnel* reflection under different *DOP* and completely preserve the phase of the considered wave without artificial interaction. Due to the limited space in this paper, the simulation includes only the *Fresnel* reflection, neglecting transmission, absorption and dispersion or surface roughness effects. Our presentation at the conference will also address these configurations in detail.

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