Homomorphisms between multidimensional constant-shape substitutions

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Abstract. We study a class of $\mathbb{Z}^d$-substitutive subshifts, including a large family of constant-length substitutions, and homomorphisms between them, i.e., factors modulo isomorphisms of $\mathbb{Z}^d$. We prove that any measurable factor map and even any homomorphism associated to a matrix commuting with the expansion matrix, induces a continuous one. We also get strong restrictions on the normalizer group, proving that any endomorphism is invertible, the normalizer group is virtually generated by the shift action and the quotient of the normalizer group by the automorphisms is restricted by the digit tile of the substitution.

1. Introduction

In this article, we study multidimensional *constant-shape substitutions* and *homomorphisms* between them, i.e., continuous maps $\phi : X \to Y$ (called *isomorphisms* for invertible ones) such that for some matrix $M \in \mathrm{GL}(d, \mathbb{Z})$ and any $\boldsymbol{n} \in \mathbb{Z}^d$, $\phi \circ S^{\boldsymbol{n}} = S^{M\boldsymbol{n}} \circ \phi$, where $(X, S, \mathbb{Z}^d)$ and $(Y, S, \mathbb{Z}^d)$ are subshifts given by substitutions with a uniform support. Such a map gives an orbit equivalence with constant orbit cocycle via a linear map. When the matrix M is the identity, surjective homomorphisms are called *factor maps*, and *conjugacies* when it is invertible. We refer to the conjugacies as *automorphisms* when the dynamical systems are the same. In the one-dimensional case, homomorphisms lead to the notion of *flip conjugacy* of dynamical systems [5] and by this fact are also called *reversing symmetries* (see [2, 24]). The relation between homomorphisms and factor maps becomes less clear in higher dimensions, since $\mathrm{GL}(d, \mathbb{Z})$ is infinite for $d \geq 2$ (see for example [1]).

The study of factors and automorphisms of a dynamical system is a classical problem. It mainly concerns their algebraic and dynamical properties in relation to the ones of the system $(X, S, \mathbb{Z}^d)$. The automorphisms can be algebraically defined as elements of the centralizer of the action group $\langle S \rangle$, seen as a subgroup of all homeomorphisms $\mathrm{Homeo}(X)$ from X to itself. With this algebraic point of view, isomorphisms can be seen as elements of the normalizer group of $\langle S \rangle$ seen as a subgroup of $\mathrm{Homeo}(X)$. The automorphism

group is always nonempty, but in general, the existence of isomorphisms for a particular matrix $M \in \mathrm{GL}(d, \mathbb{Z})$ is an open problem.

In this context, the rich family of symbolic systems exhibits rigidity properties of factor maps and automorphisms already in the one-dimensional case. For instance, the famous Curtis–Hedlund–Lyndon theorem [28], ensures that any factor map between subshifts is a *sliding block code*, showing that the automorphism group is countable. Among the simplest nontrivial zero-entropy symbolic systems, are the substitutive ones introduced by W. H. Gottschalk in [25] (see [37] for a good bibliography on this subject). They also present rigidity properties. B. Host and F. Parreau in [29] gave a complete description of factor maps between subshifts arising from certain constant-length substitutions, proving that any measurable factor map induces a continuous one, and the automorphism group is virtually generated by the shift action. Moreover, any finite group can be realized as a quotient group $\mathrm{Aut}(X, S, \mathbb{Z})/\langle S \rangle$ for these subshifts as proved by M. Lemańczyk and M. K. Mentzen in [33]. Later, I. Fagnot [19] proved that the problem of whether there exists a factor map between two constant-length substitution subshifts is decidable, using the first-order logic framework of Presburger arithmetic. Some years later, F. Durand in [15] showed that *linearly recurrent subshifts* (in particular substitutive subshifts) have finitely many symbolic factors, up to conjugacy. Using the self-induced properties of substitutive subshifts, V. Salo and I. Törmä provide in [40] a renormalization process of the factor maps to extend the description obtained in [29]. In [13] the authors proved that the automorphism group of a minimal subshift with non-superlinear complexity is virtually generated by the shift action, using the concept of *asymptotic pairs*. Next, C. Müllner and R. Yassawi [35] demonstrated that any topological factor of a constant-length substitutive shift is conjugate to a constant-length substitution via a letter-to-letter map. More recently, F. Durand and J. Leroy [17] showed the decidability of the existence problem of a factor map between two minimal substitutive subshifts.

In the multidimensional setting, substitutive systems are originally motivated by physical reasons with the discovery of the aperiodic structure of quasicrystals modelized by the Penrose tiling [36], where the symmetries play a fundamental role. Substitutions also occur in different topics such as combinatorics, diophantine approximations and theoretical computer science, with the minimal Robinson subshift of finite type being one of the most fundamental examples [23]. Characterizations of the isomorphisms of the chair tiling, together with the full shift and Ledrappier's shift, were given in [3]. The chair tiling, the table tiling and the minimal Robinson tiling belong to the class of constant-shape substitutions, which is a multidimensional analogue of the so-called constant-length substitutions. As a difference with the one-dimensional case, these substitutions may not be linearly recurrent (Example 3.1). In [7, 8], the case of bijective block substitutions was studied.

In this article, we pursue the study of isomorphisms to homomorphisms, and more general multidimensional substitutions (nondiagonal expansion matrix, nonrectangular support, and a weaker version of bijectivity. See [22] for recent results on their spectral properties). We also obtain rigidity properties about homomorphisms. First, we prove that

any aperiodic symbolic factor of a constant-shape substitution is conjugate to a constant-shape substitution via a letter-to-letter map (Theorem 3.26), extending the mentioned one-dimensional result from [35]. Then, we show that any measurable factor map and any homomorphism associated with a matrix commuting with some power of the expansion matrix of the substitution induces a continuous one, and we give an explicit bound on the radius of its block maps (Theorem 4.1 and Theorem 4.13). These are analogue results of B. Host and F. Parreau from [29]. These imply that certain constant-shape substitutions are *coalescent* (Proposition 4.8), and the automorphism group is virtually generated by the shift action (Proposition 4.9). Finally, we give algebraic and geometrical properties of the normalizer group for *polytope substitutions*, i.e., in the case where the convex hull of the digit tile generated by the expansion matrix and the support of the substitution is a polytope. To do this, we relate the *nondeterministic directions* of substitutive subshifts to the supporting hyperplanes to the convex hull of the digit tile (Theorem 5.2). We deduce that any homomorphism of the substitutive subshift is invertible, and the normalizer group is virtually generated by the shift action (Theorem 5.17). Moreover, the *linear representation group*, defined as the set of matrices associated with a homomorphism, is finite and we give explicit bounds for the norm of these matrices (Proposition 5.15). Together with the former bound on the radii of the block maps, these restrictions enable an algorithmic description of the normalizer group whenever the expansion matrix is proportional to the identity. These recover results and answer some questions in [8].

This article is organized as follows. The basic definitions and background are introduced in Section 2. In Section 3, we prove Theorem 3.26 characterizing the aperiodic symbolic factors of substitutive subshifts. For this we study a recognizability property of these symbolic factors (Proposition 3.7) and we determine their maximal equicontinuous factor (Proposition 3.19). We also give a polynomial bound on the repetitivity function for substitutive subshifts (Lemma 3.2). Section 4 is devoted to the proofs of the measurable rigidity properties of homomorphisms: Theorem 4.1 and Theorem 4.13. Then, we deduce the coalescence (Proposition 4.8) and that the automorphism group of substitutive subshifts is virtually $\mathbb{Z}^d$ (Proposition 4.9). Finally, in Section 5 we describe the nondeterministic directions of substitutive subshifts through the digit tile for *bijective on the extremities substitutions* (Theorem 5.2). Moreover, these directions are computable in terms of the combinatorics of the substitution (Corollary 5.13). This enables us to provide algebraic restrictions and to bound elements of the linear representation group (Proposition 5.15). The last theorem (Theorem 5.17) summarizes all the results of this last study.

2. General setting and notions

2.1. Basic definitions and notation

2.1.1. Notation. Throughout this article we will denote by $\boldsymbol{n} = (n_1, \ldots, n_d)$ the elements of $\mathbb{Z}^d$ and by $\boldsymbol{x} = (x_1, \ldots, x_d)$ the elements of $\mathbb{R}^d$. If $F \subseteq \mathbb{Z}^d$ is a finite set, it

will be denoted by $F \Subset \mathbb{Z}^d$ and we use the notation $\|F\| = \max_{\boldsymbol{n} \in F} \|\boldsymbol{n}\|$, where $\| \cdot \|$ is the standard Euclidean norm of $\mathbb{R}^d$. The standard cartesian product in $\mathbb{R}^d$ will be denoted by $\langle \cdot, \cdot \rangle$. If $L \in \mathcal{M}(d, \mathbb{R})$ is a matrix, we denote $\|L\| = \max_{\boldsymbol{x} \in \mathbb{R} \setminus \{\boldsymbol{0}\}} \|L(\boldsymbol{x})\|/\|\boldsymbol{x}\|$ as the *matrix norm of* L. We denote $\mathrm{GL}(d, \mathbb{Z})$ as the set of $d \times d$ matrices M with integer coefficients such that $|\det(M)| = 1$. The matrices $M \in \mathrm{GL}(d, \mathbb{Z})$ represent the automorphisms of $\mathbb{Z}^d$.

We will call a sequence of finite sets $(A_n)_{n>0} \subseteq \mathbb{Z}^d$ a *Følner sequence*[1] if for all $\boldsymbol{n} \in \mathbb{Z}^d$ we have that

$$\lim_{n \to \infty} \frac{|A_n \Delta (\boldsymbol{n} + A_n)|}{|A_n|} = 0.$$

For any $r > 0$ and $F \Subset \mathbb{Z}^d$ we denote $F^{\circ r}$ as the set of all elements $\boldsymbol{f} \in F$ such that $\boldsymbol{f} + (B(\boldsymbol{0}, r) \cap \mathbb{Z}^d) \subseteq F$, i.e.,

$$F^{\circ r} = \{\boldsymbol{f} \in F : \boldsymbol{f} + (B(\boldsymbol{0}, r) \cap \mathbb{Z}^d) \subseteq F\}.$$

Note that the Følner assumption implies that for any $r > 0$

$$\lim_{n \to \infty} \frac{|F_n^{\circ r}|}{|F_n|} = 1.$$

2.1.2. Convex geometry. A set $C \subseteq \mathbb{R}^d$ is said to be *convex* if for all $\boldsymbol{x}, \boldsymbol{y} \in C$ the set $[\boldsymbol{x}, \boldsymbol{y}] = \{\boldsymbol{z} \in \mathbb{R}^d : \boldsymbol{z} = t\boldsymbol{x} + (1 - t)\boldsymbol{y}, t \in [0, 1]\}$ is included in C. Recall that the image of a convex set under an affine map is also a convex set, and the intersection of an arbitrary family of convex sets is also a convex set. This leads to the notion of convex hull of a set.

If $A \subseteq \mathbb{R}^d$ we define the *convex hull of* A, denoted by $\mathrm{conv}(A)$, as the intersection of all convex sets containing A.

A set $S \subseteq \mathbb{R}^d$ is an *affine set* if for any $\boldsymbol{x}, \boldsymbol{y} \in S$ the line $\{t\boldsymbol{x} + (1 - t)\boldsymbol{y} : t \in \mathbb{R}\}$ is contained in S. For any set $A \subseteq \mathbb{R}^d$ we define the *affine hull of* A, denoted by $\mathrm{Aff}(A)$, as the intersection of all affine sets containing A.

A fundamental characterization of convex sets is provided by Carathéodory's theorem.

Theorem 2.1 (Carathéodory's theorem). *For any $A \subseteq \mathbb{R}^d$, any element of $\mathrm{conv}(A)$ can be represented as a convex combination of no more than $(d + 1)$ elements of A.*

We now recall some basic topological concepts associated with convex sets. A point $\boldsymbol{x} \in A$ is said to be *relative-interior* for A, if A contains the intersection of a ball centered at $\boldsymbol{x}$ with $\mathrm{Aff}(A)$, i.e., $\exists r > 0, B(\boldsymbol{x}, r) \cap \mathrm{Aff}(A) \subseteq A$. The set of all relative-interior points of A is called the *relative interior of* A and is denoted by $\mathrm{ri}(A)$. We can also define the *relative boundary* $\partial_{\mathrm{ri}}(A)$ as the set difference of the closure and the relative interior, i.e., $\partial_{\mathrm{ri}}(A) = \mathrm{cl}(A) \setminus \mathrm{ri}(A)$.

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[1] In the literature, especially group theory, it is common to also ask that the union of the sequence of sets $(F_n)_{n>0}$ is equal to $\mathbb{Z}^d$ for a sequence to be Følner, but we will not use it in this article.

An important notion for convex sets are the supporting hyperplanes. Let $C \subseteq \mathbb{R}^d$ be a closed convex set and $x \in C$ be a point in the relative boundary of C. An *affine hyperplane* $\partial H[a : c] = \{y \in \mathbb{R}^d : \langle a, y \rangle = c\}$, for some $a \in \mathbb{R}^d \setminus \{0\}$ and $c \in \mathbb{R}$ is called a *supporting hyperplane to C at x* if $x \in \partial H[a : c]$ and

$$\inf_{y \in C} \langle a, y \rangle < \langle a, x \rangle = c = \sup_{y \in C} \langle a, y \rangle.$$

We now recall some basic notions about cones and polyhedral sets. A nonempty set $C \subseteq \mathbb{R}^d$ is said to be a *cone* if for every $x \in C$, the set C contains the positive ray $\mathbb{R}_+ x = \{tx : t > 0\}$ spanned by x. A translation of a cone by a nonzero vector is called an *affine convex cone*. A cone $C \subseteq \mathbb{R}^d$ is said to be *finitely generated* if it can be written as

$$C = \left\{ \sum_{i=1}^{p} t u_i : u_i \in \mathbb{R}^d, \ t_i \geq 0, \ i = 1, \ldots, p \right\}.$$

For a given nonempty set $A \subseteq \mathbb{R}^d$, the smallest cone containing the set A is called the *positive hull* (or *conical hull*) of A. This set is given by

$$\mathrm{cone}(A) = \{tx : x \in A, \ t \geq 0\}.$$

The positive hull is also said to be the *cone generated by A*.

Convex sets can be represented, but it requires the notion of faces. A point x in a convex set C is called an *extreme point*, if it cannot be written as the convex combination of two different points in C, i.e., if x is equal to $tu + (1 - t)v$ for some $0 \leq t \leq 1$, with $u, v \in C$, then $u = v = x$. We denote by $\mathrm{Ext}(C)$ the set of the extreme points of a convex set C. A compact convex set is called a *polytope* if it has a finite number of extreme points.

Extreme points are special cases of *faces* of a convex set. A convex subset $F \subseteq C$ is called a *face* of C if for every $x \in F$ and every $y, z \in C$ such that $x = ty + (1 - t)z$, with $0 < t < 1$, we have that $y, z \in F$. The *dimension* of a face F of C is the dimension of its affine hull. The 0-dimensional faces of C are exactly the extreme points of C, and the bounded 1-dimensional faces are called *segments* or *edges*. An *extreme ray* of a convex set C is the direction of an affine half-line, that is, a face of C. A useful result about representation of closed convex sets in $\mathbb{R}^d$ is the following.

Theorem 2.2 (Krein–Milman theorem for unbounded convex sets). *If a nonempty closed convex set $C \subseteq \mathbb{R}^d$ has at least one extreme point, i.e., does not contain an affine line, then C can be written as the sum of the convex hull of its extreme points and the cone generated by its extreme rays.*

A useful relation between faces and the convex hull of a set that we will use in this article is the following. A proof can be found in [39, Section 18].

Theorem 2.3. *Let $C = \mathrm{conv}(A) \subseteq \mathbb{R}^d$ be the convex hull of a set $A \subseteq \mathbb{R}^d$ and let $F \subseteq C$ be a nonempty face of C. Then $F = \mathrm{conv}(A \cap F)$.*

Some useful notion for closed convex sets corresponds to their *normal cones*. Let F be a nonempty face of a closed convex set C. The *opposite normal cone*[2] $\widehat{N}_F(C)$ *of* C *at* F is defined as

$$\widehat{N}_F(C) = \big\{v \in \mathbb{R}^d : \min_{t \in C}\langle v, t\rangle = \langle v, p\rangle, \ \forall p \in F\big\}.$$

The *opposite normal fan of* C is the collection of all opposite normal cones of C,

$$\widehat{\mathcal{N}}(C) = \big\{\widehat{N}_F(C) \colon F \text{ is a proper face of } C\big\}.$$

The following are simple statements on the normal fan:

- $\dim(\widehat{N}_F(C)) = d - \dim(F)$.
- If F is a face of G, which is a face of C, then $\widehat{N}_G(C)$ is a face of $\widehat{N}_F(C)$.
- The set $\bigcup_{F \text{ face of } C} \widehat{N}_F(C)$ is equal to $\mathbb{R}^d$.

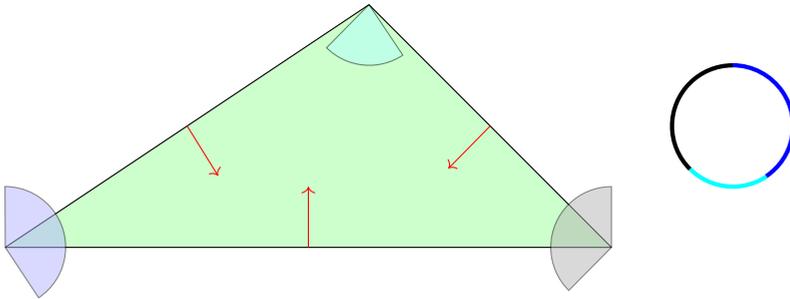Figure 1 illustrates the opposite normal cones of a triangle.

Figure 1. Example of the opposite normal cones of a triangle and the stratification of the circle $\mathbb{S}^1$ given by them.

2.1.3. Fractal Geometry. Let $\mathcal{C}(\mathbb{R}^d)$ be the collection of all nonempty compact subsets of $\mathbb{R}^d$. The *Hausdorff metric* h on $\mathcal{C}(\mathbb{R}^d)$ is defined as

$$\forall A, B \in \mathcal{C}(\mathbb{R}^d), \quad h(A, B) = \inf\{\varepsilon \colon A \subseteq B_\varepsilon \wedge B \subseteq A_\varepsilon\},$$

where $A_\varepsilon = \{t \in \mathbb{R}^d : \|t - y\| \leq \varepsilon, \text{ for some } y \in A\}$. With this metric $(\mathcal{C}(\mathbb{R}^d), h)$ is a complete metric space.

A map $f : \mathbb{R}^d \to \mathbb{R}^d$ is said to be a *contraction* if there exists $0 < c < 1$ such that $\|f(x) - f(y)\| \leq c\|x - y\|$ for all $x, y \in \mathbb{R}^d$. Let $\{f_i\}_{i=1}^N$ be a set of contraction maps

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[2]The word *opposite* comes from the fact that the usual normal cone is related to the outward normal vectors of convex sets and in this article we will use the inward normal vectors.

on $\mathbb{R}^d$ and define the map

$$F : (\mathcal{C}(\mathbb{R}^d), h) \to (\mathcal{C}(\mathbb{R}^d), h),$$

$$A \mapsto \bigcup_{i=1}^N f_i(A)$$

This map is a contraction on $(\mathcal{C}(\mathbb{R}^d), h)$. By the Banach fixed-point theorem (or particularly the IFS theorem), there exists a unique set $T \in \mathcal{C}(\mathbb{R}^d)$ (called *digit tile*) such that $T = \bigcup_{i=1}^N f_i(T)$. A way to approximate this set is by iterations

$$T = \lim_{n \to \infty} F^n(T_0), \tag{1}$$

where T_0 is an arbitrary compact set of $\mathbb{R}^d$ and the limit is with respect to the Hausdorff metric.

Since the convex hull of a compact set in $\mathbb{R}^d$ is compact, the map conv $: \mathcal{C}(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d)$, which gives for any set $A \in \mathcal{C}(\mathbb{R}^d)$ its convex hull, is well defined and is well known to be continuous.

2.2. Topological dynamical systems

A *topological dynamical system* is a triple (X, T, G), where (X, ρ) is a compact metric space, G is a group of self-homeomorphisms of the space X and $T : X \times G \to X$ is a continuous map, satisfying $T(x, e) = x$, and $T(T(x, g), h) = T(x, gh)$ for all $x \in X$ and $g, h \in G$. We denote T^g the homeomorphism $T(\cdot, g)$.

If (X, ρ) is a compact metric space, we denote Homeo(X) the group of self-homeomorphisms of X. If $T \in$ Homeo(X), we use $(X, T, \mathbb{Z})$ to denote the topological dynamical system $(X, T, \{T^n : n \in \mathbb{Z}\})$. Similarly, if $T_1, \ldots, T_d$ are d commuting homeomorphisms on X, we use $(X, T, \mathbb{Z}^d)$ to denote the topological dynamical system $(X, T, \langle \{T_1, \ldots, T_d\} \rangle)$.

For a point $x \in X$, we define its *orbit* as the set $\mathcal{O}(x, G) = \{T^g(x) : g \in G\}$. If $A \subseteq X$, we say that A is G-*invariant* if for all $x \in A$, $\mathcal{O}(x, G)$ is included in A.

If (X, T, G) is a topological dynamical system, a subset $K \subseteq X$ is called a *minimal set* if K is closed, nonempty, G-invariant and has no proper closed nonempty invariant subsets, i.e., if $N \subseteq K$ is closed and G-invariant, then $N = \emptyset$ or $N = K$. In this case, we say that $(K, T|_K, G)$ is a *minimal system*, where $T|_K : K \times G \to K$ corresponds to the restriction of T to K. It is easy to see that a system is minimal if and only if it is the closure orbit of all of its points.

Definition 2.4. Let $(X, T, \mathbb{Z}^d)$, $(Y, T, \mathbb{Z}^d)$ be two topological dynamical systems and $M \in \mathrm{GL}(d, \mathbb{Z})$. A *homomorphism associated with M* is a continuous map $\phi : X \to Y$ such that for all $\boldsymbol{n} \in \mathbb{Z}^d$, we have that $\phi \circ T^{\boldsymbol{n}} = T^{M\boldsymbol{n}} \circ \phi$. If ϕ is surjective, then ϕ is an *epimorphism* and if it is invertible, then ϕ is an *isomorphism*.

Note that a homomorphism between two minimal systems is always an epimorphism. In the following, we fix the different notations that we will use throughout this article:

- We denote the set of all homomorphisms associated with M between $(X, T, \mathbb{Z}^d)$ and $(Y, T, \mathbb{Z}^d)$ by $\text{Hom}_M(X, Y, T, \mathbb{Z}^d)$.

- The *set of homomorphisms* between two dynamical systems, is defined as the collection of all of homomorphisms, i.e.,

$$\text{Hom}(X, Y, T, \mathbb{Z}^d) = \bigcup_{M \in \text{GL}(d, \mathbb{Z})} \text{Hom}_M(X, Y, T, \mathbb{Z}^d).$$

- In the special case where M is the identity matrix, the homomorphisms are called *factor maps* and we denote $\text{Fac}(X, Y, T, \mathbb{Z}^d)$ the collection of all factor maps between $(X, T, \mathbb{Z}^d)$ and $(Y, T, \mathbb{Z}^d)$. If a factor map is invertible, then it is called a *conjugacy*.

- In the case $(X, T, \mathbb{Z}^d) = (Y, T, \mathbb{Z}^d)$, we simply denote these sets as $N_M(X, T, \mathbb{Z}^d)$ and $N(X, T, \mathbb{Z}^d)$. The last set is called the *normalizer semigroup of* $(X, T, \mathbb{Z}^d)$. A factor map is called an *endomorphism*, and a conjugacy is called an *automorphism*. We denote the set of all endomorphisms and automorphisms of a topological dynamical system as $\text{End}(X, T, \mathbb{Z}^d)$ and $\text{Aut}(X, T, \mathbb{Z}^d)$, respectively.

- We define the *linear representation semigroup* $\vec{N}(X, T, \mathbb{Z}^d)$ *of* $(X, T, \mathbb{Z}^d)$ as the collection of all matrices $M \in \text{GL}(d, \mathbb{Z})$ with $N_M(X, T, \mathbb{Z}^d) \neq \emptyset$.

- A topological dynamical system $(X, T, \mathbb{Z}^d)$ is said to be *coalescent* if every endomorphism of $(X, T, \mathbb{Z}^d)$ is an automorphism.

Note that the linear representation semigroup of a topological dynamical system is an invariant under conjugation. Now, if $\phi \in N_{M_1}(X, T, \mathbb{Z}^d)$ and $\psi \in N_{M_2}(X, T, \mathbb{Z}^d)$, then $\phi\psi$ is in $N_{M_1 M_2}(X, T, \mathbb{Z}^d)$, so the sets $N_M(X, T, \mathbb{Z}^d)$ are not semigroups (except if M is the identity matrix). Now, even though the matrices $M \in \text{GL}(d, \mathbb{Z})$ are invertible in $\mathbb{Z}^d$, the linear representation semigroup $\vec{N}(X, T, \mathbb{Z}^d)$ is not necessarily a group, since the existence of a homomorphism associated with a matrix M does not necessarily imply the existence of a homomorphism associated with the matrix M^{-1}.

The groups $\langle T \rangle$ and $\text{Aut}(X, T, \mathbb{Z}^d)$ are normal subgroups of $N^*(X, T, \mathbb{Z}^d)$ (the group of isomorphisms), and the centers of $N^*(X, T, \mathbb{Z}^d)$ and $\text{Aut}(X, T, \mathbb{Z}^d)$ are the same. In fact, we have the following short exact sequences

$$1 \to \quad \langle T \rangle \quad \to \text{Aut}(X, T, \mathbb{Z}^d) \to \text{Aut}(X, T, \mathbb{Z}^d)/\langle T \rangle \to 1,$$
$$1 \to \text{Aut}(X, T, \mathbb{Z}^d) \to N^*(X, T, \mathbb{Z}^d) \to \quad \vec{N}^*(X, T, \mathbb{Z}^d) \quad \to 1.$$

If $\pi : (X, T, \mathbb{Z}^d) \to (Y, T, \mathbb{Z}^d)$ is a factor map between two minimal systems, and there exists $y \in Y$ such that $|\pi^{-1}(\{y\})| = 1$, then this property is satisfied in a G_δ dense subset $Y_0 \subseteq Y$. In this case, we say that π is *almost* 1-*to*-1. If for some $c > 0$, $|\pi^{-1}(\{y\})| = c$ for all y in a G_δ dense subset of Y, then we say that π is *almost c-to*-1. If $|\pi^{-1}(\{y\})| \leq c < \infty$ for all $y \in Y$, we say that π is *finite-to*-1.

An important type of topological dynamical systems are the equicontinuous ones. A topological dynamical system $(X, T, \mathbb{Z}^d)$ is said to be *equicontinuous* if the set of maps

$\{T^{\boldsymbol{n}}: \boldsymbol{n} \in \mathbb{Z}^d\}$ forms an equicontinuous family of homeomorphisms. The equicontinuous systems are, in some sense, the simplest dynamical systems, in fact, there exists a complete characterization of them, and every topological dynamical system has at least one equicontinuous factor: the system given by one point. In fact, for every topological dynamical system there exists a *maximal equicontinuous factor*, i.e., a factor $\pi_{\mathrm{eq}} : (X, T, \mathbb{Z}^d) \to (X_{\mathrm{eq}}, T_{\mathrm{eq}}, \mathbb{Z}^d)$ such that $(X_{\mathrm{eq}}, T_{\mathrm{eq}}, \mathbb{Z}^d)$ is an equicontinuous system and for every equicontinuous factor $\pi : (X, T, \mathbb{Z}^d) \to (Y, T, \mathbb{Z}^d)$, there exists a factor map $\phi : (X_{\mathrm{eq}}, T_{\mathrm{eq}}, \mathbb{Z}^d) \to (Y, T_{\mathrm{eq}}, \mathbb{Z}^d)$ such that $\pi = \phi \circ \phi_{\mathrm{eq}}$.

2.3. Measure-preserving systems

A *measure-preserving system* is a 4-tuple (X, μ, T, G), where $(X, \mathcal{F}, \mu)$ is a probability space and G is a countable group of measurable and measure-preserving transformations acting on X (where the action is denoted by T), i.e., $\forall A \in \mathcal{F}, \forall g \in G, \mu(T^{g^{-1}} A) = \mu(A)$.

We say that (X, μ, T, G) is *ergodic* if for all $A \in \mathcal{F}$ we have that

$$\left[(\forall g \in G) \, \mu(T^{g^{-1}}(A) \Delta A) = 0 \right] \implies \mu(A) = 0 \vee \mu(A) = 1.$$

We now recall the notions of measurable homomorphisms in the measure-theoretic framework.

Let (X, μ, T, G) and (Y, ν, T, G) be measure-preserving systems and $M \in \mathrm{GL}(d, \mathbb{Z})$. A *measurable homomorphism associated with M* is a measure-preserving map $\phi : X' \to Y'$ where X', Y' are measurable subsets of X, Y respectively, with $\mu(X') = \nu(Y') = 1$ and for any $g \in G, T^g(X') \subseteq X', T^g(Y') \subseteq Y'$ such that for any $\boldsymbol{n} \in \mathbb{Z}^d$ we have that $\phi \circ T^{\boldsymbol{n}} = T^{M\boldsymbol{n}} \circ \phi$ in X'.

If there is a measurable factor map ϕ between X and Y, then X is said to be an *extension* of Y. If ϕ is a bi-measurable bijection, we say that ϕ is a *measurable conjugacy* and in this case (X, μ, T, G) and (Y, ν, T, G) are *metrically isomorphic*.

For topological dynamical $\mathbb{Z}^d$-actions, we always have at least one invariant probability measure (in fact, at least one ergodic probability measure). We define $\mathcal{M}(X, T, \mathbb{Z}^d)$ the set of all invariant probability measures. This set is convex and compact on the weak-* topology. We say that $(X, T, \mathbb{Z}^d)$ is *uniquely ergodic* if $|\mathcal{M}(X, T, \mathbb{Z}^d)| = 1$, and *strictly ergodic* if it is minimal and uniquely ergodic.

In the case of strictly ergodic topological dynamical systems $(X, T, \mathbb{Z}^d), (Y, T, \mathbb{Z}^d)$ we denote $m \operatorname{Hom}(X, Y, T, \mathbb{Z}^d), m \operatorname{Fac}(X, Y, T, \mathbb{Z}^d)$ the collection of all measurable homomorphisms and factor maps between $(X, T, \mathbb{Z}^d)$ and $(Y, T, \mathbb{Z}^d)$, respectively.

2.4. $\mathbb{Z}^d$-odometer systems

Let $Z_0 \geq Z_1 \geq \cdots \geq Z_n \geq Z_{n+1} \geq \cdots$ be a nested sequence of finite-index subgroups of $\mathbb{Z}^d$ such that $\bigcap_{n \geq 0} Z_n = \{\boldsymbol{0}\}$, and let $\alpha_n : \mathbb{Z}^d / Z_{n+1} \to \mathbb{Z}^d / Z_n$ be the function induced by the inclusion map. Following the notation in [10], we consider the inverse limit of these groups

$$\overleftarrow{\mathbb{Z}^d}_{(Z_n)} = \lim_{\overleftarrow{n}} (\mathbb{Z}^d / Z_n, \alpha_n),$$

i.e., $\overleftarrow{\mathbb{Z}^d}_{(Z_n)}$ is the subset of the product $\prod_{n \geq 0} \mathbb{Z}^d / Z_n$ consisting of the elements $\overleftarrow{g} = (g_n)_{n \geq 0}$ such that $\alpha_n(g_{n+1}) = g_n \pmod{Z}_n$ for all $n \geq 0$. This set is a group equipped with the addition defined coordinate-wise, i.e.,

$$\overleftarrow{g} + \overleftarrow{h} = (g_n + h_n)_{n \geq 0}.$$

Every group $\mathbb{Z}^d / Z_n$ is endowed with the discrete topology, so $\prod_{n \geq 0}(\mathbb{Z}^d / Z_n)$ is a compact metric space. The odometer $\overleftarrow{\mathbb{Z}^d}_{(Z_n)}$ is a compact topological group whose topology is spanned by the cylinder sets

$$[a]_n = \{\overleftarrow{g} \in \overleftarrow{\mathbb{Z}^d}_{(Z_n)} : g_n = a\},$$

with $a \in \mathbb{Z}^d / Z_n$, and $n \geq 0$. Now, consider the group homomorphism $\kappa_{(Z_n)} : \mathbb{Z}^d \to \prod_{n \geq 0} \mathbb{Z}^d / Z_n$ defined for $n \in \mathbb{Z}^d$ by

$$\kappa_{(Z_n)}(n) = [n \pmod{Z_n}]_{n \geq 0}.$$

The image of $\mathbb{Z}^d$ by $\kappa_{(Z_n)}$ is dense in $\overleftarrow{\mathbb{Z}^d}_{(Z_n)}$, so the $\mathbb{Z}^d$-action $n(\overleftarrow{g}) = \kappa_{(Z_n)}(n) + \overleftarrow{g}$, with $n \in \mathbb{Z}^d$, $\overleftarrow{g} \in \overleftarrow{\mathbb{Z}^d}_{(Z_n)}$, is well defined and $(\overleftarrow{\mathbb{Z}^d}_{(Z_n)}, +_{(Z_n)}, \mathbb{Z}^d)$ is a minimal equicontinuous system as proved in [10]. We call $(\overleftarrow{\mathbb{Z}^d}_{(Z_n)}, +_{(Z_n)}, \mathbb{Z}^d)$ an *odometer system*. Odometer systems have been extensively studied before (see [9, 10, 14]). The next result characterizes the factor odometer systems of a fixed odometer system.

Lemma 2.5 ([9, Lemma 1]). *Let* $(\overleftarrow{\mathbb{Z}^d}_{(Z_n^j)}, +_{(Z_n^j)}, \mathbb{Z}^d)$ *be two odometer systems ($j = 1, 2$). There exists a factor map*

$$\pi : (\overleftarrow{\mathbb{Z}^d}_{(Z_n^1)}, +_{(Z_n^1)}, \mathbb{Z}^d) \to (\overleftarrow{\mathbb{Z}^d}_{(Z_n^2)}, +_{(Z_n^2)}, \mathbb{Z}^d)$$

if and only if for every Z_n^2 there exists some Z_m^1 such that $Z_m^1 \leq Z_n^2$.

2.5. Symbolic Dynamics

Let $\mathcal{A}$ be a finite alphabet and $d \geq 1$ be an integer. We define a topology on $\mathcal{A}^{\mathbb{Z}^d}$ by endowing $\mathcal{A}$ with the discrete topology and considering in $\mathcal{A}^{\mathbb{Z}^d}$ the product topology, which is generated by cylinders. Since $\mathcal{A}$ is finite, $\mathcal{A}^{\mathbb{Z}^d}$ is a metrizable compact space. In this space, $\mathbb{Z}^d$ acts by translations (or shifts), defined for every $n \in \mathbb{Z}^d$ as

$$S^n(x)_k = x_{n+k}, \quad x \in \mathcal{A}^{\mathbb{Z}^d}, k \in \mathbb{Z}^d.$$

The $\mathbb{Z}^d$-action $(\mathcal{A}^{\mathbb{Z}^d}, S, \mathbb{Z}^d)$ is called the *fullshift*.

Let $P \subseteq \mathbb{Z}^d$ be a finite subset. A *pattern* is an element $\mathrm{p} \in \mathcal{A}^P$. We say that P is the *support* of p, and we denote $P = \mathrm{supp}(\mathrm{p})$. A pattern *occurs in* $x \in \mathcal{A}^{\mathbb{Z}^d}$, if there exists $n \in \mathbb{Z}^d$ such that $\mathrm{p} = x|_{n+P}$ (identifying $n + P$ with P by translation). In this case we denote it $\mathrm{p} \sqsubseteq x$ and we call this n an *occurrence in* x of p.

A *subshift* $(X, S, \mathbb{Z}^d)$ is given by a closed subset $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ which is invariant by the $\mathbb{Z}^d$-action. A subshift also can be defined by its *language*. For $P \Subset \mathbb{Z}^d$ we define

$$\mathcal{L}_P(X) = \{\mathrm{p} \in \mathcal{A}^P : \exists x \in X, \ \mathrm{p} \sqsubseteq x\}.$$

We define the *language* of a subshift X by

$$\mathcal{L}(X) = \bigcup_{P \Subset \mathbb{Z}^d} \mathcal{L}_P(X).$$

Let $(X, S, \mathbb{Z}^d)$ be a subshift and $x \in X$. We say that $\boldsymbol{p} \in \mathbb{Z}^d$ is a *period* of x if for all $\boldsymbol{n} \in \mathbb{Z}^d$, $x_{\boldsymbol{n}+\boldsymbol{p}} = x_{\boldsymbol{n}}$. We say that the subshift $(X, S, \mathbb{Z}^d)$ is *aperiodic* if there are no nontrivial periods.

Let $\mathcal{B}$ be another finite alphabet and $Y \subseteq \mathcal{B}^{\mathbb{Z}^d}$ be a subshift. For $P \Subset \mathbb{Z}^d$, we define a *P-block map* as a map of the form $\Phi : \mathcal{L}_P(X) \to \mathcal{B}$. This induces a factor map $\phi : X \to Y$ given by

$$\phi(x)_{\boldsymbol{n}} = \Phi(x|_{\boldsymbol{n}+P}).$$

The map ϕ is called the *sliding block code* induced by Φ and P is the support of the map ϕ. In most of the cases we may assume that the support of the sliding block codes is a ball of the form $B(\boldsymbol{0}, r)$, for $r \in \mathbb{N}$. We define the *radius* (and we denote it by $r(\phi)$) as the infimum of $r \in \mathbb{N}$ such that we can define a $B(\boldsymbol{0}, r)$-block map which induces it. The next theorem characterizes the factor maps between two subshifts.

Theorem 2.6 (Curtis–Hedlund–Lyndon). *Let $(X, S, \mathbb{Z}^d)$ and $(Y, S, \mathbb{Z}^d)$ be two subshifts. A map $\phi : (X, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ is a factor map if and only if there exists a $B(\boldsymbol{0}, r)$-block map $\Phi : \mathcal{L}_{B(\boldsymbol{0},r)}(X) \to \mathcal{L}_1(Y)$, such that $\phi(x)_{\boldsymbol{n}} = \Phi(x|_{\boldsymbol{n}+B(\boldsymbol{0},r)})$, for all $\boldsymbol{n} \in \mathbb{Z}^d$ and $x \in X$.*

For homomorphisms we have a similar characterization, but we need to make a slight variation of this theorem.

Theorem 2.7 (Curtis–Hedlund–Lyndon theorem for homomorphisms). *Let $(X, S, \mathbb{Z}^d)$ and $(Y, S, \mathbb{Z}^d)$ be two subshifts and $M \in \mathrm{GL}(d, \mathbb{Z})$. A map $\phi : (X, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ is a homomorphism associated with M if and only if there exists a $B(\boldsymbol{0}, r)$-block map $\Phi : \mathcal{L}_{B(\boldsymbol{0},r)}(X) \to \mathcal{L}_1(Y)$, such that $\phi(x)_{\boldsymbol{n}} = \Phi(x|_{M^{-1}\boldsymbol{n}+B(\boldsymbol{0},r)})$, for all $\boldsymbol{n} \in \mathbb{Z}^d$ and $x \in X$.*

Proof. We will only prove the nontrivial implication. Let $\phi : (X, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ be a homomorphism and let $r > 0$ be such that $x|_{B(\boldsymbol{0},r)} = y|_{B(\boldsymbol{0},r)}$ implies $\phi(x)_{\boldsymbol{0}} = \phi(y)_{\boldsymbol{0}}$. Then the local map $\Phi(x|_{B(\boldsymbol{0},r)}) = \phi(x)_{\boldsymbol{0}}$ is well defined by the very definition of r. Finally, note that $\phi(x)_{\boldsymbol{h}} = S^{\boldsymbol{h}}\phi(x)_{\boldsymbol{0}} = \phi(S^{M^{-1}\boldsymbol{h}}x)_{\boldsymbol{0}} = \Phi(x|_{M^{-1}\boldsymbol{h}+B(\boldsymbol{0},r)})$, which proves the claim. $\blacksquare$

This means, for any homomorphism ϕ we can define a *radius* (also denoted by $r(\phi)$), as the infimum of $r \in \mathbb{N}$ such that we can define a $B(\boldsymbol{0}, r)$-block map which induces it. In the case $r(\phi) = 0$, we say that ϕ is induced by a *letter-to-letter map*.

2.6. Nondeterministic directions of a topological dynamical system

An interesting notion in the study of higher-dimensional dynamical systems is that of so-called *nonexpansive subspaces*, introduced by M. Boyle and D. Lind in [6]. When the phase space X is infinite such subspaces always exist [6, Theorem 3.7]. We will only focus on hyperplanes in $\mathbb{R}^d$, which leads to the notion of deterministic/nondeterministic directions. Let $\mathbb{S}^{d-1}$ be the unit $(d-1)$-dimensional sphere. For $\boldsymbol{v} \in \mathbb{S}^{d-1}$ define $H_{\boldsymbol{v}} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{v} \rangle < 0\}$ to be the open half-space with outward unit normal $\boldsymbol{v}$. We identify the set $\mathbb{H}_d$ of all half-spaces in $\mathbb{R}^d$ with the sphere $\mathbb{S}^{d-1}$ using the parametrization $\boldsymbol{v} \leftrightarrow H_{\boldsymbol{v}}$.

Definition 2.8. Let $(X, S, \mathbb{Z}^d)$ be a subshift and $\boldsymbol{v}$ be a unit vector of $\mathbb{R}^d$. Then $\boldsymbol{v}$ is *deterministic* for $(X, S, \mathbb{Z}^d)$ if for all $x, y \in X$ we have that

$$x|_{H_{\boldsymbol{v}} \cap \mathbb{Z}^d} = y|_{H_{\boldsymbol{v}} \cap \mathbb{Z}^d} \implies x = y.$$

If $\boldsymbol{v}$ does not satisfy this condition, we say that $\boldsymbol{v}$ is *nondeterministic for* $(X, S, \mathbb{Z}^d)$.

For a subshift $(X, S, \mathbb{Z}^d)$ we denote $\mathrm{DD}(X, S, \mathbb{Z}^d)$ and $\mathrm{ND}(X, S, \mathbb{Z}^d)$ the sets of deterministic and nondeterministic directions for $(X, S, \mathbb{Z}^d)$, respectively. Using an analogous argument of [6, Lemma 3.4] we conclude $\mathrm{DD}(X, S, \mathbb{Z}^d)$ is an open set and $\mathrm{ND}(X, S, \mathbb{Z}^d)$ is a compact set.

In [26] was introduced the notion of *direction of determinism* for two-dimensional subshifts and in [11] these objects were used to prove a weak version of Nivat's conjecture.

The following result establishes a link between nondeterministic directions and the linear representation group of a subshift.

Proposition 2.9. *Let* $(X, S, \mathbb{Z}^d)$ *be a subshift. Then, for all* $\boldsymbol{v} \in \mathrm{ND}(X, S, \mathbb{Z}^d)$ *and* $M \in \vec{N}(X, S, \mathbb{Z}^d)$, *the vector* $(M^*)^{-1} \boldsymbol{v} / \|(M^*)^{-1} \boldsymbol{v}\|$ *is a nondeterministic direction for* $(X, S, \mathbb{Z}^d)$, *where* M^* *is the algebraic adjoint of* M.

Proof. If $\boldsymbol{v}$ is in $\mathrm{ND}(X, S, \mathbb{Z}^d)$, there exists $x \neq y \in X$ with $x|_{H_{\boldsymbol{v}} \cap \mathbb{Z}^d} = y|_{H_{\boldsymbol{v}} \cap \mathbb{Z}^d}$. Set $M \in \vec{N}(X, T, \mathbb{Z}^d)$ and $\phi \in N_M(X, S, \mathbb{Z}^d)$. Then we have that

$$\phi(x)|_{(MH_{\boldsymbol{v}}) + \boldsymbol{n}} = \phi(y)|_{(MH_{\boldsymbol{v}}) + \boldsymbol{n}},$$

where $\boldsymbol{n}$ is a vector of radius $r(\phi)$. We note that $S^{\boldsymbol{n}} \phi(x)|_{MH_{\boldsymbol{v}}} = S^{\boldsymbol{n}} \phi(y)|_{MH_{\boldsymbol{v}}}$. We conclude that $(M^*)^{-1} \boldsymbol{v} / \|(M^*)^{-1} \boldsymbol{v}\| \in (X, S, \mathbb{Z}^d)$. ∎

2.7. Multidimensional constant-shape substitutions

2.7.1. Definitions and basic properties. Let $L \in \mathcal{M}(d, \mathbb{Z})$ be an expansion matrix, i.e., $\|L\| > 1$ and $\|L^{-1}\| < 1$, such that $L(\mathbb{Z}^d) \subseteq \mathbb{Z}^d$. Let F be a fundamental domain of $L(\mathbb{Z}^d)$ in $\mathbb{Z}^d$ with $\boldsymbol{0}$ in F and $\mathcal{A}$ be a finite alphabet. A *multidimensional constant-shape substitution* ζ is a map $\mathcal{A} \to \mathcal{A}^F$. The set F is called the *support* of the substitution. These are multidimensional analogues of the so-called one-dimensional constant-length substitutions. Figure 2 shows an example of a constant-shape substitution with $L = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $F = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\}$.
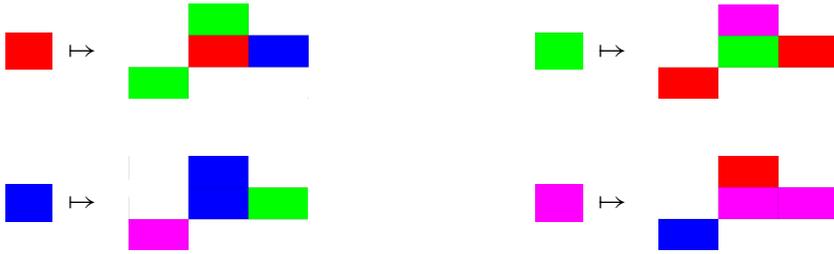
Figure 2. An example of a constant-shape substitution over a four-letter alphabet.

In the literature, constant-shape substitutions with a positive diagonal expansion matrix $L = \mathrm{diag}(l_i)_{i=1,\dots,d}$ and support equal to the standard d-dimensional parallelepiped $F_1 = \prod_{i=1}^{d} [\![0, l_i - 1]\!]$ are called *block substitutions*.

Examples of constant-shape substitutions can be generated via constant-length substitutions as follows: Let $\{\zeta_i\}_{i=1}^{d}$ be d aperiodic one-dimensional constant-length substitutions with alphabet $\mathcal{A}_i$ and length q_i for $1 \leq i \leq d$. We define the *product substitution of $\{\zeta_i\}_{i=1}^{d}$* as the constant-shape substitution ζ with alphabet $\mathcal{A} = \prod_{i=1}^{d} \mathcal{A}_i$, expansion matrix equal to $L_\zeta = \mathrm{diag}(q_i)$ and support $F_1^{\zeta} = \prod_{i=1}^{d} [\![0, q_i - 1]\!]$, defined as $\zeta(a_1, \dots, a_d)_{\boldsymbol{j}} = (\zeta_1(a_1)_{j_1}, \dots, \zeta_d(a_d)_{j_d})$.

Every element of $\mathbb{Z}^d$ can be expressed in a unique way as $\boldsymbol{p} = L(\boldsymbol{j}) + \boldsymbol{f}$, with $\boldsymbol{j} \in \mathbb{Z}^d$ and $\boldsymbol{f} \in F$, so we can consider the substitution ζ as a map from $\mathcal{A}^{\mathbb{Z}^d}$ to itself given by

$$\zeta(x)_{L(\boldsymbol{j})+\boldsymbol{k}} = \zeta(x(\boldsymbol{j}))_{\boldsymbol{k}}.$$

Given a substitution ζ, we will denote L_ζ its expansion matrix and F_1^{ζ} its support. For any $n > 0$ we define the n-th iteration of the substitution $\zeta^n : \mathcal{A} \to \mathcal{A}^{F_n^{\zeta}}$ with expansion matrix L_ζ^n and the supports of these substitutions satisfying the recurrence $F_{n+1}^{\zeta} = F_1^{\zeta} + L_\zeta(F_n^{\zeta})$ for all $n \geq 1$. From now on, we may assume that the sequence of supports $(F_n^{\zeta})_{n>0}$ is a Følner sequence.

Figure 3 shows the first three iterations of the substitution given in Figure 2.

The *language* of a substitution is the set of all patterns that appear in $\zeta^n(a)$, for some $n > 0, a \in \mathcal{A}$, i.e.,

$$\mathcal{L}_\zeta = \big\{ \mathrm{p} : \mathrm{p} \sqsubseteq \zeta^n(a), \text{ for some } n > 0, \, a \in \mathcal{A} \big\}.$$

Using the language we define the subshift X_ζ associated with a substitution as the set of all sequences $x \in \mathcal{A}^{\mathbb{Z}^d}$ such that every pattern occurring in x is in $\mathcal{L}_\zeta$. We denote $(X_\zeta, S, \mathbb{Z}^d)$ the *substitutive subshift*.

We will restrict our study to primitive substitutions. A substitution is called *primitive* if there exists a positive integer $n > 0$, such that for every $a, b \in \mathcal{A}$, b occurs in $\zeta^n(a)$. If ζ is a primitive constant-shape substitution, the substitutive system $(X_\zeta, S, \mathbb{Z}^d)$ is minimal (the proof of [37, Proposition 5.2] for the one-dimensional case can be easily adapted to
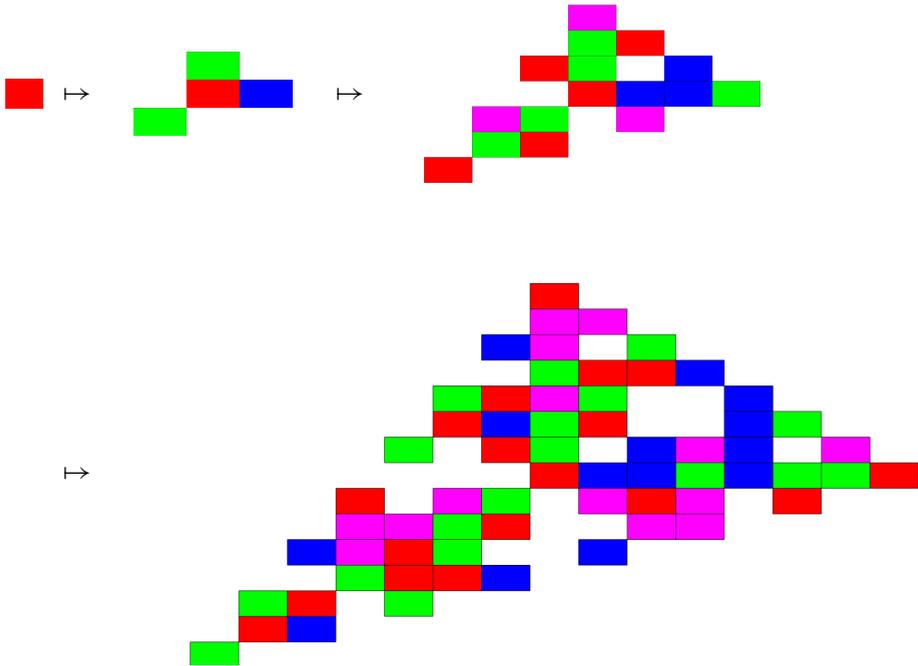
Figure 3. An example of an application of the first three iterations of the substitution illustrated in Figure 2.

our case), and the existence of ζ-*periodic points* is well known, i.e., there exists at least one point $x_0 \in X_\zeta$ such that $\zeta^p(x_0) = x_0$ for some $p > 0$. Under the primitivity assumption, the subshift is preserved replacing the substitution by a power of it, i.e., X_{ζ^n} is equal to X_ζ for all $n > 0$. Then, we may assume that the substitution possesses at least one fixed point, i.e., there exists a point $x \in X_\zeta$ such that $x = \zeta(x)$. It is well known that this subshift is strictly ergodic (in [32] can be found a proof of the unique ergodicity for substitutive tiling systems seen as substitutive Delone sets for $\mathbb{R}^d$-actions that can be adapted for our context thanks to the assumption that the sequence of supports $(F_n^\zeta)_{n>0}$ form a Følner sequence). The unique ergodic measure is characterized in terms of the expansion matrix of ζ and we denote this measure as μ_ζ. For a cylinder set $[p]_n$, where p is a pattern in $\mathcal{L}_\zeta$, the quantity $\mu_\zeta([p])$ represents the frequency of the pattern p in any sequence in X_ζ. The frequencies of patterns exist by unique ergodicity.

Since L_ζ is an expansion matrix, L_ζ^{-1} defines a contraction map in $\mathbb{R}^d$. For any $g \in F_1^\zeta$ define the map $f_g(\cdot) = L_\zeta^{-1}(\cdot + g)$. As mentioned in Section 2.1.3, by the IFS theorem, there exists a nonempty compact subset T_ζ (or denoted $T(L, F_1)$ when there is no substitution defined) in $\mathbb{R}^d$ such that

$$L_\zeta(T_\zeta) = \bigcup_{g \in F_1^\zeta} T_\zeta + g.$$

As in [44] we call this set the *digit tile* of the substitution. Using $T_0 = \{0\}$ in (1) we get that

$$T_\xi = \lim_{n\to\infty} \sum_{i=0}^{n-1} L_\xi^{-i}(F_1^\xi) = \lim_{n\to\infty} L_\xi^{-n}(F_n^\xi).$$

Figure 4 shows some approximations of some digit tiles.

The expansion matrices and fundamental domains of these examples are the following:

(a) $L_{(a)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $F_1^{(a)} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\}$.

(b) $L_{(b)} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $F_1^{(b)} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$.

(c) $L_{(c)} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $F_1^{(c)} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\}$.

(d) $L_{(d)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $F_1^{(d)} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$.

As in the one-dimensional case, the following proposition shows that for any multidimensional constant-shape substitution there exists a finite subset $K \Subset \mathbb{Z}^d$ whose iterations of the substitution fill the whole $\mathbb{Z}^d$. This set determines the ζ-periodic points of a primitive constant-shape substitution.

Proposition 2.10. *Let ζ be a multidimensional constant-shape substitution. Then, the set $K_\xi = \bigcup_{m>0}((\mathrm{id} - L_\xi^m)^{-1}(F_m^\xi) \cap \mathbb{Z}^d)$ is finite and satisfies*

$$\bigcup_{n\geq 0} L_\xi^n(K_\xi) + F_n^\xi = \mathbb{Z}^d,$$

using the notation $F_0^\xi = \{0\}$.

Proof. Set $\boldsymbol{n} \in \mathbb{Z}^d$ and consider the sequence $(\boldsymbol{a}_m)_{m\geq 0} \subseteq \mathbb{Z}^d$ given by $\boldsymbol{a}_0 = \boldsymbol{n}$ and for any $m \geq 0$, $\boldsymbol{a}_{m+1}$ is defined as the unique element in $\mathbb{Z}^d$ such that there exists an element $\boldsymbol{f}_m \in F_1^\xi$ with $\boldsymbol{a}_m = L_\xi(\boldsymbol{a}_{m+1}) + \boldsymbol{f}_m$. Note that for any $m > 0$, $\boldsymbol{n} = L_\xi^m(\boldsymbol{a}_m) + \sum_{i=1}^m L^{m-1}(\boldsymbol{f}_m)$, and $\|\boldsymbol{a}_{m+1}\| \leq \|L_\xi^{-1}\| \cdot \|\boldsymbol{a}_m\| + \|L_\xi^{-1}\| \cdot \|F_1^\xi\|$. This implies that for all $m \geq 0$,

$$\|\boldsymbol{a}_m\| \leq \|L_\xi^{-1}\|^m \cdot \|\boldsymbol{a}_0\| + \frac{\|L_\xi^{-1}\| \cdot \|F_1^\xi\|(1 - \|L_\xi^{-1}\|^m)}{1 - \|L_\xi^{-1}\|},$$
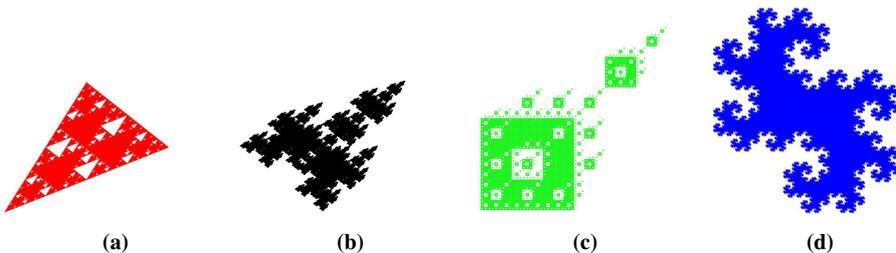

(a)   (b)   (c)   (d)

Figure 4. Approximation of some digit tiles: (a) Gasket, (b) Rocket, (c) Shooter, (d) Twin Dragon. The names of these tiles come from [44].

hence $(a_m)_{m\geq 0}$ is a bounded sequence in $\mathbb{Z}^d$. By the Pigeonhole principle there exist $m \geq 0$ and $n > 0$ such that $a_m = a_{m+n}$, i.e., $a_m = L_\zeta^n(a_m) + f$, for some $f \in F_n^\zeta$. It follows that the set $K_\zeta = \bigcup_{m>0}((\mathrm{id} - L_\zeta^m)^{-1}(F_m^\zeta) \cap \mathbb{Z}^d)$ satisfies the desired property. Now we prove that K_ζ is finite. Note that for any $m > 0$,

$$\|(\mathrm{id} - L_\zeta^m)^{-1}(F_m^\zeta)\| = \left\| \sum_{i=0}^{m-1}(\mathrm{id} - L_\zeta^m)^{-1} L_\zeta^i(F_1^\zeta) \right\|$$

$$\leq \|F_1^\zeta\| \left\| \sum_{i=0}^{m-1}(\mathrm{id} - L_\zeta^m)^{-1} L_\zeta^i \right\|$$

$$\leq \|F_1^\zeta\| \|(\mathrm{id} - L_\zeta^m)^{-1}(\mathrm{id} - L_\zeta^m)(\mathrm{id} - L_\zeta)^{-1}\|$$

$$\leq \|F_1^\zeta\| \|(\mathrm{id} - L_\zeta)^{-1}\|.$$

We conclude that K_ζ is a finite set.   ∎

Remark 2.11. The following statements can be easily verified.

(1) In the one-dimensional case, a direct computation shows that for any constant-length substitution ζ, the set K_ζ is equal to $[\![-1, 0]\!]$. Similarly, a direct computation shows that for any d-dimensional block substitution, the set K_ζ is equal to $[\![-1, 0]\!]^d$.

(2) Since K_ζ is a finite set, there exists $j > 0$ such that

$$K_\zeta = \bigcup_{m=0}^{j}((\mathrm{id} - L_\zeta^m)^{-1}(F_m^\zeta) \cap \mathbb{Z}^d),$$

thus, by the fact that the sets $(\mathrm{id} - L_\zeta^m)^{-1}(F_m^\zeta) \cap \mathbb{Z}^d$ are nested, up to considering a power of ζ, we can assume that the set K_ζ is of the form $(\mathrm{id} - L_\zeta)^{-1}(F_1^\zeta) \cap \mathbb{Z}^d$.

The argument used in the proof of Proposition 2.10 is inspired by the Euclidean division algorithm. A similar idea can be used to find different sets satisfying specific statements involving the supports $(F_n^\zeta)_{n>0}$ that will be needed for some proofs. From now on, for any $n \in \mathbb{Z}^d$ we denote $d_n \in \mathbb{Z}^d$ and $f_n \in F_1^\zeta$ the unique elements such that $n = L_\zeta(d_n) + f_n$. The following result will be useful in a series of results throughout this article.

Proposition 2.12. *Set $A \Subset \mathbb{Z}^d$ and let $F \Subset \mathbb{Z}^d$ be such that $F_1^\zeta \subseteq F$. Define $B = \{d_n\}_{n\in F+A}$. Then, there exists a finite subset C of $\mathbb{Z}^d$ satisfying the following conditions:*

(1) $B \subseteq C$.

(2) $C + F + A \subseteq L_\zeta(C) + F_1^\zeta$.

(3) $\|C\| \leq \|B\| + \|L_\zeta^{-1}\|(\|A\| + \|F\| + \|F_1^\zeta\|)/(1 - \|L_\zeta^{-1}\|)$.

Proof. We define two sequences of finite sets of $\mathbb{Z}^d$, $(B_n)_{n \geq 0}$, $(C_n)_{n \geq 0}$, with $B_0 = B$, $C_0 = B + F + A$, and for any $n \geq 0$, set $B_{n+1} \Subset \mathbb{Z}^d$ such that $B_{n+1} = \{d_n\}_{n \in C_n}$, and $C_{n+1} \Subset \mathbb{Z}^d$ such that $C_{n+1} = B_{n+1} + F + A$. Note that

$$\|B_{n+1}\| \leq \|L_\zeta^{-1}\| \left(\|C_n\| + \|F_1^\zeta\| \right)$$
$$\leq \|L_\zeta^{-1}\| \left(\|B_n\| + \|A\| + \|F\| + \|F_1^\zeta\| \right)$$
$$\leq \|L_\zeta^{-1}\| \|B_n\| + \|L_\zeta^{-1}\| \left(\|A\| + \|F\| + \|F_1^\zeta\| \right).$$

Hence, for any $n > 0$ we have that

$$\|B_n\| \leq \|B\| \|L_\zeta^{-1}\|^n + \frac{1 - \|L_\zeta^{-1}\|^n}{1 - \|L_\zeta^{-1}\|} \left(\|L_\zeta^{-1}\| (\|A\| + \|F\| + \|F_1^\zeta\|) \right).$$

Since $\|L_\zeta^{-1}\|$ is strictly smaller than 1,

$$\|B_n\| \leq \|B\| + \|L_\zeta^{-1}\| (\|A\| + \|F\| + \|F_1^\zeta\|)/(1 - \|L_\zeta^{-1}\|).$$

This implies there exists $N \in \mathbb{N}$ such that $\bigcup_{n \leq N} B_n = \bigcup_{n \leq N+1} B_n$. We conclude the proof taking $C = \bigcup_{n=0}^N B_n$. ∎

Remark 2.13. The following particular cases will be useful in the rest of the article.

(1) Condition (2) implies that $C + A + F_1^\zeta \subseteq L_\zeta(C) + F_1^\zeta$ and a direct induction proves that for all $n \geq 0$, the following inclusion holds: $L_\zeta^n(C + A) + F_n^\zeta \subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta$. So, for any set $A \Subset \mathbb{Z}^d$ and $F = F_1^\zeta$ the set C is the minimal set (in terms of cardinality) such that for all $n \geq 0$, $L_\zeta^n(A) + F_n^\zeta \subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta$, and the sets $\{L_\zeta^n(C) + F_n^\zeta : n \geq 0\}$ are nested.

(2) Using $F = F_1^\zeta + F_1^\zeta$ and $A = \{0\}$, we obtain a set $C \Subset \mathbb{Z}^d$ such that $C + F_1^\zeta + F_1^\zeta \subseteq L_\zeta(C) + F_1^\zeta$. Since $0 \in F_1^\zeta$, 0 is in C, which implies that $F_1^\zeta + F_1^\zeta \subseteq L_\zeta(C) + F_1^\zeta$. Assume for some $n > 0$ that $F_n^\zeta + F_n^\zeta \subseteq L_\zeta^n(C) + F_n^\zeta$. Then we obtain that

$$F_{n+1}^\zeta + F_{n+1}^\zeta = F_n^\zeta + L_\zeta^n(F_1^\zeta) + F_n^\zeta + L_\zeta^n(F_1^\zeta)$$
$$\subseteq L_\zeta^n(C) + F_n^\zeta + L_\zeta^n(F_1^\zeta) + L_\zeta^n(F_1^\zeta)$$
$$= L_\zeta^n(C + F_1^\zeta + F_1^\zeta) + F_n^\zeta$$
$$\subseteq L_\zeta^{n+1}(C) + L_\zeta^n(F_1^\zeta) + F_n^\zeta$$
$$= L_\zeta^{n+1}(C) + F_{n+1}^\zeta,$$

so, by induction we have proven that for all $n > 0$, $F_n^\zeta + F_n^\zeta \subseteq L_\zeta^n(C) + F_n^\zeta$.

The next result shows that for any aperiodic symbolic factor $(Y, S, \mathbb{Z}^d)$ of $(X_\zeta, S, \mathbb{Z}^d)$ we can change ζ for an appropriate substitution ζ', having the same expansion matrix and fundamental domain, such that $(X_{\zeta'}, S, \mathbb{Z}^d)$ and $(X_\zeta, S, \mathbb{Z}^d)$ are conjugate, and there exists a factor map $\pi : (X_{\zeta'}, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ induced by a letter-to-letter map.

Lemma 2.14. *Let ζ be an aperiodic primitive constant-shape substitution and ϕ: $(X_\zeta, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ be an aperiodic symbolic factor of $(X_\zeta, S, \mathbb{Z}^d)$. Then there exists a substitution ζ', having the same support and expansion matrix of ζ, such that $(X_{\zeta'}, S, \mathbb{Z}^d)$ and $(X_\zeta, S, \mathbb{Z}^d)$ are conjugate and a factor map $\pi : (X_{\zeta'}, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ induced by a letter-to-letter map.*

Proof. Suppose that $\phi : (X_\zeta, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ is a factor map via a $B(\mathbf{0}, r)$-block map. Set $A = B(\mathbf{0}, r)$, by Proposition 2.12 there exists a set $C \Subset \mathbb{Z}^d$ such that $B(0, r) + F_1^\zeta + C \subseteq L_\zeta(C) + F_1^\zeta$. Set $D = L_\zeta(C) + F_1^\zeta$. We will define a substitution $\zeta^{(D)}$ considering the set $\mathcal{L}_D(X_\zeta)$ as the alphabet, with the same expansion matrix and support of ζ in the following way: If $\mathrm{p} \in \mathcal{L}_D(X_\zeta)$, then for any $\boldsymbol{j} \in F_1^\zeta$ we set $\zeta^{(D)}(\mathrm{p})_{\boldsymbol{j}} = \zeta(\mathrm{p})|_{\boldsymbol{j}+D}$. It is straightforward to check that $x \in X_\zeta$ is a fixed point of the substitution ζ if and only if $y \in \mathcal{L}_D(X_\zeta)^{\mathbb{Z}^d}$ such that $y_{\boldsymbol{n}} = x|_{\boldsymbol{n}+D}$ for all $\boldsymbol{n} \in \mathbb{Z}^d$ is a fixed point of the substitution $\zeta^{(D)}$. With this, we can define the following sliding block codes $\psi_1 : (X_\zeta, S, \mathbb{Z}^d) \to (X_{\zeta^{(D)}}, S, \mathbb{Z}^d)$ given by the D-block map $\Psi_1(\mathrm{p}) = \mathrm{p}$ and $\psi_2 : (X_{\zeta^{(D)}}, S, \mathbb{Z}^d) \to (X_\zeta, S, \mathbb{Z}^d)$ given by the letter-to-letter map $\Psi_2(\mathrm{p}) = \mathrm{p}_{\mathbf{0}}$. These maps commute with the shift action and define a conjugacy between X_ζ and $X_{\zeta^{(D)}}$. We then define a factor map $\phi^{(D)} : (X_{\zeta^{(D)}}, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ given by the letter-to-letter map equal to $\psi_2\phi$. ∎

3. Recognizability property of aperiodic substitutions and dynamical consequences

In this section, we will study the recognizability property of aperiodic constant-shape substitutions and some dynamical consequences. Every aperiodic substitution satisfies it. To prove this result we first prove that there is a polynomial growth of the repetitivity function for substitutive subshifts (Lemma 3.2) that allows us to prove that aperiodic symbolic factors of substitutive subshifts also satisfy a recognizability property (Proposition 3.7 and Proposition 3.8). Then we will present here several consequences of the recognizability property: there exist a finite number of orbits in X_ζ which are invariant by the substitution map (Proposition 3.9) and we determine the maximal equicontinuous factor of substitutive subshifts and their aperiodic symbolic factors (Proposition 3.19). Thanks to these last descriptions, we get that any aperiodic symbolic factor of a substitutive subshift is conjugate to a substitutive subshift (Theorem 3.26).

3.1. The repetitivity function of a substitutive subshift

In order to prove that the recognizability property is satisfied, we study first the repetitivity function of a substitutive subshift. Let ζ be an aperiodic primitive constant-shape substitution and assume that $x \in X_\zeta$ is a fixed point of the substitution. The minimality property implies that the substitutive subshift is *repetitive*, i.e., for every pattern $\mathrm{p} \sqsubseteq x$ there is a radius $r > 0$ such that for every $\boldsymbol{n} \in \mathbb{Z}^d$, the ball $B(\boldsymbol{n}, r)$ contains an occurrence

of p in x. The *repetitivity function* is the map $M_{X_\xi} : \mathbb{R}_+ \to \mathbb{R}_+$ defined for $r > 0$ as the smallest radius such that every ball $B(\boldsymbol{n}, M_{X_\xi}(r))$ contains an occurrence in x of every pattern with $\mathrm{diam}(\mathrm{supp}(\mathrm{p})) \leq r$. We say that the substitution is *linearly recurrent* or *linearly repetitive* if the repetitivity function has a linear growth, i.e., there exists $C > 0$ such that $M_{X_\xi}(r) \leq C \cdot r$. It is well known that aperiodic primitive one-dimensional substitutions are linearly recurrent [15], but in the multidimensional case this is no longer true, as we can see in Example 3.1.

Example 3.1 (A non-linearly repetitive constant-shape substitution). Consider the block substitution σ_2, given by $L_{\sigma_2} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ and $F_1^{\sigma_2} = [\![0, 1]\!] \times [\![0, 2]\!]$ defined by

$$
\sigma_2 : \quad a \mapsto \begin{matrix} b & c \\ c & b \\ a & b \end{matrix}, \quad b \mapsto \begin{matrix} a & c \\ c & b \\ b & c \end{matrix}, \quad c \mapsto \begin{matrix} c & b \\ a & c \\ c & b \end{matrix}.
$$

For any $p \geq 1$, we consider the pattern $\mathtt{w}_p = \sigma_2^p(a)|_{[\![0,2^{p-1}]\!] \times \{0\}} \in \mathcal{L}_{[\![0,2^{p-1}]\!] \times \{0\}}(X_\xi)$. Observe that the pattern ab appears horizontally only in the inferior corner of $\sigma_2(a)$ between the three images of the substitution. So, a direct induction enables to prove that if for some $p \geq 1$ the pattern $\mathtt{w}_p$ occurs in $\sigma_2^p\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, for $\alpha, \beta, \gamma, \delta \in \{a, b, c, \varepsilon\}$ (where ε denotes the empty pattern), then one of the letters must be a. Moreover, $\mathtt{w}_p$ only appears in the lower left corner of the pattern $\sigma_2^p(a)$. These properties imply that there is only one occurrence of $\mathtt{w}_p$ in $\sigma_2^p(\mathtt{w}_p)$, which is in the lower left corner of $\sigma_2^p(\mathtt{w}_p)$ as seen in Figure 5.

Then there is a ball of radius $3^p/2$ in the support of $\sigma_2^p(\mathtt{w}_p)$ with no occurrences of $\mathtt{w}_p$. Since this is true for any p, this implies that this substitution is not linearly recurrent.

However, the repetitivity function has at most polynomial growth, with the exponent depending only on the expansion matrix of the substitution.
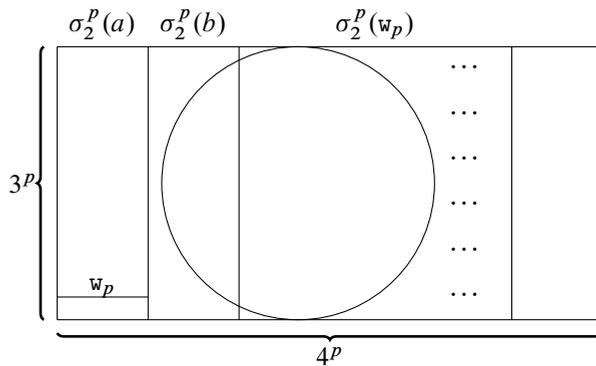

Figure 5. Decomposition of $\sigma_2^p(\mathtt{w}_p)$.

Lemma 3.2. *Let ζ be an aperiodic primitive constant-shape substitution. Then, the repetitivity function $M_{X_\zeta}(r)$ is $\mathcal{O}(r^{-\log(\|L_\zeta\|)/\log(\|L_\zeta^{-1}\|)})$.*

Proof. Let $x \in X_\zeta$ be a fixed point of ζ. Using $A = \{\mathbf{0}\}$ and $F = F_1^\zeta + F_1^\zeta$ in Proposition 2.12, we get a finite set $C_1 \subseteq \mathbb{Z}^d$ such that for every $n \geq 0$, $F_{n+1}^\zeta + F_{n+1}^\zeta \subseteq L_\zeta^{n+1}(C_1) + F_{n+1}^\zeta$.

Claim 1. *For any $r > 0$, we have that*

$$L_\zeta^n(B(\mathbf{0}, r)) \cap \mathbb{Z}^d \subseteq L_\zeta^n(B(\mathbf{0}, r + \|\mathrm{id} - L_\zeta^{-1}\| \cdot \|F_1^\zeta\|) \cap \mathbb{Z}^d) + F_n^\zeta.$$

Proof of Claim 1. Set $\boldsymbol{n} \in L_\zeta^n(B(\mathbf{0}, r)) \cap \mathbb{Z}^d$. Then, there exist $\boldsymbol{m}_1 \in \mathbb{Z}^d$ and $\boldsymbol{f} \in F_n^\zeta$ such that $\boldsymbol{m} = L_\zeta^n(\boldsymbol{m}_1) + \boldsymbol{f}$, which implies that $\|\boldsymbol{m}_1 + L_\zeta^{-n}(\boldsymbol{f})\| \leq r$. We then get that

$$\|\boldsymbol{m}_1\| \leq r + \|L_\zeta^{-n}(\boldsymbol{f})\| \leq r + \|\mathrm{id} - L_\zeta^{-1}\| \cdot \|F_1^\zeta\|. \qquad \blacksquare$$

Consider $R_1 > 0$ as the maximum radius such that for any $F_1^\zeta + (B(\mathbf{0}, R_1) \cap \mathbb{Z}^d) \subseteq L_\zeta(C_1) + F_1^\zeta$. Set $R_2 = R_1 + \|\mathrm{id} - L_\zeta^{-1}\| \cdot \|F_1^\zeta\|$. Note that, by the definition of C_1 and Claim 1, for every $n > 0$,

$$F_n^\zeta + (L_\zeta^n(B(\mathbf{0}, R_1)) \cap \mathbb{Z}^d) \subseteq L_\zeta^n(C_1 + (B(\mathbf{0}, R_2) \cap \mathbb{Z}^d)) + F_n^\zeta.$$

Consider $T = M_{X_\zeta}(\mathrm{diam}(C_1 + (B(\mathbf{0}, R_2) \cap \mathbb{Z}^d)))$ and for any $n > 0$, let $T_n > 0$ be such that every ball of radius T_n contains an occurrence of any pattern of the form $\zeta^n(\mathtt{w})$, with $\mathtt{w} \in \mathcal{L}_{C_1 + (B(\mathbf{0}, R_2) \cap \mathbb{Z}^d)}(X_\zeta)$.

Claim 2. *We have that $T_n \leq \|L_\zeta\|^n(T + 1/2)$.*

Proof of Claim 2. We recall that the lattice $L_\zeta^n(\mathbb{Z}^d)$ is $\|L_\zeta\|^n/2$-relatively dense, i.e., any ball of radius $\|L_\zeta\|^n/2$ contains an element of $L_\zeta^n(\mathbb{Z}^d)$. Set $\boldsymbol{n} \in \mathbb{Z}^d$. Consider $\boldsymbol{m} = L_\zeta^n(\boldsymbol{p}) \in \mathbb{Z}^d$ such that $\|\boldsymbol{n} - \boldsymbol{m}\| \leq \|L_\zeta\|^n/2$. Then, the ball $B(\boldsymbol{p}, T) \cap \mathbb{Z}^d$ contains an occurrence for any pattern $\mathtt{w} \in \mathcal{L}_{C_1 + (B(\mathbf{0}, R_2) \cap \mathbb{Z}^d)}(X_\zeta)$. Since x is a fixed point, the set $L_\zeta^n(B(\boldsymbol{m}, T)) \cap \mathbb{Z}^d$ contains an occurrence of any pattern of the form $\zeta^n(\boldsymbol{w})$, with $\mathtt{w} \in \mathcal{L}_{C_1 + (B(\mathbf{0}, R_2) \cap \mathbb{Z}^d)}(X_\zeta)$. The fact that $L_\zeta^n(B(\boldsymbol{m}, T)) \subseteq B(\boldsymbol{m}, \|L_\zeta\|^n T)$ and the Cauchy–Schwarz inequality let us conclude that the ball $B(\boldsymbol{n}, \|L_\zeta\|^n(T + 1/2)) \cap \mathbb{Z}^d$ contains an occurrence of any pattern $\zeta^n(\mathtt{w})$, for $\mathtt{w} \in \mathcal{L}_{C_1 + (B(\mathbf{0}, R_2) \cap \mathbb{Z}^d)}(X_\zeta)$. $\blacksquare$

Let $r > 0$, $\mathrm{p} \in \mathcal{L}(X_\zeta)$ a pattern such that $\mathrm{diam}(\mathrm{supp}(\mathrm{p})) \leq r$ and $\boldsymbol{n} \in \mathbb{Z}^d$ be an occurrence of p. Consider $n \geq 0$ such that $R_1/\|L_\zeta^{-1}\|^{n-1} \leq \mathrm{diam}(\mathrm{supp}(\mathrm{p})) \leq R_1/\|L_\zeta^{-1}\|^n$. Then there exist $\boldsymbol{n}_1 \in \mathbb{Z}^d$ and $\boldsymbol{f} \in F_n^\zeta$ such that $\boldsymbol{n} = L_\zeta^n(\boldsymbol{n}_1) + \boldsymbol{f}$. Set

$$\mathtt{w} = x|_{\boldsymbol{n}_1 + C_1 + (B(\mathbf{0}, R_2) \cap \mathbb{Z}^d)}.$$

Noting that $B(\mathbf{0}, R_1/\|L_\zeta^{-1}\|) \subseteq L_\zeta^k(B(\mathbf{0}, R_1))$, by Claim 1 we have that $\mathrm{p} \sqsubseteq \zeta^n(\mathtt{w})$. By Claim 2, any ball of radius $\|L_\zeta\|^n(T + 1/2)$ contains an occurrence of $\zeta^n(\mathtt{w})$ in x, so it

also contains an occurrence of p. Set $t = -\log(\|L_\zeta\|)/\log(\|L_\zeta^{-1}\|)$. Hence

$$\frac{M_X(r)}{r^t} \le \frac{\|L_\zeta^{-1}\|^{t(n-1)}\|L_\zeta\|^n(T+1/2)}{R_1^t} = \|L_\zeta\|(T+1/2)R_1^{-t} =: C.$$

We finally conclude that $M_{X_\zeta}(r) \le Cr^t$, with $t = -\log(\|L_\zeta\|)/\log(\|L_\zeta^{-1}\|)$. ∎

Remark 3.3. The following statements can be easily verified.

(1) In the case of a symmetric expansion matrix for the substitution, a bound for $M_{X_\zeta}(R)$ is given by its eigenvalues, that is, the repetitivity function $M_{X_\zeta}(R)$ is $\mathcal{O}(R^{(\log(|\lambda_1|))/(\log(|\lambda_d|))})$, where $|\lambda_1|, |\lambda_d|$ are the maximum and minimum of the absolute values of the eigenvalues of L_ζ, respectively.

(2) In the case of a self-similar substitution (where the expansion matrix satisfies $\|L_\zeta(t)\| = \lambda\|t\|$, for some $\lambda > 0$), the norm matrix satisfies $\|L_\zeta\| = (\|L_\zeta^{-1}\|)^{-1} = \lambda$, so the repetitivity function has a sublinear growth. Hence self-similar substitutions are linearly recurrent, as it was already proved in [41].

(3) The sufficiency of the previous case is not true, there exist constant-shape substitutions that are not self-similar, but are linearly recurrent.

3.2. Recognizability of a constant-shape substitution and their aperiodic symbolic factors

The substitution ζ seen as a map from X_ζ to $\zeta(X_\zeta)$ is continuous. Moreover, when the substitution is aperiodic and primitive, this map is actually a homeomorphism. This property is satisfied, even in the case where the substitution $\zeta : \mathcal{A} \to \mathcal{A}^{F_1^\zeta}$ is not *injective on letters*, i.e., when there exists a pair of letters $a, b \in \mathcal{A}$ such that $\zeta(a) = \zeta(b)$. This comes from the notion of *recognizability* of a substitution.

Definition 3.4. Let ζ be a primitive substitution and $x \in X_\zeta$ be a fixed point. We say that ζ is *recognizable on x* if there exists some constant $R > 0$ such that for all $i, j \in \mathbb{Z}^d$,

$$x|_{B(L_\zeta(i),R)\cap\mathbb{Z}^d} = x|_{B(j,R)\cap\mathbb{Z}^d} \implies (\exists k \in \mathbb{Z}^d)((j = L_\zeta(k)) \wedge (x_i = x_k)).$$

This implies that for every $x \in X_\zeta$ there exist a unique $x' \in X_\zeta$ and a unique $j \in F_1^\zeta$ such that $x = S^j\zeta(x')$. With this, the set $\zeta(X_\zeta)$ is a clopen subset of X_ζ and $\{S^j\zeta(X_\zeta): j \in F_1^\zeta\}$ is a clopen partition of X_ζ (in [37, Section 5.6] can be found a summary of these statements for the one-dimensional case. The proofs can be found in [37, Proposition 5.17], [37, Proposition 5.20], and [20, Corollary 7.2.3] that can be easily adapted to our case). These properties are also true for the iterations ζ^n, for all $n > 0$. The recognizability property was first proved for any aperiodic primitive substitution by B. Mossé in [34] for the one-dimensional case, and in the multidimensional case, B. Solomyak in [41] proved a recognizability property for aperiodic self-affine tilings with $\mathbb{R}^d$-actions.

In this section, we will prove it for aperiodic symbolic factors of substitutive subshifts $(X_\zeta, S, \mathbb{Z}^d)$ (Proposition 3.7). Like in Mossé's original proofs, the proof of the recognizability will go in two steps. This property will allows us to determine the maximal equicontinuous factors of aperiodic symbolic factors of substitutive subshifts.

For the first step of the proof of the recognizability property we use the following propositions. The first one, called repulsion property, is a direct consequence of the growth of the repetitivity function. The proof is an adaptation of the proof of [41, Lemma 2.4].

Proposition 3.5 (Repulsion property). *Let ζ be an aperiodic primitive constant-shape substitution, $x \in X_\zeta$ and set $t = -\log(\|L_\zeta\|)/\log(\|L_\zeta^{-1}\|)$. Then, there exists $N > 0$ such that, if a pattern $p \sqsubseteq x$ with $B(s, r) \cap \mathbb{Z}^d \subseteq \operatorname{supp}(p)$, for some $s \in \mathbb{Z}^d$ and $r > 0$, has two occurrences $j_1, j_2 \in \mathbb{Z}^d$ in x such that $r \geq N\|j_1 - j_2\|^t$, then j_1 is equal to j_2 (see Figure 6).*
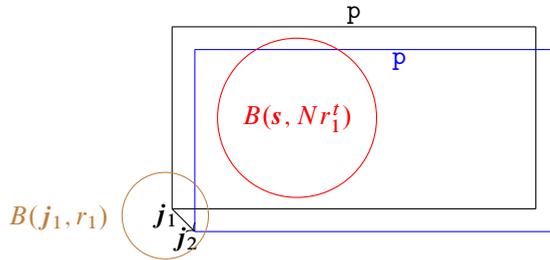

Figure 6. Illustration of a forbidden situation given by the repulsion property (Proposition 3.5).

Proof. For any $k \in \mathbb{Z}^d$, we consider the pattern $w_k = x|_{k \cup (k+j_2-j_1)}$. Note that

$$\operatorname{diam}(\operatorname{supp}(w_k)) = \|j_2 - j_1\|.$$

We are going to prove that the statement is true for $N > 0$ such that $M_{X_\zeta}(\|j_2 - j_1\|) \leq N\|j_2 - j_1\|^t \leq r$. By Lemma 3.2 such $N > 0$ exists. Indeed, since $r \geq M_{X_\zeta}(\|j_2 - j_1\|)$, the support of the pattern p contains an occurrence in x of any pattern w_k. Since j_1 is an occurrence of p in x, we get that for any $k \in \mathbb{Z}^d$, there exists $n_k \in \mathbb{Z}^d$ such that $x_{j_1+n_k+k} = x_k$ and $x_{j_1+n_k+(j_2-j_1+k)} = x_{j_2-j_1+k}$, which implies that $x_{j_2+n_k+k} = x_{j_2-j_1+k}$. The fact that j_2 is an occurrence of p in x let us conclude that for any $k \in \mathbb{Z}^d$, $x_{j_2-j_1+k}$ is equal to x_k, i.e., $j_2 - j_1$ is a period of x. Since ζ is aperiodic, we conclude that $j_1 = j_2$. ∎

As mentioned in [16], a key argument of the proof of the recognizability property is to prove the existence of an integer $p > 0$ such that for all $a, b \in \mathcal{A}$, if $\zeta^n(a) = \zeta^n(b)$ for some $n > 0$, then $\zeta^p(a) = \zeta^p(b)$. This was proved in the one-dimensional case in [18]. We extend this result in our context.

Proposition 3.6. *Let ζ be a constant-shape substitution over an alphabet $\mathcal{A}$. For any patterns $\mathtt{w}_1, \mathtt{w}_2 \in \mathcal{L}(X_\zeta)$, we have that*

$$\zeta^{|\mathcal{A}|-1}(\mathtt{w}_1) \neq \zeta^{|\mathcal{A}|-1}(\mathtt{w}_2) \implies \forall n \geq 0, \zeta^n(\mathtt{w}_1) \neq \zeta^n(\mathtt{w}_2).$$

Proof. We proceed by induction on $m = |\mathcal{A}|$. The statement is obviously true when $|\mathcal{A}| = 1$. Assume now that the statement is true for $|\mathcal{B}| \leq m - 1$.

Suppose that $\mathtt{w}_1 \neq \mathtt{w}_2$ (if $\mathtt{w}_1 = \mathtt{w}_2$ there is nothing to prove). Then ζ is not injective on letters. Consider the equivalence relation $a \sim b$ in $\mathcal{A}$, such that $a \sim b$ if $\zeta(a) = \zeta(b)$ and consider $\pi_\sim : \mathcal{A} \to \mathcal{A}/\sim$ the canonical projection. We define a morphism $\sigma : \mathcal{A}/\sim \to \mathcal{A}^F$ given by $\sigma([a]) = \zeta(a)$. This morphism is well defined by the very definition of $\sim$. Note that $\zeta = \sigma \circ \pi_\sim$. Then

$$(\exists n \geq 1)\, \zeta^n(\mathtt{w}_1) = \zeta^n(\mathtt{w}_2) \iff (\pi_\sim\sigma)^n \pi_\sim(\mathtt{w}_1) = (\pi_\sim\sigma)^n \pi_\sim(\mathtt{w}_2),$$

where $\pi_\sim\sigma : \mathcal{A}/\sim \to (\mathcal{A}/\sim)^F$ is the substitution with alphabet $\mathcal{A}/\sim$ such that for any $f \in F_1^\zeta$ and $[a] \in \mathcal{A}/\sim$. Thus, $\pi_\sim\sigma([a])_f$ is equal to $[\zeta(a)_f]$. Note that $(\exists n \geq 1)\zeta^n(\mathtt{w}_1) = \zeta^n(\mathtt{w}_2)$ if and only if $(\exists n \geq 0)(\pi_\sim\sigma)^n \pi_\sim(\mathtt{w}_1) = (\pi_\sim\sigma)^n \pi_\sim(\mathtt{w}_2)$. Since the cardinality of $\mathcal{A}/\sim$ is smaller than $m - 1$, by the inductive assumption we have that

$$(\exists n \geq 1)\, \zeta^n(\mathtt{w}_1) = \zeta^n(\mathtt{w}_2) \iff (\pi_\sim\sigma)^{m-2} \pi_\sim(\mathtt{w}_1) = (\pi_\sim\sigma)^{m-2} \pi_\sim(\mathtt{w}_2)$$
$$\iff \pi_\sim\zeta^{m-2}(\mathtt{w}_1) = \pi_\sim\zeta^{m-2}(\mathtt{w}_2).$$

Now, $\pi_\sim\zeta^{m-2}(\mathtt{w}_1) = \pi_\sim\zeta^{m-2}(\mathtt{w}_2)$ implies that $\sigma\pi_\sim\zeta^{m-2}(\mathtt{w}_1) = \sigma\pi_\sim\zeta^{m-2}(\mathtt{w}_2)$, which is equivalent to $\zeta^{m-1}(\mathtt{w}_1) = \zeta^{m-1}(\mathtt{w}_2)$. ∎

We then proceed to prove the first step of the recognizability property for aperiodic symbolic factors of constant-shape substitutions. As proved in Lemma 2.14, we may assume that an aperiodic symbolic factor of a substitutive subshift is induced by a letter-to-letter map.

Proposition 3.7 (First step of the recognizability property of aperiodic symbolic factors of substitutive subshifts). *Let $\mathcal{A}$, $\mathcal{B}$ be two finite alphabets, ζ be an aperiodic primitive constant-shape substitution from the alphabet $\mathcal{A}$ and let $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ be a map such that $(\tau(X_\zeta), S, \mathbb{Z}^d)$ is an aperiodic subshift. Let $x \in X_\zeta$ be a fixed point of ζ and $y = \tau(x)$. Then, there exists $R > 0$ such that if $y|_{i+B(\mathbf{0},R)} = y|_{j+B(\mathbf{0},R)}$ and $i \in L_\zeta(\mathbb{Z}^d)$, then $j \in L_\zeta(\mathbb{Z}^d)$.*

Proof. By Proposition 2.12 there exists two finite sets $C, D \subseteq \mathbb{Z}^d$ such that for every $n > 0$, $F_n^\zeta + F_n^\zeta \subseteq L_\zeta^n(C) + F_n^\zeta$ and $F_n^\zeta - F_n^\zeta \subseteq L_\zeta^n(D) + F_n^\zeta$. Set

$$r = N\|L_\zeta^{-1}\|^{|\mathcal{A}|-1}\|L_\zeta\|^{t(|\mathcal{A}|-1)}\|C + C + D\|^t + \|\mathrm{id} - L_\zeta^{-1}\| \cdot \|F_1^\zeta\|,$$

where $t = -\log(\|L_\zeta\|)/\log(\|L_\zeta^{-1}\|)$ and N is one given by Proposition 3.5. We prove the statement by contradiction. Assume the contrary, then for every $n > 0$ there exist

$i_n \in L_\zeta(\mathbb{Z}^d)$, $j_n \notin L_\zeta(\mathbb{Z}^d)$ such that

$$y|_{i_n + L_\zeta^n(D + B(0,r) \cap \mathbb{Z}^d) + F_n^\zeta} = y|_{j_n + L_\zeta^n(D + B(0,r) \cap \mathbb{Z}^d) + F_n^\zeta}.$$

For any $n > 0$, we consider $a_n \in \mathbb{Z}^d$ and $f_n \in F_n^\zeta$ such that $L_\zeta^n(a_n) + f_n = i_n$. Note that

$$L_\zeta^n(a_n) + L_\zeta^n(B(0,r) \cap \mathbb{Z}^d) + F_n^\zeta \subseteq i_n + L_\zeta^n(D + B(0,r) \cap \mathbb{Z}^d) + F_n^\zeta.$$

Let $u_n \in \mathcal{L}_{B(0,r) \cap \mathbb{Z}^d}(X_\zeta)$ be such that (see Figure 7)

$$y|_{i_n + L_\zeta^n(B(0,r) \cap \mathbb{Z}^d) + F_n^\zeta} = \tau(\zeta^n(u_n)) = y|_{j_n + L_\zeta^n(B(0,r) \cap \mathbb{Z}^d) + F_n^\zeta}.$$

Note that $j_n - f_n$ is not necessarily in $L_\zeta^n(\mathbb{Z}^d)$, so we set $b_n \in \mathbb{Z}^d$ and $g_n \in F_n^\zeta$ such that $L_\zeta^n(b_n) + g_n = j_n - f_n$. Now, for any $n > 0$ and $E \subseteq \mathbb{Z}^d$ we define the following sets:

$$G_{n,E} = \{n \in \mathbb{Z}^d : (L_\zeta^n(n) + F_n^\zeta) \cap (j_n - f_n) + L_\zeta^n(E) + F_n^\zeta \neq \emptyset\},$$

$$H_{n,E} = \{n \in \mathbb{Z}^d : (L_\zeta^n(n) + F_n^\zeta) \subseteq (j_n - f_n) + L_\zeta^n(E) + F_n^\zeta\}.$$

Since $x = \zeta(x)$, there exist a pattern $v_n \in \mathcal{L}_{G_{n,B(0,r) \cap \mathbb{Z}^d} - b_n}(X_\zeta)$, with $L_\zeta^n(b_n)$ being an occurrence of $\zeta^n(v_n)$ in x, such that $j_n + L_\zeta^n(B(0,r) \cap \mathbb{Z}^d) \subseteq L_\zeta^n(G_{n,B(0,r) \cap \mathbb{Z}^d}) + F_n^\zeta$. In particular, $\zeta^n(u_n) \sqsubseteq \zeta^n(v_n)$ (hence $\tau(\zeta^n(u_n)) \sqsubseteq \tau(\zeta^n(v_n))$), as illustrated in Figure 8.

Claim 3. *For any $n > 0$, $b_n \in H_{n,B(0,r) \cap \mathbb{Z}^d}$ and $(G_{n,(B(0,r) \cap \mathbb{Z}^d)} - b_n)$ is a bounded set.*

Proof of Claim 3. Let $h_n \in F_n^\zeta$. Note that $b_n \in H_{n,B(0,r) \cap \mathbb{Z}^d}$ if and only if there exists $r_n \in B(0,r) \cap \mathbb{Z}^d$ and $l_n \in F_n^\zeta$ such that $L_\zeta^n(b_n) + h_n = L_\zeta^n(b_n) + g_n + L_\zeta^n(r_n) + l_n$, i.e., $h_n = L_\zeta^n(r_n) + g_n + l_n$, which is true since $r \geq \|D\|$.

Now, set $m \in (G_{n,B(0,r) \cap \mathbb{Z}^d} - b_n)$, i.e., there exists $h_n \in F_n^\zeta$, $r_n \in B(0,r) \cap \mathbb{Z}^d$ and $l_n \in F_n^\zeta$ such that $L_\zeta^n(m) + h_n = L_\zeta^n(b_n) + g_n + L_\zeta^n(r_n) + l_n$, i.e., $m - b_n = r_n + L_\zeta^{-n}(g_n + l_n - h_n)$, which implies that $\|m - b_n\| \leq r + \|L_\zeta^{-n}(g_n + l_n - h_n)\|$. Note that $\|L_\zeta^{-n}(g_n + l_n - h_n)\| \leq 3\|\mathrm{id} - L_\zeta^{-1}\|\|F_1^\zeta\|$, which let us conclude that $\|m - b_n\| \leq r + 3\|\mathrm{id} - L_\zeta^{-1}\|\|F_1^\zeta\|$. ∎
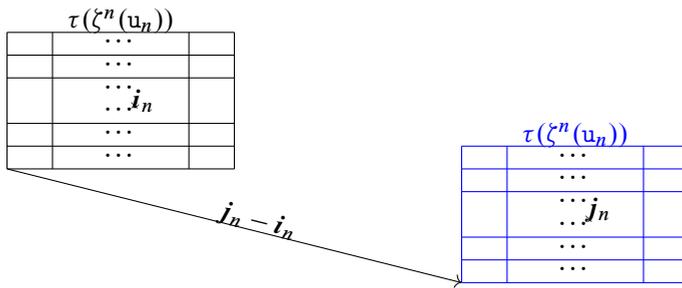

Figure 7. Illustration of the pattern $\tau(\zeta^n(u_n))$ around the coordinates i_n (black) and j_n (blue).
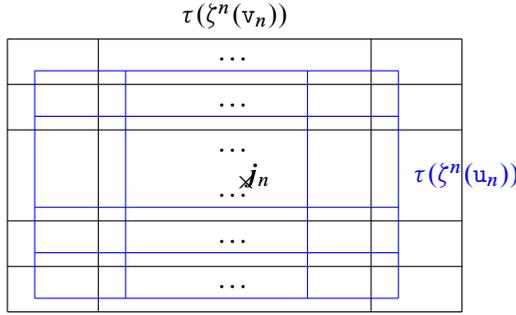
$$\tau(\zeta^n(\mathtt{v}_n))$$

Figure 8. Illustration of the patterns $\tau(\zeta^n(\mathtt{v}_n))$ and $\tau(\zeta^n(\mathtt{u}_n))$ around $\boldsymbol{j}_n$.

By the Pigeonhole principle, there are an infinite set $J \subseteq \mathbb{N}$, a finite set $G \Subset \mathbb{Z}^d$ such that $G = (G_{n,(B(\mathbf{0},r)\cap\mathbb{Z}^d)} - \boldsymbol{b}_n)$ and $H = (H_{n,(B(\mathbf{0},r)\cap\mathbb{Z}^d)} - \boldsymbol{b}_n)$ for all $n \in J$ and patterns $\mathtt{u} \in \mathcal{L}_{B(\mathbf{0},r)\cap\mathbb{Z}^d}(X_\zeta)$, $\mathtt{v} \in \mathcal{L}_G(X_\zeta)$ such that for all $n \in J$, $\mathtt{u}_n = \mathtt{u}$ and $\mathtt{v}_n = \mathtt{v}$. Consider $\mathtt{w} \in \mathcal{L}_H(X_\zeta)$ such that $x|_{L^n_\zeta(\boldsymbol{b}_n)+L^n_\zeta(H)+F^\zeta_n} = \zeta^n(\mathtt{w})$. Note that $L^n_\zeta(\boldsymbol{b}_n)$ is an occurrence of $\zeta^n(\mathtt{w})$ in x and set $\mathtt{a}_n = x|_{(\boldsymbol{j}_n-\boldsymbol{f}_n+(L^n_\zeta(B(\mathbf{0},r)+F^\zeta_n\cap\mathbb{Z}^d)))\setminus(L^n_\zeta(\boldsymbol{b}_n)+L^n_\zeta(H)+F^\zeta_n)}$ as illustrated in Figure 9.

Set $m > n \geq |\mathcal{A}| \in J$. Applying ζ^{m-n} to $\zeta^n(\mathtt{u})$, we obtain the patterns $\zeta^m(\mathtt{a}_n)$ and $\zeta^{m-n}(\zeta^n(\mathtt{w})) = \zeta^m(\mathtt{w})$.

Claim 4. *For any $n > 0$ and any $E \subseteq \mathbb{Z}^d$, we have that $G_{n,E} \subseteq H_{n,E} + C + C + D$.*

Proof of Claim 4. First, we are going to prove that for any $n > 0$ and $E \Subset \mathbb{Z}^d$, we have that $G_{n,E} \subseteq H_{n,E+C+C} + D$. Set $\boldsymbol{m} \in G_{n,E}$. Then, there exists $\boldsymbol{h}_n \in F^\zeta_n$, $\boldsymbol{e}_n \in E$, $\boldsymbol{l}_n \in F^\zeta_n$ such that

$$L^n_\zeta(\boldsymbol{m}) + \boldsymbol{h}_n = L^n_\zeta(\boldsymbol{b}_n) + \boldsymbol{g}_n + L^n_\zeta(\boldsymbol{e}_n) + \boldsymbol{l}_n.$$

Set $\boldsymbol{d}_n \in D$ such that $\boldsymbol{l}_n - \boldsymbol{h}_n + \boldsymbol{g}_n = L^n_\zeta(\boldsymbol{d}_n)$. Hence $\boldsymbol{m} = \boldsymbol{b}_n + \boldsymbol{e}_n + \boldsymbol{d}_n$. We prove that $\boldsymbol{m} - \boldsymbol{d}_n \in H_{n,E+C+C}$.

$$\tau(\zeta^n(\mathtt{v}))$$
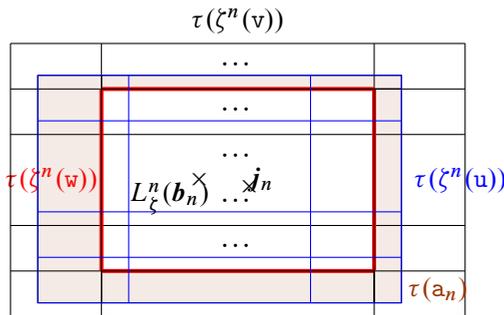
Figure 9. Illustration of the patterns $\zeta^n(\mathtt{w})$ and $\mathtt{a}_n$ in $\boldsymbol{j}_n$.

Set $o_n \in F_n^\zeta$. Then

$$
\begin{aligned}
L_\zeta^n(\boldsymbol{m} - \boldsymbol{d}_n) + \boldsymbol{o}_n &= L_\zeta^n(\boldsymbol{m}) + \boldsymbol{h}_n - L_\zeta^n(\boldsymbol{d}_n) - \boldsymbol{h}_n + \boldsymbol{o}_n \\
&= L_\zeta^n(\boldsymbol{b}_n) + \boldsymbol{g}_n + L_\zeta^n(\boldsymbol{e}_n) + \boldsymbol{l}_n - L_\zeta^n(\boldsymbol{d}_n) - \boldsymbol{h}_n + \boldsymbol{o}_n \\
&= L_\zeta^n(\boldsymbol{b}_n) + L_\zeta^n(\boldsymbol{e}_n) + \boldsymbol{o}_n
\end{aligned}
$$

Let $\boldsymbol{q}_n \in F_n^\zeta$ and $\boldsymbol{c}_n \in C$ be such that $\boldsymbol{g}_n + \boldsymbol{q}_n = L_\zeta^n(\boldsymbol{c}_n)$. We get that

$$
L_\zeta^n(\boldsymbol{m} - \boldsymbol{d}_n) + \boldsymbol{o}_n = L_\zeta^n(\boldsymbol{b}_n) + \boldsymbol{g}_n + L_\zeta^n(\boldsymbol{e}_n + \boldsymbol{c}_n) + (\boldsymbol{o}_n + \boldsymbol{q}_n).
$$

Since $F_n^\zeta + \boldsymbol{q}_n \subseteq L_\zeta^n(C) + F_n^\zeta$, we conclude that $\boldsymbol{m} \in H_{n,E+C+C} + D$.

To finish the proof, we note that a straightforward computation shows that for any $n > 0$ and $A, B \Subset \mathbb{Z}^d$, we have that $H_{n,A+B} \subseteq H_{n,A} + B$. We then, conclude that $G_{n,E} \subseteq H_{n,E} + C + C + D$. ∎

If $\zeta^{m-n}(\mathtt{a}_n)$ and $\mathtt{a}_m$ are different, there are two occurrences of $\zeta^m(\mathtt{w})$. By Proposition 3.6, these patterns come from two patterns $\mathtt{w}_1, \mathtt{w}_2 \in \mathscr{L}_H(X_\zeta)$ such that $\zeta^{|\mathcal{A}|-1}(\mathtt{w}_1) = \zeta^{|\mathcal{A}|-1}(\mathtt{w}_2) = \zeta^{|\mathcal{A}|-1}(\mathtt{w})$, occurring in $\zeta^{|\mathcal{A}|-1}(\mathtt{v})$. The distance between these two occurrences is smaller than $\max_{\boldsymbol{t} \in C+C+D} \|L_\zeta^{|\mathcal{A}|-1}(\boldsymbol{t})\| \le \|L_\zeta\|^{|\mathcal{A}|-1}\|C + C + D\|$, and by Claim 1 we have that $L_\zeta^{|\mathcal{A}|-1}(B(\boldsymbol{0}, r)) \subseteq \mathrm{supp}(\zeta^{|\mathcal{A}|-1}(\mathtt{w}))$, so $\mathrm{supp}(\zeta^{|\mathcal{A}|-1}(\mathtt{w}))$ contains a ball of radius $1/\|L_\zeta^{-1}\|^{|\mathcal{A}|-1}r$. By the repulsion property (Proposition 3.5), this is a contradiction, so $\zeta^{m-n}(\mathtt{a}_n) = \mathtt{a}_m$ as illustrated in Figure 10.

To finish the proof, we note that since $\zeta^n(\mathtt{u}) \sqsubseteq \zeta^n(\mathtt{v})$, there exists $\boldsymbol{p}_m \in L_\zeta^n(\boldsymbol{b}_m) + L_\zeta^n(G) + F_n^\zeta$ such that $x|_{\boldsymbol{p}_m + L_\zeta^n(B(\boldsymbol{0},r) \cap \mathbb{Z}^d) + F_n^\zeta} = \zeta^n(\mathtt{u})$, which implies that

$$
x|_{L_\zeta^{m-n}(\boldsymbol{p}_m) + L_\zeta^m(B(\boldsymbol{0},r) \cap \mathbb{Z}^d) + F_m^\zeta} = \zeta^m(\mathtt{u}).
$$

Using the fact that $\zeta^{m-n}(\mathtt{a}_n) = \mathtt{a}_m$, we get that

$$
L_\zeta^{m-n}(\boldsymbol{p}_m) + L_\zeta^m(B(\boldsymbol{0},r) \cap \mathbb{Z}^d) + F_m^\zeta = \boldsymbol{j}_m - \boldsymbol{f}_m + L_\zeta^m(B(\boldsymbol{0},r) \cap \mathbb{Z}^d) + F_m^\zeta,
$$

i.e., $\boldsymbol{j}_m - \boldsymbol{f}_m = L_\zeta^{m-n}(\boldsymbol{p}_m) \in L_\zeta^m(\mathbb{Z}^d)$. Since $\boldsymbol{i}_m \in L_\zeta(\mathbb{Z}^d)$, we have $\boldsymbol{f}_m \in L_\zeta(\mathbb{Z}^d)$ and we conclude that $\boldsymbol{j}_m \in L_\zeta(\mathbb{Z}^d)$. ∎
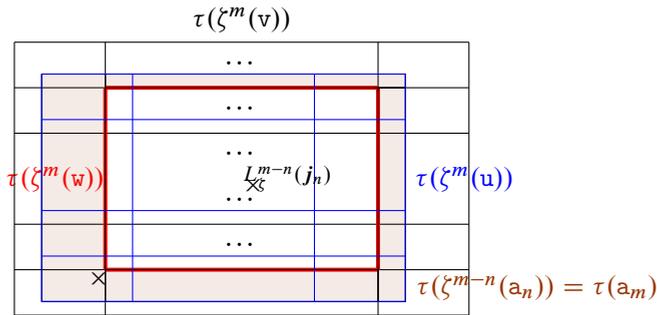

Figure 10. Illustration of the patterns $\zeta^{m-n}(\mathtt{a}_n)$ in $L_\zeta^{m-n}(\boldsymbol{j}_n)$.

We now prove the second part of the recognizability property. Note that if the substitution is injective on letters, the second step is a direct consequence of the first one.

Proposition 3.8 (Recognizability property of aperiodic symbolic factors of substitutive subshifts). *Let ζ be an aperiodic primitive substitution from the alphabet $\mathcal{A}$ and let $\mathcal{T}$: $\mathcal{A} \to \mathcal{B}$ be a map such that $(\tau(X_\zeta), S, \mathbb{Z}^d)$ is an aperiodic subshift. Let $x \in X_\zeta$ be a fixed point of ζ and $y = \tau(x)$. Consider $R > 0$ as the recognizability radius from Proposition* 3.7 *for $\zeta^{|\mathcal{A}|}$ and $M = R + 2\|F_1^\zeta\|(\|L_\zeta\|^{|\mathcal{A}|} - 1)/(\|L_\zeta\|)$. Then, for any $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^d$*

$$y|_{B(L(\boldsymbol{i}),M)} = y|_{B(L(\boldsymbol{j}),M)} \implies y_{\boldsymbol{i}} = y_{\boldsymbol{j}}.$$

Proof. Let $\boldsymbol{k} \in \mathbb{Z}^d$ and $\boldsymbol{f} = \sum_{i=1}^{|\mathcal{A}|} L_\zeta^i(\boldsymbol{f}_i) \in F_{|\mathcal{A}|}^\zeta$ be such that $L_\zeta^{|\mathcal{A}|}(\boldsymbol{k}) + \boldsymbol{f} = L_\zeta(\boldsymbol{i})$. Hence, we have that

$$L_\zeta^{|\mathcal{A}|-1}(\boldsymbol{k}) + \sum_{i=1}^{|\mathcal{A}|} L_\zeta^{i-1}(\boldsymbol{f}_i) = \boldsymbol{i}.$$

By the definition of $R > 0$, we have the existence of $\boldsymbol{m} \in \mathbb{Z}^d$ such that $L_\zeta^{|\mathcal{A}|}(\boldsymbol{m}) + \boldsymbol{f} = L_\zeta(\boldsymbol{j})$, which implies that $\boldsymbol{j} = L_\zeta^{|\mathcal{A}|-1}(\boldsymbol{m}) + \sum_{i=1}^{|\mathcal{A}|} L_\zeta^{i-1}(\boldsymbol{f}_i)$. Note that, by the definition of $M > 0$, we get that

$$y|_{L_\zeta^{|\mathcal{A}|}(\boldsymbol{k}) + F_{|\mathcal{A}|}^\zeta} = y|_{L_\zeta^{|\mathcal{A}|}(\boldsymbol{m}) + F_{|\mathcal{A}|}^\zeta}.$$

Hence $\tau(\zeta^{|\mathcal{A}|}(x_{\boldsymbol{k}})) = \tau(\zeta^{|\mathcal{A}|})(x_{\boldsymbol{m}})$. By Proposition 3.6 for the substitution $\tau\zeta$ on $\mathcal{B}$, we get that $\tau(\zeta^{|\mathcal{A}|-1}(x_{\boldsymbol{k}})) = \tau(\zeta^{|\mathcal{A}|-1})(x_{\boldsymbol{m}})$, which then implies that $y_{\boldsymbol{i}} = y_{\boldsymbol{j}}$. ∎

3.3. Invariant orbits of substitutive subshifts

As mentioned in the previous subsection, we assume that aperiodic primitive constant-shape substitutions admit at least one fixed point for the map $\zeta : X_\zeta \to X_\zeta$. The orbits of these fixed points lead to the notion of ζ-*invariant orbits*. An orbit $\mathcal{O}(x, \mathbb{Z}^d)$ is called ζ-*invariant* if there exists $\boldsymbol{j} \in \mathbb{Z}^d$ such that $\zeta(x) = S^{\boldsymbol{j}} x$, i.e., the orbit is invariant under the action of ζ in X_ζ. Since for every $\boldsymbol{n} \in \mathbb{Z}^d$ we have $\zeta \circ S^{\boldsymbol{n}} = S^{L_\zeta \boldsymbol{n}} \circ \zeta$, the definition is independent of the choice of the point in the $\mathbb{Z}^d$-orbit of x. The orbit of a fixed point of the substitution map is an example of an invariant orbit. In the following, we will prove that for aperiodic primitive constant-shape substitutions there exist finitely many ζ-invariant orbits. This property will be used to prove other properties about some constant-shape substitutions such as coalescence (Proposition 4.8) and that the automorphism group is virtually generated by the shift action (Proposition 4.9).

Proposition 3.9. *Let ζ be an aperiodic primitive constant-shape substitution. Then, there exist finitely many ζ-invariant orbits in the substitutive subshift X_ζ. The bound is explicit and depends only on d, $|\mathcal{A}|$, $\|L_\zeta^{-1}\|$, $\|F_1^\zeta\|$ and $\det(L_\zeta - \mathrm{id})$.*

Proof. Let $x \in X_\zeta$ be such that $\zeta(x) = S^{\boldsymbol{j}_x} x$, for some $\boldsymbol{j}_x \in \mathbb{Z}^d$. For any $\boldsymbol{m} \in \mathbb{Z}^d$, we have that

$$\zeta(S^{\boldsymbol{m}} x) = S^{L_\zeta \boldsymbol{m}} \zeta(x) = S^{L_\zeta \boldsymbol{m} + \boldsymbol{j}_x} x = S^{(L_\zeta - \mathrm{id})\boldsymbol{m} + \boldsymbol{j}_x} S^{\boldsymbol{m}} x,$$

thus $j_x - j_{S^m x} \in (L_\zeta - \mathrm{id})\mathbb{Z}^d$. Let $H \Subset \mathbb{Z}^d$ be a fundamental domain of $(L_\zeta - \mathrm{id})(\mathbb{Z}^d)$ in $\mathbb{Z}^d$ with $\mathbf{0} \in H$. We may assume that $x \in X_\zeta$ is in a ζ-invariant orbit with $j_x \in H$. Let $K_\zeta \Subset \mathbb{Z}^d$ be from Proposition 2.10. Using $-H$ as the set A and $F = F_1^\zeta$ in Proposition 2.12 we obtain a set $C \Subset \mathbb{Z}^d$ such that $L_\zeta^n(-H + C) + F_n^\zeta \subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta$ for all $n > 0$. Define $D \Subset \mathbb{Z}^d$ as $D = C + K_\zeta - H$. Suppose that there are more than $|\mathcal{A}|^{|D|} \cdot |H|$ ζ-invariant orbits. By the Pigeonhole principle there exist $j \in H$ and two different points $x \neq y \in X_\zeta$ such that $x|_D = y|_D$ and $\zeta(x) = S^j x$, $\zeta(y) = S^j y$. Note that

$$\zeta(x|_D) = \zeta(x)|_{L_\zeta(D)+F_1^\zeta} = x|_{j+L_\zeta(D)+F_1^\zeta}.$$

Hence, the patterns $x|_{j+L_\zeta(D)+F_1^\zeta}$, $y|_{j+L_\zeta(D)+F_1^\zeta}$ are the same. Inductively, we obtain that for every $n \geq 0$

$$x|_{\left(\sum_{k=0}^n L_\zeta^k j\right)+L_\zeta^{n+1}(D)+F_{n+1}^\zeta} = y|_{\left(\sum_{k=0}^n L_\zeta^k j\right)+L_\zeta^{n+1}(D)+F_{n+1}^\zeta}.$$

Let E_0 be equal to D and for all $n > 0$, define $E_n = \left(\sum_{k=0}^{n-1} L_\zeta^k j\right) + L_\zeta^n(D) + F_n^\zeta$. We will prove that $\bigcup_{n \geq 0} E_n = \mathbb{Z}^d$. This implies that $x = y$, which is a contradiction. To do this, we will prove that for every $n \geq 0$ the set $L_\zeta^n(K_\zeta) + F_n^\zeta$ is included in E_{n+1} and we conclude by Proposition 2.10. Note that $L_\zeta^n(K_\zeta) + F_n^\zeta \subseteq E_{n+1}$ if and only if $L_\zeta^n(K_\zeta - j) + \left(\sum_{k=0}^{n-1} L_\zeta^k(F_1^\zeta - j)\right) \subseteq L_\zeta^{n+1}(D) + F_{n+1}^\zeta$.

Claim 5. *For every $n \geq 0$ the set $\sum_{k=0}^{n-1} L_\zeta^k(F_1^\zeta - j)$ is included in $L_\zeta^n(C) + F_n^\zeta$.*

Proof of Claim 5. For $n = 0$, note that $F_1^\zeta - j$ is included in $L_\zeta(C) + F_1^\zeta$ by Proposition 2.12. Assume that for some $n \geq 0$, $\sum_{k=0}^n L_\zeta^k(F_1^\zeta - j) \subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta$. We have that

$$\begin{aligned}
\sum_{k=0}^{n+1} L_\zeta^k(F_1^\zeta - j) &= \left(\sum_{k=0}^n L_\zeta^k(F_1^\zeta - j)\right) + L_\zeta^{n+1}(F_1^\zeta - j) \\
&\subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta + L_\zeta^{n+1}(F_1^\zeta - j) \\
&\subseteq L_\zeta^{n+1}(C + F_1^\zeta - j) + F_{n+1}^\zeta \\
&\subseteq L_\zeta^{n+2}(C) + L_\zeta^{n+1}(F_1^\zeta) + F_{n+1}^\zeta \qquad \text{(by Proposition 2.12)} \\
&= L_\zeta^{n+2}(C) + F_{n+2}^\zeta.
\end{aligned}$$

We conclude that for every $n \geq 0$, $\sum_{k=0}^n L_\zeta^k(F_1^\zeta - j) \subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta$. ∎

By Claim 5, the set $L_\zeta^n(K_\zeta - j) + \left(\sum_{k=0}^{n-1} L_\zeta^k(F_1^\zeta - j)\right)$ is included in $L_\zeta^n(K_\zeta - j) + L_\zeta^n(C) + F_n^\zeta$ and $L_\zeta^n(K_\zeta + C - j)$ is a subset of $L_\zeta^{n+1}(D) + L_\zeta^n(F_1^\zeta)$, by definition of D. We have that

$$L_\zeta^n(K_\zeta - j) + L_\zeta^n(C) + F_n^\zeta \subseteq L_\zeta^{n+1}(D) + L_\zeta^n(F_1^\zeta) + F_n^\zeta = L_\zeta^{n+1}(D) + F_{n+1}^\zeta$$

and we conclude the proof. ∎

Remark 3.10. Let ζ be an aperiodic primitive substitution with an expansion matrix L_ζ such that $|\det(L_\zeta - \mathrm{id}_{\mathbb{R}^d})| = 1$. This implies that $(L_\zeta - \mathrm{id}_{\mathbb{R}^d})(\mathbb{Z}^d) = \mathbb{Z}^d$. Let $x \in X_\zeta$ be a point in a ζ-invariant orbit, i.e., there exists $\boldsymbol{j} \in \mathbb{Z}^d$ such that $\zeta(x) = S^{\boldsymbol{j}}x$, and set $\boldsymbol{m} \in \mathbb{Z}^d$ such that $(L_\zeta - \mathrm{id}_{\mathbb{R}^d})(\boldsymbol{m}) = -\boldsymbol{j}$. Then

$$\zeta(S^{\boldsymbol{m}}x) = S^{L_\zeta \boldsymbol{m}+\boldsymbol{j}}\zeta(x) = S^{\boldsymbol{m}+(L_\zeta-\mathrm{id}_{\mathbb{R}^d})(\boldsymbol{m})+\boldsymbol{j}}x = S^{\boldsymbol{m}}x.$$

Hence $S^{\boldsymbol{m}}x$ is a fixed point of ζ. We conclude that the only ζ-invariant orbits in this case are the ones given by the fixed points of the substitution.

In the following, we show an example of a ζ-invariant orbit that it is not the orbit of a fixed point of the substitution map.

Example 3.11 (An example of a nonfixed point orbit, which is ζ-invariant). Consider the one-dimensional substitution σ_2 of length 3 given by

$$\sigma_2 : \quad \begin{aligned} 0 &\mapsto 032, \\ 1 &\mapsto 123, \\ 2 &\mapsto 013, \\ 3 &\mapsto 102. \end{aligned}$$

This is an aperiodic primitive constant-length substitution. The set of words of length 2 (or with support $K_{\sigma_2} = \{-1, 0\}$) is $\{01, 02, 03, 10, 12, 13, 20, 21, 23, 30, 31, 32\}$. The words $20, 21, 30, 31$ generate the 4 fixed points of the square of the substitution σ_2^2, as illustrated in Figure 11.

$$\ldots 032123102.032102013 \ldots$$
$$\ldots 032123102.123013102 \ldots$$
$$\ldots 123032013.032102013 \ldots$$
$$\ldots 123032013.123013102 \ldots$$
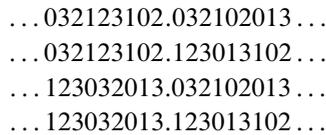
Figure 11. The four fixed points of σ_2^2. The dot in the center represents the origin.

In [29] it was proved that the substitutive subshift $(X_{\sigma_2}, S, \mathbb{Z})$ has an automorphism ϕ induced by a sliding block code of length 2, given by:

$$\Phi : \quad \begin{aligned} 01 &\mapsto 2, & 02 &\mapsto 0, & 03 &\mapsto 2, & 10 &\mapsto 3, & 12 &\mapsto 3, & 13 &\mapsto 1, \\ 20 &\mapsto 3, & 21 &\mapsto 1, & 23 &\mapsto 1, & 30 &\mapsto 0, & 31 &\mapsto 2, & 32 &\mapsto 0, \end{aligned}$$

and also that $S\phi\sigma_2$ is equal to $\sigma_2\phi$. This implies that

$$S^4\phi\sigma_2^2 = \sigma_2^2\phi. \tag{2}$$

Let $x \in X_{\sigma_2}$ be a fixed point of σ_2^2. We have that $S^4 \phi(x) = \sigma_2^2(\phi(x))$, so the orbit of $\phi(x)$ is σ_2^2-invariant. The orbit of $\phi(x)$ does not contain a fixed point of σ_2^2. Indeed, suppose that there exists $j \in \mathbb{Z}$ such that $\sigma_2^2(S^j(\phi(x))) = S^j(\phi(x))$. By (2), we have that $S^{9j+4}\phi(x) = S^j \phi(x)$. The aperiodicity of $(X_{\sigma_2}, S, \mathbb{Z})$ implies that $8j + 4 = 0$, which is a contradiction since $j \in \mathbb{Z}$. We conclude that in the orbit of $\phi(x)$ there is no fixed point of σ_2^2.

3.4. Substitutive subshifts as extensions of d-dimensional odometers

The recognizability property establishes a factor map from the substitutive subshift to a d-dimensional odometer as follows: For every $n > 0$, let $\pi_n : X_\xi \to F_n^\xi$ be the map satisfying $x \in S^{\pi_n(x)} \zeta^n(X_\xi)$. This map is well defined (mod $L_\xi^n(\mathbb{Z}^d)$) thanks to the recognizability property. Using basic group theory arguments, $\pi_{n+1}(x) = \pi_n(x)$ (mod $L_\xi^n(\mathbb{Z}^d)$).

Consider the odometer system $(\overleftarrow{\mathbb{Z}^d}_{(L_\xi^n(\mathbb{Z}^d))}, +_{(L_\xi^n(\mathbb{Z}^d))}, \mathbb{Z}^d)$. The map

$$\pi : (X_\xi, S, \mathbb{Z}^d) \to \big(\overleftarrow{\mathbb{Z}^d}_{(L_\xi^n(\mathbb{Z}^d))}, +_{(L_\xi^n(\mathbb{Z}^d))}, \mathbb{Z}^d\big)$$

given by $(\pi_n(x))_{n>0}$ is a factor map between $(X_\xi, S, \mathbb{Z}^d)$ and $(\overleftarrow{\mathbb{Z}^d}_{(L_\xi^n(\mathbb{Z}^d))}, +_{(L_\xi^n(\mathbb{Z}^d))}, \mathbb{Z}^d)$. Moreover, it satisfies the following property.

Lemma 3.12. *Let $(X_\xi, S, \mathbb{Z}^d)$ be the substitutive subshift from an aperiodic primitive constant-shape substitution ζ. Then, for any $\overleftarrow{g} = (g_n)_{n>0}$ in $\overleftarrow{\mathbb{Z}^d}_{(L_\xi^n(\mathbb{Z}^d))}$, two different cases occur:*

- (K1) *Assume that $\bigcup_{n>0}(-g_n + F_n^\xi) = \mathbb{Z}^d$. Then, there are at most $|\mathcal{A}|$ elements in $|\pi^{-1}(\{\overleftarrow{g}\})|$.*
- (K2) *On the other hand, $|\pi^{-1}(\{\overleftarrow{g}\})|$ is not greater than $|\mathcal{A}|^{|\overline{K}_\xi|}$, where $\overline{K}_\xi = K_\xi + C$, with $C + F_1^\xi + F_1^\xi \subseteq L_\xi(C) + F_1^\xi$ is obtained by Proposition 2.12 using $A = \{\mathbf{0}\}$ and $F = F_1^\xi + F_1^\xi$, and K_ξ is given by Proposition 2.10.*

In particular, the factor map $\pi : (X_\xi, S, \mathbb{Z}^d) \to (\overleftarrow{\mathbb{Z}^d}_{(L_\xi^n(\mathbb{Z}^d))}, +_{(L_\xi^n(\mathbb{Z}^d))}, \mathbb{Z}^d)$ is finite-to-1.

Proof. We separate the proof in these two cases.

- (K1) Assume that $|\pi^{-1}(\{\overleftarrow{g}\})| > |\mathcal{A}|$. Let $x_0, \ldots, x_{|\mathcal{A}|}$ be in $\pi^{-1}(\{\overleftarrow{g}\})$. For all $n > 0$ and all $j \in \{0, \ldots, |\mathcal{A}|\}$, there exist $y_j^n \in X_\xi$ such that $x_j = S^{g_n} \zeta^n(y_j^n)$. By the Pigeonhole principle, there exist $j_1 \neq j_2 \in \{0, \ldots, |\mathcal{A}|\}$ and an infinite set $E \subseteq \mathbb{N}$ with $y_{j_1}^n|_\mathbf{0} = y_{j_2}^n|_\mathbf{0}$ for $n \in E$. This implies that for any $n \in E$, $x_{j_1}|_{-g_n+F_n^\xi} = x_{j_2}|_{-g_n+F_n^\xi}$. By definition of $\overleftarrow{g}$, the points x_{j_1}, x_{j_2} are the same, which is a contradiction.
- (K2) Suppose that $\overleftarrow{g}$ does not satisfy (K1). Proceeding as in the previous case, we assure the existence of two indices $j_1 \neq j_2$ such that $y_{j_1}^n|_{\overline{K}_\xi} = y_{j_2}^n|_{\overline{K}_\xi}$ for any n in an infinite subset $E \subseteq \mathbb{N}$. This will imply that

$$x_{j_1}|_{-g_n+L_\xi^n(\overline{K}_\xi)+F_n} = x_{j_2}|_{-g_n+L_\xi^n(\overline{K}_\xi)+F_n^\xi}$$

for any $n \in E$. Now, note that for any $n \in \mathbb{N}$, $g_{n+1} - g_n \in L^n(F_1^\zeta)$, and a direct induction prove, $-g_n + L^n(\overline{K}_\zeta) + F_n^\zeta$ is included in $-g_{n+1} + L^{n+1}(\overline{K}_\zeta) + F_{n+1}^\zeta$. Since $K_\zeta \subseteq \overline{K}_\zeta$, we conclude by Proposition 2.10 that $x_{j_1} = x_{j_2}$ which will be a contradiction. ∎

Remark 3.13. Note that the set of points $\overleftarrow{g} \in \overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n(\mathbb{Z}^d))}$ satisfying (K1) is a G_δ-set. Indeed, for any $M > 0$ define

$$U_M = \{\overleftarrow{g} \in \overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n(\mathbb{Z}^d))} : [\![-M, M]\!]^d \subseteq -g_n + F_n^\zeta, \text{ for some } n > 0\}.$$

Note that U_M is an open subset of $\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n(\mathbb{Z}^d))}$ and the set $\bigcap_{M>0} U_M$ contains exactly the points satisfying (K1).

Now, for any n in $\mathbb{Z}^d$ and $M > 0$ the set U_M is included in $U_{M+|n|}$. Hence the set of points on the odometer satisfying (K1) is invariant by the $\mathbb{Z}^d$-action.

In some particular cases, we can compute explicitly the fibers of the factor map. For any $f \in F_1^\zeta$ we define p_f the restriction of ζ in f, i.e., for $a \in \mathcal{A}$, we have that $p_f(a) = \zeta(a)_f$. We say the substitution is *bijective* if for all $f \in F_1^\zeta$ the maps p_f are bijective. We have the following result about bijective substitutions

Proposition 3.14. *Let ζ be an aperiodic bijective primitive constant-shape substitution with alphabet $\mathcal{A}$. Then, the factor map $\pi : (X_\zeta, S, \mathbb{Z}^d) \to (\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n(\mathbb{Z}^d))}, +_{(L_\zeta^n(\mathbb{Z}^d))}, \mathbb{Z}^d)$ is almost $|\mathcal{A}|$-to-1.*

Proof. The map p_0 is a permutation of $\mathcal{A}$, so we consider a power of p_0 such that p_0^n is equal to the identity. Since ζ is primitive, we replace it by ζ^n and with this we may assume that ζ possesses at least $|\mathcal{A}|$ fixed points (one given by each letter of the alphabet). Now, let $\overleftarrow{g} \in \overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n(\mathbb{Z}^d))}$ satisfy (K1). For any $a \in \mathcal{A}$, consider a fixed point of ζ, denoted as x_a, with $x_a(0) = a$ and define $x_a^{\overleftarrow{g}} = \lim_{n_m \to \infty} S^{g_{n_m}} x_a$ for some convergent subsequence. Since the sets $\{S^{g_n}[\zeta^n(a)]\}_{a \in \mathcal{A}}$ are disjoint, for any $a \neq b \in \mathcal{A}$, $x_a^{\overleftarrow{g}}$ is different from $x_b^{\overleftarrow{g}}$. Finally, noticing that $\pi(x_a^{\overleftarrow{g}}) = \overleftarrow{g}$, we have that $\pi^{-1}(\{\overleftarrow{g}\}) \geq |\mathcal{A}|$. By Lemma 3.12 we conclude that $\pi^{-1}(\{\overleftarrow{g}\}) = |\mathcal{A}|$. Since the set of points satisfying (K1) form a G_δ set, we conclude that the factor map $\pi : (X_\zeta, S, \mathbb{Z}^d) \to (\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n(\mathbb{Z}^d))}, +_{(L_\zeta^n(\mathbb{Z}^d))}, \mathbb{Z}^d)$ is almost $|\mathcal{A}|$-to-1. ∎

In general, this d-dimensional odometer is not the maximal equicontinuous factor of aperiodic constant-shape substitutions.

3.5. The maximal equicontinuous factor of substitutive subshifts

For completeness, we will describe the maximal equicontinuous factor of substitutive subshifts (see [21] for the case with diagonal expansion matrices and [12] for the one-dimensional case). The results are well known in the literature but nowhere written.

A subgroup $\mathfrak{L} \leq \mathbb{Z}^d$ is called a *lattice* if it is isomorphic to $\mathbb{Z}^d$, i.e., it has finite index. For a lattice $\mathfrak{L}$ of $\mathbb{Z}^d$, we define the *dual lattice* of $\mathfrak{L}$, as the subgroup

$$\mathfrak{L}^* = \left\{ x \in \mathbb{R}^d : \langle x, n \rangle \in \mathbb{Z}, \ \forall n \in \mathfrak{L} \right\}.$$

We have that $(\mathbb{Z}^d)^* = \mathbb{Z}^d$, and for any $R \in \mathcal{M}(d, \mathbb{Z})$ the set $R(\mathbb{Z}^d)$ is a lattice of $\mathbb{Z}^d$ with dual lattice equal to $(R^*)^{-1}(\mathbb{Z}^d)$, where R^* stands for the *algebraic adjoint* of R. Let $\mathfrak{L}_1, \mathfrak{L}_2$ be two lattices of $\mathbb{Z}^d$. We denote by $\mathfrak{L}_1 \vee \mathfrak{L}_2$ the smallest lattice that contains $\mathfrak{L}_1$ and $\mathfrak{L}_2$, i.e., if a lattice $\mathfrak{L}$ contains $\mathfrak{L}_1$ and $\mathfrak{L}_2$, then it must contain $\mathfrak{L}_1 \vee \mathfrak{L}_2$.

Fix $x \in X_\zeta$. We define the *set of return times* as

$$\mathcal{R}(X_\zeta) = \left\{ j \in \mathbb{Z}^d : \exists k \in \mathbb{Z}^d, \ x_{k+j} = x_k \right\}.$$

By minimality of X_ζ this set is well defined independently of $x \in X_\zeta$ and it is syndetic, i.e., there exists a finite subset $A \Subset \mathbb{Z}^d$ such that $\mathcal{R}(X_\zeta) + A = \mathbb{Z}^d$. We define $\mathfrak{L}(\mathcal{R}(X_\zeta))$ as the smallest lattice containing $\mathcal{R}(X_\zeta)$. The *height lattice* $\mathcal{H}(X_\zeta)$ of a substitutive subshift ζ is the smallest lattice containing $\mathfrak{L}(\mathcal{R}(X_\zeta))$ such that $\mathcal{H}(X_\zeta) \cap L_\zeta(\mathbb{Z}^d) \leq L_\zeta(\mathcal{H}(X_\zeta))$. Notice that the last property is equivalent to $\mathcal{H}(X_\zeta) \cap L_\zeta^n(\mathbb{Z}^d) \leq L_\zeta^n(\mathcal{H}(X_\zeta))$, for any $n > 0$. The height lattice is *trivial* whenever $\mathcal{H}(X_\zeta) = \mathbb{Z}^d$.

Example 3.15 (An example of a height lattice). Consider the product substitution σ_3 of the one-dimensional substitutions $\sigma_4 : 0 \mapsto 010, 1 \mapsto 201, 2 \mapsto 102$ (with height 2) and $\sigma_5 : a \mapsto abcd, b \mapsto bcde, c \mapsto cdef, d \mapsto defa, e \mapsto efab, f \mapsto fabc$ (with height 3). The substitution σ_3 has an alphabet of cardinality 18 with $L_{\sigma_3} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ and $F_1^{\sigma_3} = [\![0, 2]\!] \times [\![0, 3]\!]$. A direct computation shows that the height lattice $\mathcal{H}((X_{\sigma_3}, S, \mathbb{Z}^2))$ is equal to $2\mathbb{Z} \times 3\mathbb{Z}$.

In the following, we will give a description for the height lattice, adapting the proof for the one-dimensional case [12,37]. For $k \in \mathbb{Z}^d$, we define $\mathcal{R}_k(X_\zeta) = \{ j \in \mathbb{Z}^d : x_{j+k} = x_k \}$. Let $\mathfrak{L}(\mathcal{R}_k(X_\zeta))$ be the smallest lattice containing $\mathcal{R}_k(X_\zeta)$ and $\mathcal{H}_k(X_\zeta)$ be the smallest lattice containing $\mathfrak{L}(\mathcal{R}_k(X_\zeta))$ such that $\mathcal{H}_k(X_\zeta) \cap L_\zeta(\mathbb{Z}^d) \leq L_\zeta(\mathcal{H}_k(X_\zeta))$.

Lemma 3.16. *Let ζ be an aperiodic primitive constant-shape substitution. Then, for any $k_1, k_2 \in \mathbb{Z}^d$ the sets $\mathcal{H}_{k_1}(X_\zeta), \mathcal{H}_{k_2}(X_\zeta)$ are the same. In particular, for any $k \in \mathbb{Z}^d$, $\mathcal{H}_k(X_\zeta)$ is equal to $\mathcal{H}(X_\zeta)$.*

Proof. Let $x \in X_\zeta$ be a fixed point of the substitution, $k_1, k_2 \in \mathbb{Z}^d$ and N be large enough such that $x_{k_2} \sqsubseteq \zeta^N(x_{k_1})$. Set m such that $x_{k_2} = \zeta^N(x_{k_1})_m$. Since $x = \zeta^N(x)$ for any $j \in \mathcal{R}_{k_1}(X_\zeta)$, the set $L_\zeta^N(k_1 + j) + m$ is included in $\mathcal{R}_{k_2}(X_\zeta)$. Hence $L_\zeta^N(j)$ is in $\mathfrak{L}(\mathcal{R}_{k_2}(X_\zeta))$ and therefore in $\mathcal{H}_{k_2}(X_\zeta)$. By definition of $\mathcal{H}_{k_2}(X_\zeta)$ and invertibility of L_ζ^N, we conclude that $j \in \mathcal{H}_{k_2}(X_\zeta)$. Since it is the smallest lattice satisfying the property, $\mathcal{H}_{k_1}(X_\zeta)$ is a subgroup of $\mathcal{H}_{k_2}(X_\zeta)$, and by reciprocity of the arguments these sets are the same. We conclude the second equality by observing that $\mathcal{H}(X_\zeta) = \sum_{k \in \mathbb{Z}^d} \mathcal{H}_k(X_\zeta)$. $\blacksquare$

As in the one-dimensional case, to study the maximal equicontinuous factor of a substitutive subshift, we study their *eigenvalues*. A vector $\boldsymbol{x} \in \mathbb{R}^d$ is said to be an eigenvalue for the topological dynamical system $(X_\zeta, S, \mathbb{Z}^d)$ (the measure-preserving system $(X_\zeta, \mu, S, \mathbb{Z}^d)$) if there exists a continuous function $f : X_\zeta \to \mathbb{C}$ ($f \in L^2(X_\zeta, \mu_\zeta)$, respectively) such that for every $\boldsymbol{n} \in \mathbb{Z}^d$, $f \circ S^{\boldsymbol{n}} = e^{2\pi i \langle \boldsymbol{x}, \boldsymbol{n} \rangle} f$ in X_ζ ($f \circ S^{\boldsymbol{n}} = e^{2\pi i \langle \boldsymbol{x}, \boldsymbol{n} \rangle} f$ in X_ζ, μ_ζ-a.e. in X_ζ, respectively). In [42] the following result was proved, generalizing the characterization of eigenvalues for the one-dimensional case [12].

Theorem 3.17 ([42]). *Let ζ be an aperiodic primitive constant-shape substitution which has a fixed point in X_ζ. Then the following are equivalent for $\boldsymbol{x} \in \mathbb{R}^d$:*

(1) *The vector $\boldsymbol{x}$ is a continuous eigenvalue for the topological dynamical system $(X_\zeta, S, \mathbb{R}^d)$.*

(2) *The vector $\boldsymbol{x}$ is a measurable eigenvalue for the measure-preserving system $(X_\zeta, \mu_\zeta, S, \mathbb{R}^d)$.*

(3) *The vector $\boldsymbol{x}$ satisfies the following condition:*
$$\lim_{n\to\infty} e^{2\pi i \langle L_\zeta^n \boldsymbol{j}, \boldsymbol{x} \rangle} = 1, \quad \forall \boldsymbol{j} \in \mathcal{R}(X_\zeta).$$

(4) *The vector $\boldsymbol{x}$ satisfies the following condition:*
$$\lim_{n\to\infty} e^{2\pi i \langle L_\zeta^n \boldsymbol{j}, \boldsymbol{x} \rangle} = 1, \quad \forall \boldsymbol{j} \in \mathcal{H}(X_\zeta).$$

Remark 3.18. The same results are satisfied for $\mathbb{Z}^d$-topological actions adapting the arguments given in [42]. Note that condition (4) in Theorem 3.17 is not proved in [42] but it can be easily checked noticing the set of points satisfying $\langle (L_\zeta^*)^N \boldsymbol{x}, \boldsymbol{j} \rangle \in \mathbb{Z}$ is a lattice.

This implies that the set of continuous (and measurable) eigenvalues of $(X_\zeta, S, \mathbb{Z}^d)$ corresponds to the set $\bigcup_{n\geq 0} (L_\zeta^*)^{-n}(\mathcal{H}^*(X_\zeta))$, where $\mathcal{H}^*(X_\zeta)$ is the dual lattice of $\mathcal{H}(X_\zeta)$. In particular, the set of eigenvalues $E(X_\zeta, S, \mathbb{Z}^d)$ of $(X_\zeta, S, \mathbb{Z}^d)$ is a subset of $\mathbb{Q}^d$. A direct consequence of Theorem 3.17 is a description of the maximal equicontinuous factor of $(X_\zeta, S, \mathbb{Z}^d)$.

Proposition 3.19. *Let ζ be an aperiodic primitive constant-shape substitution. The maximal equicontinuous factor of the substitutive subshift of $(X_\zeta, S, \mathbb{Z}^d)$ is the odometer system*
$$\left(\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n(\mathcal{H}(X_\zeta)))}, +_{L_\zeta^n(\mathcal{H}(X_\zeta))}, \mathbb{Z}^d \right) = \left(\lim_{\overleftarrow{n}} (\mathbb{Z}^d / L_\zeta^n(\mathcal{H}(X_\zeta)), \alpha_n), +_{(L_\zeta^n(\mathcal{H}(X_\zeta)))}, \mathbb{Z}^d \right).$$

For completeness, we provide another description for the maximal equicontinuous factor in a general setting. Let $(X, T, \mathbb{Z}^d)$ be a topological dynamical system, where X is a Cantor set, and $\Gamma \leq \mathbb{Z}^d$ a finite-index subgroup. We say that $(X, T, \mathbb{Z}^d)$ admits a Γ-*minimal partition* if there exists a closed partition $\bigcup_{\boldsymbol{g} \in \mathbb{Z}^d / \Gamma} X_{\boldsymbol{g}} = X$ such that for all

$g \in \mathbb{Z}^d / \Gamma$ the set of return times $\mathcal{RT}(X_g) = \{n \in \mathbb{Z}^d : T^{-n}(X_g) \cap X_g \neq \emptyset\}$ is equal to Γ and the topological dynamical system (X_g, Γ) is a minimal system. Note that, since it is a finite partition, the sets X_g are clopen. The minimality of the induced actions implies that there is at most one Γ-minimal partition, up to a permutation of the sets $\{X_g\}_{g \in \mathbb{Z}^d / \Gamma}$.

A Γ-minimal partition is associated with an equicontinuous factor of $(X, T, \mathbb{Z}^d)$ in the following way: We enumerate X_g such that for all $g \in \mathbb{Z}^d / \Gamma$ and $n \in \mathbb{Z}^d$ we have that $T^n X_g = X_{g+n \bmod \Gamma}$. Then, the map $\pi : (X, T, \mathbb{Z}^d) \to (\mathbb{Z}^d / \Gamma, +_\Gamma, \mathbb{Z}^d)$, such that $\pi(x) = g$ if and only if $x \in X_g$, is a factor map onto $(\mathbb{Z}^d / \Gamma, +_\Gamma, \mathbb{Z}^d)$, where $\mathbb{Z}^d$ acts by quotient translations onto $\mathbb{Z}^d / \Gamma$. The following proposition shows the connection between Γ-minimal partitions and eigenvalues of a topological dynamical system.

Proposition 3.20. *Let $(X, T, \mathbb{Z}^d)$ be a minimal topological dynamical system and $\Gamma \leq \mathbb{Z}^d$ be a finite-index subgroup. The system $(X, T, \mathbb{Z}^d)$ admits a Γ-minimal partition, if and only if $\Gamma^* \subseteq E(X, T, \mathbb{Z}^d)$.*

Proof. Let $L \in \mathcal{M}(d, \mathbb{Z})$ be such that $\Gamma = L(\mathbb{Z}^d)$. We recall that $L([0, 1)^d) \cap \mathbb{Z}^d$ is a fundamental domain of $L(\mathbb{Z}^d)$ in $\mathbb{Z}^d$.

Let $\{X_g\}_{g \in L([0,1)^d) \cap \mathbb{Z}^d}$ be a Γ-minimal partition satisfying $X_g = T^g(X_0)$ for all $g \in L([0, 1)^d) \cap \mathbb{Z}^d$. Let x be in $(L^*)^{-1}(\mathbb{Z}^d)$. Define f as the map

$$f = \sum_{g \in L([0,1)^d) \cap \mathbb{Z}^d} e^{2\pi i \langle x, g \rangle} \mathbb{1}_{X_g}.$$

Since the sets $\{X_g\}_{g \in L([0,1)^d) \cap \mathbb{Z}^d}$ are clopen, the map f is continuous. Let $x \in X_0$, $m \in \mathbb{Z}^d$ and $m_1 \in \mathbb{Z}^d$, $m_2 \in L([0, 1)^d) \cap \mathbb{Z}^d$ be such that $m = L(m_1) + m_2$. Note that $T^m x$ is in X_{m_2} and since x is in $(L^*)^{-1}(\mathbb{Z}^d)$, we have that

$$e^{2\pi i \langle x, m \rangle} = e^{2\pi i \langle x, R(m_1) + m_2 \rangle} = e^{2\pi i (\langle x, R(m_1) \rangle + \langle x, m_2 \rangle)} = e^{2\pi i \langle x, m_2 \rangle}. \tag{3}$$

We found a continuous map such that $f(T^m x) = e^{2\pi i \langle x, m \rangle} f(x)$ for all $x \in X$ and $m \in \mathbb{Z}^d$. We conclude that $x \in E(X, T, \mathbb{Z}^d)$.

On the other hand, let y be in X. For $j \in \{1, \ldots, d\}$ we denote $x_j = (L^*)^{-1}(e_j)$. Since $\Gamma^* \subseteq E(X, T, \mathbb{Z}^d)$ there exists a map $f_j : X \to \mathbb{C}$ such that $f_j(T^m y) = e^{2\pi i \langle x_j, m \rangle} f_j(y)$ for all $m \in \mathbb{Z}^d$. Since the eigenspaces are one-dimensional we choose f_j such that $f_j(y) = 1$. By (3), the values of $e^{2\pi i \langle x_j, m \rangle}$ only depend on $m \in L([0, 1)^d) \cap \mathbb{Z}^d$. Now, for any $j \in \{1, \ldots, d\}$ and $m \in L([0, 1)^d) \cap \mathbb{Z}^d$ we denote $X_m^j = f_j^{-1}(\{e^{2\pi i \langle x_j, m \rangle}\})$. By minimality, for each $j \in \{1, \ldots, d\}$, the set X is equal to $\bigcup_{m \in L([0,1)^d) \cap \mathbb{Z}^d} X_m^j$. We define $X_m = \bigcap_{j \in \{1,\ldots,d\}} X_m^j$. We will prove that $\{X_m\}_{m \in L([0,1)^d) \cap \mathbb{Z}^d}$ is a Γ-minimal partition.

Indeed, note that X_m is Γ-invariant. Now, assume that there exist $n_1, n_2 \in L([0, 1)^d) \cap \mathbb{Z}^d$ such that $n_1 \neq n_2$ and $X_{n_1} \cap X_{n_2} \neq \emptyset$. Using the $\mathbb{Z}^d$-action on X, this is equivalent to the existence of $m \in R([0, 1)^d) \cap \mathbb{Z}^d$ with $m \neq 0$ and $X_m \cap X_0 \neq \emptyset$. This means that for all $j \in \{1, \ldots, d\}$, $e^{2\pi i \langle x_j, m \rangle}$ is equal to 1, i.e., $\langle x_j, m \rangle \in \mathbb{Z}$. Since $\Gamma^* = \langle \{x_1, \ldots, x_d\} \rangle$, we have that $m \in (\Gamma^*)^* = \Gamma$ which is a contradiction, so all of these sets are disjoint.

By minimality, we have that $X = \bigcup_{\boldsymbol{m} \in R([0,1)^d) \cap \mathbb{Z}^d} \overline{\mathcal{O}(T^{\boldsymbol{m}} y, \Gamma)}$. Since $\overline{\mathcal{O}(T^{\boldsymbol{m}} y, \Gamma)}$ is included in $X_{\boldsymbol{m}}$, we conclude that $\{X_{\boldsymbol{m}}\}_{\boldsymbol{m} \in R([0,1)^d) \cap \mathbb{Z}^d}$ is a clopen partition of X and $\overline{\mathcal{O}(T^{\boldsymbol{m}} y, \Gamma)} = X_{\boldsymbol{m}}$, so the action of Γ on $X_{\boldsymbol{m}}$ is minimal. A direct computation shows that the set of return times of each $X_{\boldsymbol{m}}$ is Γ. Hence $\{X_{\boldsymbol{m}}\}_{\boldsymbol{m} \in R([0,1)^d) \cap \mathbb{Z}^d}$ is a Γ-minimal partition. ∎

Remark 3.21. The following statements can be easily verified.

(1) In the case of an aperiodic primitive constant-shape substitution ζ, the recognizability property implies that for all $n > 0$, the sets $\{S^{\boldsymbol{j}} \zeta^n(X_\zeta)\}_{\boldsymbol{j} \in F_n^\zeta}$ form a $L_\zeta^n(\mathbb{Z}^d)$-minimal partition.

(2) By Theorem 3.17 and Proposition 3.20 for any aperiodic primitive constant-shape substitution there exists a $\mathcal{H}(X_\zeta)$-minimal partition.

3.6. Aperiodic symbolic factors of substitutive subshifts are conjugate to substitutive subshifts

In the following, we will extend the results of C. Müllner and R. Yassawi in [35], by proving that aperiodic symbolic factors of a substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$ are conjugate to substitutive subshifts where the substitutions have the same expansion matrix and same support as ζ. First we have the following consequence of Proposition 3.7.

Remark 3.22. It is straightforward to check that Proposition 3.7 implies that if $x, x' \in X_\zeta$ are such that $\tau(x) = \tau(x')$, then $\pi(x)$ is equal to $\pi(x')$, where

$$\pi : (X_\zeta, S, \mathbb{Z}^d) \to \left(\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n(\mathbb{Z}^d))}, +_{(L_\zeta^n(\mathbb{Z}^d))}, \mathbb{Z}^d \right)$$

is the factor map.

To prove the result we will introduce some notions as in [35].

Let ζ be an aperiodic primitive constant-shape substitution. We consider a labeled directed graph G_ζ with vertex set $E = \mathcal{A}^2$ and we put an edge (a, b) to (c, d) with label $\boldsymbol{f} \in F_1^\zeta$ if $\zeta(a)_{\boldsymbol{f}} = c, \zeta(b)_{\boldsymbol{f}} = d$. Note that the diagonal $\Delta_{\mathcal{A}} = \{(a, a) : a \in \mathcal{A}\}$ is a stable set, i.e., $E(\Delta_{\mathcal{A}}) = \Delta_{\mathcal{A}}$. Let $P = (a_0, b_0)(a_1, b_1)(a_2, b_2)$ be a path, by definition, there is an edge from (a_0, b_0) to (a_1, b_1) with some label $\boldsymbol{f}_0$ and an edge (a_1, b_1) to (a_2, b_2) with some label $\boldsymbol{f}_1$, i.e., $\zeta(a_0)_{\boldsymbol{f}_0} = a_1, \zeta(b_0)_{\boldsymbol{f}_0} = b_1$, and $\zeta(a_1)_{\boldsymbol{f}_1} = a_2, \zeta(b_1)_{\boldsymbol{f}_1} = b_2$. Then, we have that $\zeta^2(a_0)_{L_\zeta \boldsymbol{f}_0 + \boldsymbol{f}_1} = a_2, \zeta^2(b_0)_{L_\zeta \boldsymbol{f}_0 + \boldsymbol{f}_1} = b_2$. This means that the paths indicate the simultaneous positions of the letters in the iterates of the substitution.

The following definitions were introduced in [35].

Definition 3.23. Let ζ be an aperiodic primitive constant-shape substitution.

(1) We say that a pair $(a, b) \in \mathcal{A}^2 \setminus \Delta_{\mathcal{A}}$ is a *periodic-pair* if there is a cycle in G_ζ which starts and ends in (a, b). We define $n(a, b) = \min\{|P| : P$ is a cycle in $(a, b)\}$ and we denote

$$n(\zeta) = \operatorname{lcm}\{n(a, b) : (a, b) \text{ is a periodic pair}\},$$

we call the substitution *pair-aperiodic* if $n(\zeta) = 1$.

(2) We call a pair $(a, b) \in \mathcal{A} \setminus \Delta_{\mathcal{A}}$ an *asymptotically disjoint pair* if for any $k > 0$, there exists a path $P = (a_0, b_0) \cdots (a_k, b_k)$ in G_ζ of length k with $(a_0, b_0) = (a, b)$ and $(a_k, b_k) \notin \Delta_{\mathcal{A}}$.

Remark 3.24. The following statements can be easily verified:

(1) As for the case of ζ-periodic points for the substitution, we can replace the substitution ζ for an appropriate power, i.e., $\zeta^{n(\zeta)}$, so we may assume that the substitution is pair-aperiodic.

(2) If the substitution ζ is bijective, every $(a, b) \in \mathcal{A}^2 \setminus \Delta_{\mathcal{A}}$ is an asymptotically disjoint pair.

(3) Assume that ζ is pair-aperiodic. If $(a, b) \in \mathcal{A}^2$ is not an asymptotically disjoint pair, let $\overline{k}$ be the minimum length of a path from (a, b) such that any path of length $\overline{k}$ has an end in $\Delta_{\mathcal{A}}$. Since the shortest path has no cycle, we have that $\overline{k} \leq |\mathcal{A}|^2$. Indeed, if $\overline{k} > |A|^2$, there exists a cycle as a subpath in P, i.e., one of the vertex (c, d) is a periodic-pair. Since ζ is pair-aperiodic, there exists $f \in F_1^\zeta$ such that $\zeta(c)_f = c \neq d = \zeta(d)_f$. So we can create a path of length arbitrarily large from (a, b) that does not reach Δ_A, which is a contradiction. This implies that $\zeta^{|\mathcal{A}|^2}(a) = \zeta^{|\mathcal{A}|^2}(b)$.

Definition 3.25. Let $\mathcal{A}, \mathcal{B}$ be two finite alphabets, ζ be an aperiodic primitive constant-shape substitution with alphabet $\mathcal{A}$ and $\tau : \mathcal{A} \to \mathcal{B}$. We say that $a, b \in \mathcal{A}$ with $a \neq b$ are *indistinguishable* (by (ζ, τ)) if for all $n \geq 0$ we have that $\tau(\zeta^n(a)) = \tau(\zeta^n(b))$.

With these definitions we are ready to prove the next result which is a multidimensional analogue of [35, Theorem 22].

Theorem 3.26. *Let $(Y, S, \mathbb{Z}^d)$ be an aperiodic symbolic factor (with alphabet $\mathcal{B}$) of a substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$, with ζ being an aperiodic primitive constant-shape substitution with alphabet $\mathcal{A}$. Then there exists an aperiodic primitive constant-shape substitution ζ' with alphabet $\mathcal{C}$ having the same expansion matrix and support of a power of ζ and a conjugacy $\tau' : (X_{\zeta'}, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ via a letter-to-letter map.*

Proof. By means of Lemma 2.14 and Remark 3.24 we can assume that the factor map $\tau : (X_\zeta, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ is induced via a letter-to-letter map and ζ is a pair-aperiodic substitution.

We define an equivalence relation $a \sim b$ in $\mathcal{A}$, such that $a \sim b$ if a, b are indistinguishable. We define a substitution ζ' on $\mathcal{A}/\sim$ as $\zeta'([a])_f = [\zeta(a)_f]$, for $f \in F_1^\zeta$. Note that the map $\mathcal{T}' : \mathcal{A}/\sim \to \mathcal{B}$, given by $\mathcal{T}'([a]) = \tau(a)$ is well defined. This map satisfies the following property:

$$\text{Every pair in } \mathcal{A}/\sim \text{ is distinguishable.} \qquad (4)$$

It is straightforward to check that primitivity of ζ implies primitivity of ζ'.

We will prove that every periodic pair in $\mathcal{A}/\sim$ by ζ' is the quotient of a periodic pair in $\mathcal{A}$ by ζ. Assume now that $([a], [b])$ is a periodic pair, i.e., there exists a cycle $P_1 = ([a_0], [b_0]) \cdots ([a_k], [b_k])$ in $G_{\zeta'}$ with $([a_0], [b_0]) = ([a_k], [b_k]) = ([a], [b])$. Let $P_2 = (c_0, d_0) \cdots (c_k, d_k)$ be a path in G_ζ with $[c_i] = [a_i]$ and $[d_i] = [b_i]$ for $0 \leq i \leq k$ having the same label of edges as P_1. Now, repeating this process we get a path P_3 in G_ζ of length $(\max\{|[a]|, |[b]|\} + 1)k$ from (c, d) repeating the labels of the path P_1 with a period k. By the Pigeonhole principle, there exist two subpaths $P_4 = (e_0, f_0) \cdots (e_{l_1k}, f_{l_1k})$, $P_5 = (g_0, h_0) \cdots (g_{l_2}k, h_{l_2k})$ of P_3, having the same labels of the edges of P_1 repeating them with period k, such that $e_0 = e_{l_1k}$, $h_0 = h_{l_2k}$ and $[e_0] = [a_0]$, $[h_0] = [b_0]$. Now consider the cycle in G_ζ, $(u_0, v_0) \cdots (u_{l_1l_2k^2}, v_{l_1l_2k^2})$ where $u_{l_1kj} \cdots u_{l_1k(j+1)} = e_0 \cdots e_{l_1k}$ and $v_{l_2km} \cdots v_{l_2k(m+1)} = h_0 \cdots h_{l_2k}$ for all $0 \leq j < l_2k$, $0 \leq m < l_1k$. Since ζ is pair-aperiodic, there exists $f \in F_1^\zeta$ such that $\zeta(e_0)_f = e_0$, $\zeta(h_0)_f = h_0$. We then conclude that $\zeta'([a])_f = [a]$, $\zeta'([b])_f = [b]$, i.e., ζ' is pair-aperiodic.

On the other hand, for all $n > 0$, we have that $\tau(\zeta^n(a)) = \tau'(\zeta'^n([a]))$, hence Y has the same language as $\tau'(X_{\zeta'})$, so they are equal, since subshifts are uniquely determined by their language.

Finally, we prove that $\tau' : (X_{\zeta'}, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ is a conjugacy. Let $x, x' \in X_{\zeta'}$, with $\tau'(x) = \tau'(x')$. By the recognizability property of $X_{\zeta'}$ we can write $x = S^{f_1} \zeta'^{2|\mathcal{A}|^2}(y)$, $x' = S^{f_2} \zeta'^{2|\mathcal{A}|^2}(y')$. By Remark 3.22 we have that $f_1 = f_2$.

If there exists $n \in \mathbb{Z}^d$ such that $(y_n, y'_n) \in \mathcal{A}^2$ is an asymptotically disjoint pair, we can find a periodic pair $([a], [b])$ and a path $P = ([a_0], [b_0]) \cdots ([a_k], [b_k])$ in $G_{\zeta'}$ with $([a_0], [b_0]) = (y_n, y'_n)$ and $([a_k], [b_k]) = ([a], [b])$, with $k \leq |A|^2$. Since ζ' is pair-aperiodic, we have that there exists $f \in F_{2|A|^2}^\zeta$ such that $(\zeta')^{2|\mathcal{A}|^2}(y_n)_f = [a]$, $(\zeta')^{2|\mathcal{A}|^2}(y'_n)_f = [b]$. Note that

$$\tau'(\zeta'^{2|A|^2}(y_n)) = \zeta'^{2|\mathcal{A}|^2}(\tau'(y_n)) = \zeta'^{2|\mathcal{A}|^2}(\tau'(y_n)) = \tau'(\zeta'^{2|A|^2}(y'_n)),$$

so $\tau'(a)$ is equal to $\tau'(b)$. This implies that $([a], [b])$ are indistinguishable, which contradicts (4).

If (y_n, y'_n) is not an asymptotically disjoint pair for any $n \in \mathbb{Z}^d$, by Remark 3.24 we have that $\zeta^{2|\mathcal{A}|^2}(y_n) = \zeta^{2|\mathcal{A}|^2}(y'_n)$, i.e., $x = x'$. We conclude τ' is bijective and then a conjugacy. ∎

4. Measurable morphisms between constant-shape substitutions

In this section, we study different types of homomorphisms between substitutive subshifts. First, we extend a result of B. Host and F. Parreau from [29] to the multidimensional case (Theorem 4.1), which establishes that any measurable factor between two substitutive subshifts given by aperiodic constant-shape substitutions, under some combinatorial property, induces a continuous one. Then, we deduce some consequences: Every substitutive subshift given by an aperiodic constant-shape substitution satisfying the combinatorial

property is coalescent (Proposition 4.8), the quotient group $\mathrm{Aut}(X_\xi, S, \mathbb{Z}^d)/\langle S\rangle$ is finite, generalizing the results in [8] and we give some conditions where the automorphism group of a substitutive subshift is isomorphic to a direct product of $\mathbb{Z}^d$ with a finite group (Corollary 4.10). Finally, we extend Theorem 4.1 to homomorphisms associated with matrices commuting with a power of the expansion matrix of the substitution (Theorem 4.13). This leads to the same rigidity properties about these homomorphisms (Proposition 4.17) and for some restricted normalizer group. Notice that in Section 5 we will give sufficient conditions to ensure that the former result is a complete characterization of the normalizer group.

4.1. Measurable factors imply continuous ones for substitutive subshifts

Let ζ be a constant-shape substitution. Recall that the substitutive subshift $(X_\xi, S, \mathbb{Z}^d)$ is uniquely ergodic and we denote μ_ξ the unique ergodic measure. For any $n > 0$, the image measure of μ_ξ by ζ^n is equal to $|F_n^\xi| \cdot \mu_\xi|_{\zeta^n(X_\xi)}$, and as in the one-dimensional case [37], the unique ergodic measure satisfies

$$\forall U \in \mathscr{F}_{X_\xi}, \quad \mu_\xi(U) = \frac{1}{|F_n^\xi|} \int_{X_\xi} \left|\{f \in F_n^\xi : S^f \zeta^n(x) \in U\}\right| d\mu_\xi(x),$$

where $\mathscr{F}_{X_\xi}$ corresponds to the Borel sets of X_ξ.

As in [29] we use the notion of *reducibility* of a constant-shape substitution. For any pair of distinct letters $a \neq b \in \mathcal{A}$ and $n > 0$ we consider the sequence

$$d_n(\zeta^n(a), \zeta^n(b)) = \frac{|\{f \in F_n^\xi : \zeta^n(a)_f \neq \zeta^n(b)_f\}|}{|F_n^\xi|}.$$

This sequence is decreasing for all of the pairs $a, b \in \mathcal{A}$. We say that the constant-shape substitution is *reduced* if $\min_{n \in \mathbb{N}, a \neq b \in \mathcal{A}} d_n(\zeta^n(a), \zeta^n(b)) > 0$. For instance, every bijective constant-shape substitution is reduced. The following theorem is a multidimensional analogue of [29, Theorem 1.3].

Theorem 4.1. *Let $(X_{\xi_1}, S, \mathbb{Z}^d)$, $(X_{\xi_2}, S, \mathbb{Z}^d)$ be two substitutive subshifts from two aperiodic primitive constant-shape substitutions ζ_1, ζ_2 from finite alphabets $\mathcal{A}$ and $\mathcal{B}$, with the same expansion matrix L and same support F_1. If ζ_2 is reduced, then for every measurable factor $\phi : (X_{\xi_1}, \mu_{\xi_1}, S, \mathbb{Z}^d) \to (X_{\xi_2}, \mu_{\xi_2}, S, \mathbb{Z}^d)$, there exists $j \in \mathbb{Z}^d$ such that $S^j \phi$ is equal μ_{ξ_1}-a.e. to a continuous factor map $\psi : (X_{\xi_1}, S, \mathbb{Z}^d) \to (X_{\xi_2}, S, \mathbb{Z}^d)$, satisfying the following two properties:*

(1) *The factor map ψ is a sliding block code of radius*

$$\|F_1^\xi\|(1 + \|L_\xi^{-1}\|(2 + 1/(1 - \|L_\xi^{-1}\|))).$$

(2) *There exist an integer $n > 0$ and $p \in F_n^\xi$ such that, $S^p \psi \zeta_1^n = \zeta_2^n \psi$.*

Remark 4.2. The following statements can be easily verified.

(1) If L_ζ is a diagonal matrix, then $\|L_\zeta^{-1}\| \leq 1/2$, so the radius of ψ is bounded by $3\|F_1^\zeta\|$.

(2) As the set of sliding block codes of radius $\|F_1^\zeta\|(1 + \|L_\zeta^{-1}\|(2 + 1/(1 - \|L_\zeta^{-1}\|)))$ between X_{ζ_1} and X_{ζ_2} is finite, we may consider an appropriate iteration of ζ_1 and ζ_2 such that any factor ψ satisfying (2) in Theorem 4.1 satisfies $S^p \psi \zeta_1 = \zeta_2 \psi$.

(3) As also mentioned in [29], we may assume that $p \notin (L_\zeta^n - \mathrm{id})(\mathbb{Z}^d)$, because it is equivalent to find a factor map commuting with the substitution map, i.e., $\psi \zeta^n = \zeta^n \psi$, with $p \in F_1^\zeta$.

If a substitution is not reduced, we consider an equivalence relation calling two letters a, b equivalent when $d_n(\zeta^n(a), \zeta^n(b)) \to 0$. If two letters $a, b \in \mathcal{A}$ are equivalent, then $(\zeta(a))_f \sim (\zeta(b))_f$ for all $f \in F_1^\zeta$. We define a substitution $\tilde{\zeta}$ on $\mathcal{A}/\sim$ given by $(\tilde{\zeta}([a]))_f = [\zeta(a)_f]$ for $f \in F_1^\zeta$. We have a natural letter-to-letter factor map $\tilde{\phi}$: $(X_\zeta, S, \mathbb{Z}^d) \to (X_{\tilde{\zeta}}, S, \mathbb{Z}^d)$, and it is called the *reduced substitution* of ζ.

In the one-dimensional case, if $(X_\zeta, \mu_\zeta, S, \mathbb{Z})$ does not have purely discrete spectrum, it can be proved, using the results in [12], that $(X_{\tilde{\zeta}}, S, \mathbb{Z})$ is aperiodic. An ergodic system $(X, \mu, T, \mathbb{Z})$ has *purely discrete spectrum* if the vector space spanned by eigenfunctions of the Koopman operator $U_T(f) = f \circ T \in L^2(X, \mu)$ is dense in $L^2(X, \mu)$. It is well known that any ergodic system with purely discrete spectrum is conjugate to an ergodic rotation on a compact abelian group [27].

In the multidimensional case this is not true in general, as we can see in Example 4.3.

Example 4.3 (An aperiodic constant-shape substitution, with a periodic reduced substitution). Consider the substitution σ_6 with $L_{\sigma_6} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and $F_1^{\sigma_6} = [\![0, 1]\!]^2$, given by

$$\sigma_6 : \qquad 0 \mapsto \begin{matrix} 1 & 3 \\ 0 & 2 \end{matrix}, \qquad 1 \mapsto \begin{matrix} 0 & 2 \\ 0 & 2 \end{matrix}, \qquad 2 \mapsto \begin{matrix} 3 & 1 \\ 2 & 0 \end{matrix}, \qquad 3 \mapsto \begin{matrix} 2 & 0 \\ 2 & 0 \end{matrix}.$$

This substitution corresponds to the product substitution between the Thue–Morse substitution ($\sigma_7 : 0 \mapsto 01, 1 \mapsto 10$) and the doubling sequence substitution ($\sigma_8 : a \mapsto ab$, $b \mapsto aa$). Note that $a \sim b$, since for all $n > 0$, $\sigma_8^n(a)$ and $\sigma_8^n(b)$ differ only in their last letters. Figure 12 is a pattern of σ_6.

The system $(X_{\sigma_7} \times \overleftarrow{\mathbb{Z}}_{(2^n\mathbb{Z})}, S \times +_{(2^n\mathbb{Z})}, \mathbb{Z}^2)$ is a topological factor of $(X_{\sigma_6}, S, \mathbb{Z}^2)$, hence the substitutive subshift $(X_{\sigma_6}, S, \mathbb{Z}^2)$ does not have purely discrete spectrum. The reduced substitution of σ_6 is defined with the same expansion matrix and support, given by

$$\tilde{\sigma}_6 : \qquad a \mapsto \begin{matrix} a & b \\ a & b \end{matrix}, \qquad b \mapsto \begin{matrix} b & a \\ b & a \end{matrix},$$

where every element in $\{0\} \times \mathbb{Z}$ is a nontrivial period of $\tilde{\sigma}_6$.

However, as proved in [29] for the one-dimensional case, if the reduced substitution system is aperiodic, then $(X_{\tilde{\zeta}}, \mu_\zeta, S, \mathbb{Z}^d)$ is metrically isomorphic to $(X_\zeta, \mu_{\tilde{\zeta}}, S, \mathbb{Z}^d)$.

1331311331131331
0220200220020220
0220200220020220
0220200220020220
1331311331131331
0220200220020220

Figure 12. A pattern of the substitution σ_6.

Proposition 4.4. *Let ζ be an aperiodic primitive constant-shape substitution. If the reduced substitution $\tilde{\zeta}$ is aperiodic, then the natural factor map between $(X_\zeta, \mu_\zeta, S, \mathbb{Z}^d)$ and $(X_{\tilde{\zeta}}, \mu_{\tilde{\zeta}}, S, \mathbb{Z}^d)$ is a metric isomorphism.*

Proof. Let $\pi : (X_\zeta, S, \mathbb{Z}^d) \to (\overleftarrow{\mathbb{Z}^d}_{(L^n_\zeta)}, +_{(L^n_\zeta)}, \mathbb{Z}^d)$ as defined in Section 3.4. By Proposition 3.7, for every $n > 0$, $\pi_n(x)$ is equal to $\pi_n(\tilde{\phi}(x))$. In particular, if $x, y \in X_\zeta$ satisfies $\tilde{\phi}(x) = \tilde{\phi}(y)$, then $\pi(x), \pi(y)$ are equal.

Set $U = \{x \in X_\zeta : \exists y \in X_\zeta, \tilde{\phi}(x) = \tilde{\phi}(y), x_{\mathbf{0}} \neq y_{\mathbf{0}}\}$. It is enough to prove that U is a null-set.

Let $n > 0$, $f \in F^\zeta_n$ and $x \in X_\zeta$ be such that $S^f \zeta^n(x) \in U$. Then, there exists $y \in X_\zeta$ with $\tilde{\phi}(y) = \tilde{\phi}(S^f \zeta^n(x))$ and $y_0 \neq \zeta^n(x_0)_f$. Then $\pi_n(y) = \pi_n(S^f \zeta^n(x))$ and is equal to f. Moreover, there exists $z \in X_\zeta$ with $y = S^f \zeta^n(z)$, so $\tilde{\phi}(x)$ is equal to $\tilde{\phi}(z)$. This implies that $(\zeta^n z_{\mathbf{0}})_j, (\zeta^n x_{\mathbf{0}})_j$ are equivalent for all $j \in F^\zeta_n$. Note that $(\zeta^n z_{\mathbf{0}})_f = y_{\mathbf{0}}$, so it is different from $(\zeta^n x_{\mathbf{0}})_f$. We define the set

$$G_n = \bigcup_{a,b \in \mathcal{A}} \{f \in F^\zeta_n : [(\zeta^n a)_f] = [(\zeta^n b)_f], (\zeta^n a)_f \neq (\zeta^n b)_f\}.$$

We deduce from the previous paragraph that

$$\mu_\zeta(U) = \frac{1}{|F^\zeta_n|} \int |\{f \in F^\zeta_n : S^f \zeta^n(x) \in U\}| d\mu(x)$$
$$\leq \frac{|G_n|}{|F^\zeta_n|}.$$

For any $a, b \in \mathcal{A}$ we denote

$$\mathcal{D}^{a,b}_n = \{(c,d) \in \mathcal{A}^2 : \exists f \in F^\zeta_n, (\zeta^n a)_f = c, (\zeta^n b)_f = d, [c] \neq [d]\}$$

and

$$\mathcal{E}^{a,b}_n = \{(c,d) \in \mathcal{A}^2 : \exists f \in F^\zeta_n, (\zeta^n a)_f = c, (\zeta^n b)_f = d, [c] = [d]\}.$$

Set $\varepsilon > 0$ and let $j > 0$ be large enough such that for any $a, b \in \mathcal{A}$

$$d_j(\zeta^j(a), \zeta^j(b)) \leq \lim_{k \to \infty} d_k(\zeta^k(a), \zeta^k(b)) + \varepsilon.$$

Fix $a, b \in \mathcal{A}$. Note that

$$
\begin{aligned}
&d_{n+j}(\zeta^{n+j}(a), \zeta^{n+j}(b)) \\
&= \frac{1}{|F_{n+j}^{\zeta}|} \sum_{(c,d) \in \mathcal{D}_n^{a,b}} |F_j^{\zeta}| d_j(\zeta^j(c), \zeta^j(d)) + \sum_{(c,d) \in \mathcal{E}_n^{a,b}} |F_j^{\zeta}| d_j(\zeta^j(c), \zeta^j(d)) \\
&= \frac{1}{|F_n^{\zeta}|} \left(\sum_{(c,d) \in \mathcal{D}_n^{a,b}} d_j(\zeta^j(c), \zeta^j(d)) + \sum_{(c,d) \in \mathcal{E}_n^{a,b}} d_j(\zeta^j(c), \zeta^j(d)) \right) \\
&\leq \frac{1}{|F_n^{\zeta}|} \left(\sum_{(c,d) \in \mathcal{D}_n^{a,b}} \left(\lim_{k \to \infty} d_k(\zeta^k(c), \zeta^k(d)) + \varepsilon \right) + \sum_{(c,d) \in \mathcal{E}_n^{a,b}} \varepsilon \right) \\
&\leq \frac{\varepsilon(|\mathcal{D}_n^{a,b}| + |\mathcal{E}_n^{a,b}|)}{|F_n^{\zeta}|} + \frac{1}{|F_n^{\zeta}|} \sum_{(c,d) \in \mathcal{D}_n^{a,b}} \lim_{k \to \infty} d_k(\zeta^k(c), \zeta^k(d)).
\end{aligned}
$$

Since this is true for every $\varepsilon > 0$ and $\lim_{k \to \infty} d_k(\zeta^k(c), \zeta^k(d)) \leq 1$, we have that $d_{n+j}(\zeta^{n+j}(a), \zeta^{n+j}(b)) \leq |\mathcal{D}_n^{a,b}| / |F_n^{\zeta}|$ and this is true for every j large enough, so

$$\lim_{k \to \infty} d_k(\zeta^k(a), \zeta^k(b)) \leq \frac{|\mathcal{D}_n^{a,b}|}{|F_n^{\zeta}|},$$

hence

$$\mu_{\zeta}(U) \leq \sum_{a,b \in \mathcal{A}} \left(d_n(\zeta^n(a), \zeta^n(b)) - \lim_{k \to \infty} d_k(\zeta^k(a), \zeta^k(b)) \right).$$

When $n \to \infty$, the right expression goes to zero, and we conclude that $\mu_{\zeta}(U) = 0$. ∎

Now, to prove Theorem 4.1, we assume that ζ_2 is an aperiodic primitive reduced constant-shape substitution. We denote by $\eta = \min_{n \in \mathbb{N}, a \neq b \in \mathcal{B}} d_n(\zeta_2^n(a), \zeta_2^n(b))$ and R the radius from the recognizability property of ζ_2. Let ϕ be in $m \operatorname{Fac}(X_{\xi_1}, X_{\xi_2}, S, \mathbb{Z}^d)$. The map $\pi_n(x) - \pi_n(\phi x) \pmod{L^n(\mathbb{Z}^d)}$ is invariant under the $\mathbb{Z}^d$-action, so it is constant μ_{ξ_1}-a.e.. We denote this constant by $\boldsymbol{p}_n(\phi)$ in $F_n^{\zeta_1}$. The set $S^{\boldsymbol{p}_n(\phi)} \phi \zeta_1^n(X_{\xi_1})$ is included in $\zeta_2^n(X_{\xi_2})$ up to a μ_{ξ_2}-null set. Recall that the recognizability property implies that ζ_1^n is a homeomorphism from X_{ξ_1} to $\zeta_1^n(X_{\xi_1})$, so for μ_{ξ_1}-almost all $x \in X_{\xi_1}$ there exists a unique point $y \in X_{\xi_2}$ such that $S^{\boldsymbol{p}_n(\phi)} \phi \zeta_1^n(x) = \zeta_2^n(y)$, which we denote $\phi_n(x)$. So, for every $\phi \in m \operatorname{Fac}(X_{\xi_1}, X_{\xi_2}, S, \mathbb{Z}^d)$ we consider a sequence $(\boldsymbol{p}_n(\phi))_{n \geq 0}$ and a sequence of maps $(\phi_n)_n \in m \operatorname{Fac}(X_{\xi_1}, X_{\xi_2}, S, \mathbb{Z}^d)$ such that

$$\boldsymbol{p}_n(\phi) \in F_n^{\xi_1}, \quad S^{\boldsymbol{p}_n(\phi)} \phi \zeta_1^n(x) = \zeta_2^n(\phi_n(x)).$$

It is straightforward to check that the sequence $(p_n(\phi))_{n>0}$ satisfies the recurrence

$$p_{n+1}(\phi) = p_n(\phi) + L_{\zeta_1}^n p_1(\phi_n) \quad (\mathrm{mod}\ L_{\zeta_1}^{n+1}(\mathbb{Z}^d)).$$

We also have the recurrence $(\phi_n)_1 = \phi_{n+1}$.

Now, for $\phi, \theta \in m\,\mathrm{Fac}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$, we denote $d(\phi, \theta) = \mu_{\zeta_1}(\{x \in X_{\zeta_1} : (\phi x)_0 \neq (\theta x)_0\})$. We also denote, for any $r > 0$, the quantity $C(r) = |B(0, r) \cap \mathbb{Z}^d|$.

Lemma 4.5. *If $d(\phi, \theta)$ is smaller than $\eta/C(R)$, then ϕ, θ are equal μ_{ζ_1}-a.e. in X_{ζ_1}.*

Proof. For all $n \geq 0$, we denote $U_n = \{x \in X_{\zeta_1} : (\phi_n x)_0 \neq (\theta_n x)_0\}$. We will prove by induction on $n \geq 0$ that $p_n(\phi) = p_n(\theta)$ and $\mu_{\zeta_1}(U_n) < 1/C(R)$. By hypothesis, this is true for $n = 0$. Now, suppose that $p_n(\phi) = p_n(\theta)$ and $\mu_{\zeta_1}(U_n) < 1/C(R)$ for some $n \geq 0$. We note that the map $\pi_1(\phi_n x) - \pi_1(\theta_n x)$ is equal to $(p_1(\theta_n) - p_1(\phi_n)) \,(\mathrm{mod}\ L_{\zeta_1}^n(\mathbb{Z}^d))$ for μ_{ζ_1}-a.e x in X_{ζ_1}. By the recognizability property, this map vanishes on the set $\{x \in X_{\zeta_1} : (\phi_n x)|_{B(0,R)} = (\theta_n x)|_{B(0,R)}\}$, which has a positive measure by hypothesis. We conclude that $p_{n+1}(\phi) = p_{n+1}(\theta)$.

Now, let x be in U_{n+1}. Then, there exist at least $\eta|F_{n+1}^{\zeta_1}|$ indices $f \in F_{n+1}^{\zeta_1}$ such that $(\zeta_2^{n+1} \phi_{n+1}(x))_f \neq (\zeta_2^{n+1} \theta_{n+1}(x))_f$, i.e.,

$$(S^{p_{n+1}(\phi)} \phi \zeta_1^{n+1}(x))_f \neq (S^{p_{n+1}(\theta)} \theta \zeta_1^{n+1}(x))_f,$$

so we have that

$$\eta|F_{n+1}^{\zeta_1}|\mu_{\zeta_1}(U_{n+1}) \leq \int |\{g \in F_{n+1}^{\zeta_1} + p_{n+1}(\phi) : S^g \zeta_1^{n+1} x \in U_0\}| d\mu_{\zeta_1}(x)$$

$$= |F_{n+1}^{\zeta_1}|\mu_{\zeta_1}(U_0).$$

Hence, $\mu_{\zeta_1}(U_{n+1})$ is less than $1/C(R)$.

Now, we will prove that $\phi = \theta$ for μ_{ζ_1}-a.e. $x \in X_{\zeta_1}$. Let $r > 0$ be an integer. Since $(F_n^{\zeta_1})_{n>0}$ is Følner, we choose $n > 0$ large enough such that $|(F_n^{\zeta_1})^{\circ r}|/|\{F_n^{\zeta_1}\}| \geq 1/2$. Set $p = p_n(\phi)$. If $x \in U_0^c$, then $(S^p \phi \zeta_1^n x)|_{F_n^{\zeta_1}} = (S^p \theta \zeta_1^n x)|_{F_n^{\zeta_1}}$, so we have that

$$(S^{p+f} \phi \zeta_1^n x)_{B(0,r) \cap \mathbb{Z}^d} = (S^{p+f} \theta \zeta_1^n x)_{B(0,r) \cap \mathbb{Z}^d}$$

for $f \in (F_n^{\zeta_1})^{\circ r}$. This implies that

$$\mu_{\zeta_1}\big(\{x \in X : (\phi x)|_{B(0,r)} = (\theta x)_{B(0,r)}\}\big) \geq \frac{|(F_n^{\zeta_1})^{\circ r}|}{|F_n^{\zeta_1}|} \mu_{\zeta_1}(U_n^c)$$

$$\geq \frac{1}{2}\Big(1 - \frac{1}{C(R)}\Big) > 0.$$

Finally, the set $\{x \in X : \phi(x) = \theta(x)\}$ is the decreasing intersection of these sets, so it has a positive measure. By ergodicity, ϕ, θ are equal μ_{ζ_1}-a.e. in X_{ζ_1}. ∎

Lemma 4.6. *Let ϕ be in $m \operatorname{Fac}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$. Then there exists a sequence (ψ_n) of sliding block codes of radius $\|F_1^{\zeta_1}\|(1 + \|L_{\zeta_1}^{-1}\|(2 + 1/(1 - \|L_{\zeta_1}^{-1}\|)))$ such that $d(\phi_n, \psi_n) \to 0$.*

Proof. Set $\varepsilon > 0$. By Lusin's theorem, there exist an integer $\ell > 0$ and a continuous map $f : X_{\zeta_1} \to \mathcal{B}$ such that $f(x)$ only depends on $x|_{B(\mathbf{0}, \ell)}$, and the measure of the set $V = \{x \in X_{\zeta_1} : (\phi x)_\mathbf{0} \neq f(x)\}$ is less than $\varepsilon \eta / 6$. Since $(F_n^{\zeta_1})_{n>0}$ is Følner, we choose $n > 0$ large enough such that

$$\frac{|(F_n^{\zeta_1})^{\circ \ell}|}{|F_n^{\zeta_1}|} > 1 - \frac{\eta}{3}.$$

Set $\boldsymbol{p} = \boldsymbol{p}_n(\phi)$. For every $x \in X_{\zeta_1}$, we denote $J(x) = \{\boldsymbol{f} \in (F_n^{\zeta_1})^{\circ \ell} : S^{\boldsymbol{f} + \boldsymbol{p}} \zeta_1^n x \notin V\}$ and $W = \{x \in X_{\zeta_1} : |J(x)| > (1 - \eta/2)|F_n^{\zeta_1}|\}$. Note that

$$\frac{\varepsilon \eta}{6} > \mu_{\zeta_1}(V)$$

$$\geq \frac{1}{|F_n^{\zeta_1}|} \int_{W^c} (|(F_n^{\zeta_1})^{\circ \ell}| - |J(x)|) d\mu(x)$$

$$\geq (1 - \mu_{\zeta_1}(W))\frac{\eta}{6}.$$

So we have that $\mu_{\zeta_1}(W) > 1 - \varepsilon$.

Using Proposition 2.12 with $F = F_1^{\zeta} + F_1^{\zeta}$ and $A = \{\mathbf{0}\}$, we find a set $C \Subset \mathbb{Z}^d$ such that $\mathbf{0} \in C$, $\|C\| \leq \|F_1^{\zeta}\|(1 + \|L_\zeta^{-1}\|(2 + 1/(1 - \|L_\zeta^{-1}\|)))$ and by Remark 2.13 we have that $F_n^{\zeta} + F_n^{\zeta} \subseteq L_\zeta^n(C) + F_n^{\zeta}$. If $x, y \in W$ with $x|_C = y|_C$, then for every $\boldsymbol{f} \in (F_n^{\zeta_1})^{\circ \ell}$, we have that $(S^{\boldsymbol{f}+\boldsymbol{p}}\zeta_1^n x)|_{B(\mathbf{0},\ell) \cap \mathbb{Z}^d} = (S^{\boldsymbol{f}+\boldsymbol{p}}\zeta_1^n y)|_{B(\mathbf{0},\ell) \cap \mathbb{Z}^d}$, and so $f(S^{\boldsymbol{f}+\boldsymbol{p}}\zeta_1^n x) = f(S^{\boldsymbol{f}+\boldsymbol{p}}\zeta_1^n y)$. Moreover, we note that for $\boldsymbol{f}$ in $J(x) \cap J(y)$ that,

$$(\zeta_2^n \phi_n x)_{\boldsymbol{f}} = (S^{\boldsymbol{p}}\phi\zeta_1^n x)_{\boldsymbol{f}} = f(S^{\boldsymbol{f}+\boldsymbol{p}}\phi\zeta_1^n x) = f(S^{\boldsymbol{f}+\boldsymbol{p}}\phi\zeta_1^n y) = (\zeta_2^n \phi_n y)_{\boldsymbol{f}}.$$

Since $x, y \in W$, there are strictly more than $(1 - \eta)|F_n^{\zeta_1}|$ elements in $J(x) \cap J(y)$, so $(\phi_n x)_\mathbf{0}$ is equal to $(\phi_n y)_\mathbf{0}$ by definition of η. Hence, for every x in W, $(\phi_n x)_\mathbf{0}$ only depends on $x|_C$. ∎

Finally, to prove Theorem 4.1 we use similar arguments to those given in [29] that we describe for completeness.

Proof of Theorem 4.1. For fixed alphabets $\mathcal{A}$ and $\mathcal{B}$, there exist a finite number of sliding block codes of radius $\|F_1^{\zeta}\|(1 + \|L_\zeta^{-1}\|(2 + 1/(1 - \|L_\zeta^{-1}\|)))$. By Lemma 4.6 there exist two different integers $m, k \geq 0$ such that $d(\phi_m, \phi_{m+k}) < \eta/C(R)$, so by Lemma 4.5 we have that $\phi_m = \phi_{m+k}$, μ_{ζ_1}-a.e..

Let $n \geq m$ be a multiple of k. Note that

$$(\phi_n)_k = \phi_{n+k} = (\phi_{m+k})_{n-m} = (\phi_m)_{n-m} = \phi_n,$$

μ_{ξ_1}-a.e.. This implies that for all $r \in \mathbb{N}$, ϕ_n is equal to $(\phi_n)_{rk}$, μ_{ξ_1}-a.e. and then ϕ_n is equal to a sliding block code of radius $\|F_1^\zeta\|(1 + \|L_\zeta^{-1}\|(2 + 1/(1 - \|L_\zeta^{-1}\|)))$, μ_{ξ_1}-a.e. in X_{ξ_1}. Since ϕ_n is equal to ϕ_{2n}, μ_{ξ_1}-a.e., we denote $\psi = \phi_n$ and $p = p_n(\psi)$. By definition of p we have that $S^p \psi \zeta_1^n = \zeta_2^n \psi$.

Set $j = p_n(\phi) - p$, then

$$S^j \phi \zeta_1^n = S^{p_n(\phi) - p} \phi \zeta_1^n = S^{-p} \zeta_2^n \psi = \psi \zeta_1^n, \quad \mu_{\xi_1}\text{-a.e.,}$$

this implies that $S^j \phi$ and ψ coincides in $\zeta_1^n(X_{\xi_1})$, μ_{ξ_1}-a.e., and by ergodicity in the whole set X_{ξ_1}, μ_{ξ_1}-a.e.. ∎

In [15] it was proved that linearly recurrent subshifts (in particular substitutive subshifts) have a finite number of topological Cantor factors, up to conjugacy. In our context, Theorem 3.26 together with Theorem 4.1 implies that a substitutive subshift has finitely many reduced substitutive factors, up to conjugacy. However, the reduced hypothesis does not cover all the aperiodic symbolic factors a substitutive subshifts may have, and this property is not invariant by conjugation, leaving the following question.

Question 4.7. *Does all aperiodic substitutive subshifts have finitely many aperiodic symbolic factors, up to conjugacy?*

4.2. Applications of rigidity results on factors

As applications of Theorem 4.1, we get some results on the coalescence and the automorphism group of substitutive subshifts. Since the set of sliding block codes $\|F_1^\zeta\|(1 + \|L_\zeta^{-1}\|(2 + 1/(1 - \|L_\zeta^{-1}\|)))$ is finite, we will assume here (up to considering a power of ζ) that if a factor map $\psi \in \mathrm{End}(X_\zeta, S, \mathbb{Z}^d)$ satisfies property (2) in Theorem 4.1, then it does so for $n = 1$, i.e., there exists $p \in F_1^\zeta$ such that $S^p \psi \zeta = \zeta \psi$.

4.2.1. Coalescence of substitutive subshifts.
In [15] it was proved that one-dimensional linearly recurrent subshifts (in particular substitutive subshifts) are coalescent, i.e., any endomorphism is an automorphism. Here we use Theorem 4.1 to obtain that substitutive subshifts are also coalescent, for aperiodic primitive reduced constant-shape substitutions.

Proposition 4.8. *Let ζ be an aperiodic primitive reduced constant-shape substitution. Then the substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$ is coalescent.*

Proof. Set $\phi \in \mathrm{End}(X_\zeta, S, \mathbb{Z}^d)$. Theorem 4.1 ensures that there exists $j \in \mathbb{Z}^d$ such that $S^j \phi$ is equal to a sliding block code ψ of a fixed radius satisfying $S^p \psi \zeta = \zeta \psi$, for some $p \in F_1^\zeta$. Let $\overline{x} \in X_\zeta$ be in a ζ-invariant orbit, i.e., there exists $j \in \mathbb{Z}^d$ such that $\zeta(\overline{x}) = S^j \overline{x}$. Note that

$$S^p \psi \zeta(\overline{x}) = S^{p+j} \psi \overline{x} = \zeta \psi(\overline{x}),$$

so, if the orbit of x is in a ζ-invariant orbit, then $\psi(x)$ is also in a ζ-invariant orbit. By Proposition 3.9, there exist finitely many ζ-invariant orbits, hence for n large enough, we

can find $x \in X_\zeta$ with x and $\psi^n(x)$ being in the same orbit, i.e., there exists $\boldsymbol{m} \in \mathbb{Z}^d$ such that $S^{\boldsymbol{m}} \psi^n(x) = x$. The minimality of $(X_\zeta, S, \mathbb{Z}^d)$ allows to conclude that $\psi^n = S^{-\boldsymbol{m}}$. Hence ψ is invertible, which implies that ϕ is invertible. $\blacksquare$

4.2.2. The automorphism group of substitutive subshifts.
Because the set of sliding block codes of radius $\|F_1^\zeta\|(1 + \|L_\zeta^{-1}\|(2 + 1/(1 - \|L_\zeta^{-1}\|)))$ between X_ζ to itself is finite, we get the following result as a direct corollary of Theorem 4.1.

Proposition 4.9. *Let $(X_\zeta, S, \mathbb{Z}^d)$ be a substitutive subshift from an aperiodic primitive reduced constant-shape substitution ζ. Then, the quotient group $\mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle$ is finite. A bound for $|\mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle|$ is given by an explicit formula depending only on d, $|\mathcal{A}|$, $\|L_\zeta^{-1}\|$, $\|F_1^\zeta\|$.*

In the special case, where any automorphism of $(X_\zeta, S, \mathbb{Z}^d)$ satisfies property (2) of Theorem 4.1 with $\boldsymbol{p} = \boldsymbol{0}$, i.e., commutes with the substitution map (like in bijective substitutions as we will prove in Section 5), we have a more rigid result.

Corollary 4.10. *Let ζ be an aperiodic primitive reduced substitution. If any automorphism $\psi \in \mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d)$ satisfying property (2) in Theorem 4.1 commutes with the substitution map, i.e., $\psi\zeta = \zeta\psi$, then the automorphism group is isomorphic to a direct product of $\mathbb{Z}^d$ (generated by the shift action) with a finite group.*

Proof. Note that an automorphism ϕ commutes with the substitution map if and only if ϕ is equal to ϕ_n, for all $n > 0$.

Now, by Lemma 4.6, that property (2) implies property (1) of Theorem 4.1. Therefore, the group of automorphisms commuting with the substitution map is finite. To conclude we just need to observe that the pair $(\boldsymbol{j}_\phi, \psi_\phi)$ in Theorem 4.1 associated with any automorphism ϕ is unique. Indeed, set $\phi \in \mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d)$ and $\boldsymbol{j}_1, \boldsymbol{j}_2 \in \mathbb{Z}^d$, $\psi_1, \psi_2 \in \mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d)$ commuting with the substitution map such that $S^{\boldsymbol{j}_i}\phi = \psi_i$, for $i \in \{1, 2\}$. Then $S^{\boldsymbol{j}_2 - \boldsymbol{j}_1}\psi_1$ is equal to ψ_2. Hence, for any $n > 0$

$$S^{\boldsymbol{j}_2 - \boldsymbol{j}_1}\psi_1\zeta^n = \zeta^n(S^{\boldsymbol{j}_2 - \boldsymbol{j}_1}\psi_1)$$
$$= S^{L_\zeta^n(\boldsymbol{j}_2 - \boldsymbol{j}_1)}\zeta^n\psi_1,$$

which implies that $(\mathrm{id} - L_\zeta^n)(\boldsymbol{j}_2 - \boldsymbol{j}_1) = 0$, so $\boldsymbol{j}_2 = \boldsymbol{j}_1$ and then $\psi_1 = \psi_2$. $\blacksquare$

4.3. Rigidity properties for homomorphisms between substitutive subshifts and applications

This subsection is devoted to homomorphisms between substitutive subshifts. We recall that for $M \in \mathrm{GL}(d, \mathbb{Z})$, a map $\phi : (X, T, \mathbb{Z}^d) \to (Y, T, \mathbb{Z}^d)$ between two topological dynamical systems is said to be a homomorphism associated with M if for all $\boldsymbol{m} \in \mathbb{Z}^d$ we have that $\phi \circ S^{\boldsymbol{m}} = S^{M\boldsymbol{m}} \circ \phi$. We can also define measurable homomorphisms in the measure-theoretic setting. First, we establish a necessary condition for the matrices M

with $m \operatorname{Hom}_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$ being nonempty whenever ζ_1, ζ_2 are two aperiodic primitive constant-shape substitutions with the same expansion matrix and support. Then we prove an analogue of Theorem 4.1 (Theorem 4.13) establishing that measurable homomorphisms induce continuous ones when the matrix M commutes with some power of the expansion matrix. Finally, we give an explicit bound on the norm of these matrices for the quotient group of a restricted normalizer semigroup with respect to the shift action (Proposition 4.17).

Lemma 4.11. *Let ζ_1, ζ_2 be two aperiodic primitive constant-shape substitutions having the same expansion matrix L and same support F. If $M \in \mathrm{GL}(d, \mathbb{Z})$ is such that $m \operatorname{Hom}_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d) \neq \emptyset$, then for all $n > 0$ there exists $m(n) > 0$ such that*

$$ML^{m(n)}(\mathcal{H}(X_{\zeta_1})) \leq L^n(\mathcal{H}(X_{\zeta_2})) \qquad \text{(Normalizer condition)}.$$

Proof. Let ϕ be in $m \operatorname{Hom}_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$, and $\boldsymbol{x} \in E(X_{\zeta_2}, \mu_{\zeta_2}, S, \mathbb{Z}^d)$. We will prove that $M^* \boldsymbol{x} \in E(X_{\zeta_1}, \mu_{\zeta_1}, S, \mathbb{Z}^d)$. Indeed, let $f \in L^2(X_{\zeta_2}, \mu_{\zeta_2})$ be such that for all $\boldsymbol{m} \in \mathbb{Z}^d$, $f \circ S^{\boldsymbol{m}} = e^{2\pi i \langle \boldsymbol{x}, \boldsymbol{m} \rangle} \cdot f$, μ_{ζ_2}-a.e. in X_{ζ_2}. Then, we have that

$$(f \circ \phi) \circ S^{\boldsymbol{m}} = (f \circ S^{M\boldsymbol{m}}) \circ \phi = e^{2\pi i \langle \boldsymbol{x}, M\boldsymbol{m} \rangle} \cdot f \circ \phi$$
$$= e^{2\pi i \langle M^* \boldsymbol{x}, \boldsymbol{m} \rangle} \cdot f \circ \phi, \qquad \mu_{\zeta_1}\text{-a.e. in } X_{\zeta_1}.$$

By Theorem 3.17 and Proposition 3.20 the system $(\underleftarrow{\mathbb{Z}^d}/M^{-1}L^n(\mathcal{H}(X_{\zeta_2})), +, \mathbb{Z}^d)$, for any $n > 0$, is a finite factor of the odometer system $(\underleftarrow{\mathbb{Z}^d}_{(L^n(\mathcal{H}(X_{\zeta_1})))}, +_{(L^n(\mathcal{H}(X_{\zeta_1})))}, \mathbb{Z}^d)$, which implies that the odometer system $(\underleftarrow{\mathbb{Z}^d}_{(M^{-1}L^n(\mathcal{H}(X_{\zeta_2})))}, +_{(M^{-1}L^n(\mathcal{H}(X_{\zeta_1})))}, \mathbb{Z}^d)$ is a factor of $(\underleftarrow{\mathbb{Z}^d}_{(L^n(\mathcal{H}(X_{\zeta_1})))}, +_{(L^n(\mathcal{H}(X_{\zeta_1})))}, \mathbb{Z}^d)$. By Lemma 2.5, we conclude that for any $n > 0$, there exists $m(n) > 0$ such that $L^{m(n)}(\mathcal{H}(X_{\zeta_1})) \leq M^{-1}L^n(\mathcal{H}(X_{\zeta_2}))$. ∎

A consequence of Proposition 4.8 is that if a homomorphism is associated with a matrix with finite order, then it is an isomorphism.

Lemma 4.12. *Let ζ be an aperiodic primitive reduced constant-shape substitution. If $M \in \mathrm{GL}(d, \mathbb{Z})$ has finite order, then any homomorphism $\phi \in N_M(X_{\zeta}, S, \mathbb{Z}^d)$ is invertible.*

Proof. Since M has finite order, there exists $n > 0$ such that $M^n = \mathrm{id}_{\mathbb{R}^d}$. This implies that $\phi^n \in \operatorname{End}(X_{\zeta}, S, \mathbb{Z}^d)$. By Proposition 4.8, ϕ^n is invertible, so ϕ is also invertible. ∎

Now, we will prove an analogue of Theorem 4.1 for homomorphisms associated with matrices commuting with a power of the expansion matrix L. As mentioned before, a priori this does not cover all the homomorphisms between substitutive subshifts.

Theorem 4.13. *Let $(X_{\zeta_1}, S, \mathbb{Z}^d)$, $(X_{\zeta_2}, S, \mathbb{Z}^d)$ be two substitutive subshifts from two aperiodic primitive constant-shape substitutions ζ_1, ζ_2 from finite alphabets $\mathcal{A}$ and $\mathcal{B}$, with the same support F_1 and expansion matrix L. Let $M \in \mathrm{GL}(d, \mathbb{Z})$ be a matrix commuting with a power of L, i.e., there exists $n > 0$ such that $ML^n = L^n M$. If ζ_2 is reduced, then for every measurable homomorphism ϕ, associated with M, there exists $\boldsymbol{j} \in \mathbb{Z}^d$ such that*

$S^{\boldsymbol{j}}\phi$ is equal μ_{ξ_1}-a.e. to a continuous homomorphism ψ, associated with M, satisfying the following two properties:

(1) The homomorphism ψ is given by a block map of radius

$$\|F_1^{\zeta}\|\|L_{\xi_1}^{-1}\|(1+\|M\|)(2-\|L_{\xi}^{-1}\|)/(1-\|L_{\xi}^{-1}\|).$$

(2) There exist an integer $n > 0$ and $\boldsymbol{q} \in F_n^{\zeta}$ such that $S^{\boldsymbol{q}}\psi\zeta_1^n = \zeta_2^n\psi$.

Remark 4.14. Let $\psi \in \mathrm{Hom}_M(X_{\xi_1}, X_{\xi_2}, S, \mathbb{Z}^d)$ satisfying property (2) of Theorem 4.13. For any $\boldsymbol{m}$ in $\mathbb{Z}^d$, we have that $S^{\boldsymbol{q}}\psi(\zeta_1^n(S^{\boldsymbol{m}}x)) = \zeta_2^n(\psi(S^{\boldsymbol{m}}x))$, and

$$S^{\boldsymbol{q}}\psi(\zeta_1^n(S^{\boldsymbol{m}}x)) = S^{\boldsymbol{q}+ML_{\xi_1}^n\boldsymbol{m}}\psi(\zeta_1^n(x)), \quad \zeta_2^n(\psi(S^{\boldsymbol{m}}x)) = S^{L_{\xi_1}^n M\boldsymbol{m}}\zeta_2^n(\psi(x)),$$

it follows that $ML_{\xi_1}^n\boldsymbol{m} = L_{\xi_1}^n M\boldsymbol{m}$, i.e., M and $L_{\xi_1}^n$ commute. Hence this hypothesis is optimal to obtain property (2). Note that if L is an integer multiple of the identity, then any matrix $M \in \mathrm{GL}(d, \mathbb{Z})$ commutes with L.

The proof of Theorem 4.13 follows the same strategy as the one of Theorem 4.1, except for some small modifications. Since the substitution ζ_1 is primitive, we can replace it by some power ζ_1^n, so we may assume that M commutes with the expansion matrix of ζ_1. We replace the term $\boldsymbol{p}_n(\phi)$ by the map $\pi_n(x) - M^{-1}\pi_n(\phi x) \pmod{L_{\zeta}^n(\mathbb{Z}^d)}$, with $\pi_n(x)$ and $M^{-1}\pi_n(\phi x)$ being the representative classes in F_n^{ζ}. The commutation assumption implies that, for any $n > 0$ the map M defines a bijection in $\mathbb{Z}^d/L^n(\mathbb{Z}^d)$, also denoted by M, i.e., $\boldsymbol{n} = \boldsymbol{m} \pmod{L^n(\mathbb{Z}^d)}$, if and only if $M\boldsymbol{n} = M\boldsymbol{m} \pmod{L^n(\mathbb{Z}^d)}$. With this, the map $\boldsymbol{p}_n(\phi)$ is invariant under the shift action. Since $(X_{\xi_1}, \mu_{\xi_1}, S, \mathbb{Z}^d)$ is ergodic, the map $\boldsymbol{p}_n(\phi) \in F_n^{\zeta}$ is a constant map μ_{ξ_1}-a.e. in X_{ξ_1} and the set $S^{M\boldsymbol{p}_n(\phi)}\phi\zeta_1^n(X_{\xi_1})$ is included, up to a μ_{ξ_2}-null set, in $\zeta_2^n(X_{\xi_2})$. We can define the map ϕ_n for μ_{ξ_1}-a.e. in X_{ξ_1} as the unique point $y \in X_{\xi_2}$ such that $S^{M\boldsymbol{p}_n(\phi)}\phi\zeta_1^n(x) = \zeta_2^n(y)$, where $M\boldsymbol{p}_n(\phi)$ is the representative element in F_n^{ζ}. It is straightforward to check that $\phi_n \circ S^{\boldsymbol{n}} = S^{M\boldsymbol{n}} \circ \phi_n$ for all $\boldsymbol{n} \in \mathbb{Z}^d$, so ϕ_n is in $m\,\mathrm{Hom}_M(X_{\xi_1}, X_{\xi_2}, S, \mathbb{Z}^d)$. The sequences $\boldsymbol{p}_n(\phi)$ and (ϕ_n) satisfy the same recurrences given in Section 4.1,

$$\boldsymbol{p}_{n+1}(\phi) = \boldsymbol{p}_n(\phi) + L_{\zeta}^n\boldsymbol{p}_1(\phi_n), \quad (\phi_n)_1 = \phi_{n+1}.$$

As in Theorem 4.1 we need the following adaptations of Lemma 4.5 and Lemma 4.6 for homomorphisms. The proof are the same, so we omit them.

Lemma 4.15. If $\phi, \psi \in mN_M(X_{\xi_1}, X_{\xi_2}, S, \mathbb{Z}^d)$ are such that $d(\phi, \psi)$ is smaller than $\eta/C(R)$, then ϕ, ψ are equal μ_{ξ_1}-a.e in X_{ξ_1}.

Lemma 4.16. Let $\phi \in mN_M(X_{\xi_1}, X_{\xi_2}, S, \mathbb{Z}^d)$. Then there exists a sequence (ψ_n) of homomorphisms associated with M of radius $\|F_1^{\zeta}\|\|L_{\xi_1}^{-1}\|(1+\|M\|)(2-\|L_{\zeta}^{-1}\|)/(1-\|L_{\zeta}^{-1}\|)$ such that $d(\phi_n, \psi_n) \to 0$.

To finish the proof of Theorem 4.13, we proceed exactly as in the proof of Theorem 4.1

Proof of Theorem 4.13. For fixed alphabets $\mathcal{A}$ and $\mathcal{B}$, there exists a finite number of homomorphisms associated with M of radius $\|F_1^\zeta\|\|L_{\zeta_1}^{-1}\|(1+\|M\|)(2-\|L_\zeta^{-1}\|)/(1-\|L_\zeta^{-1}\|)$. By Lemma 4.16 there exist two different integers $m, k \geq 0$ such that $d(\phi_m, \phi_{m+k}) < \eta/C(R)$, so by Lemma 4.15 we have that $\phi_m = \phi_{m+k}$, μ_{ζ_1}-a.e..

Let $n \geq m$ be a multiple of k. We have $(\phi_n)_k = \phi_{n+k} = (\phi_{m+k})_{n-m} = (\phi_m)_{n-m} = \phi_n$, μ_{ζ_1}-a.e.. This implies for all $r \in \mathbb{N}$, ϕ_n is equal to $(\phi_n)_{rk}$, μ_{ζ_1}-a.e., and then ϕ_n is equal to a homomorphism associated with M of radius $\|F_1^\zeta\|\|L_{\zeta_1}^{-1}\|(1+\|M\|)(2-\|L_\zeta^{-1}\|)/(1-\|L_\zeta^{-1}\|)$, μ_{ζ_1}-a.e. in X_{ζ_1}. Since ϕ_n is equal to ϕ_{2n} μ_{ζ_1}-a.e., we denote $\psi = \phi_n$ and $\boldsymbol{p} = \boldsymbol{p}_n(\psi)$. By definition of $\boldsymbol{p}$, we have $S^{M\boldsymbol{p}}\psi\zeta_1^n = \zeta_2^n\psi$.

Set $\boldsymbol{j} = M(\boldsymbol{p}_n(\phi) - \boldsymbol{p})$, then

$$S^{\boldsymbol{j}}\phi\zeta_1^n = S^{M(\boldsymbol{p}_n(\phi)-\boldsymbol{p})}\phi\zeta_1^n = S^{-M\boldsymbol{p}}\zeta_2^n\psi = \psi\zeta_1^n, \quad \mu_{\zeta_1} - \text{a.e},$$

this implies that $S^{\boldsymbol{j}}\phi$ and ψ coincides in $\zeta_1^n(X_{\zeta_1})$ μ_{ζ_1}-a.e.. Further, the ergodicity of $(X_{\zeta_1}, \mu_{\zeta_1}, S, \mathbb{Z}^d)$ lets us conclude that ϕ and ψ are equal μ_{ζ_1}-a.e.. ∎

In the case $\zeta_1 = \zeta_2$, we can consider a restricted normalizer group: the group of isomorphisms associated with matrices commuting with some power of the expansion matrix L_{ζ_1},

$$NC(X_{\zeta_1}, S, \mathbb{Z}^d) = \bigcup_{\substack{M \in \mathrm{GL}(d,\mathbb{Z}), \\ ML_{\zeta_1}^n = L_{\zeta_1}^n M, \text{ for some } n}} (N_M(X, T, \mathbb{Z}^d) \cap \mathrm{Homeo}(X)).$$

This set is a group under composition and $\langle S \rangle$, $\mathrm{Aut}(X_{\zeta_1}, S, \mathbb{Z}^d)$ are normal subgroups of $NC(X_{\zeta_1}, S, \mathbb{Z}^d)$. We obtain a similar result on this restricted normalizer group as for the automorphism group (Proposition 4.9).

Proposition 4.17. *Let $(X_\zeta, S, \mathbb{Z}^d)$ be a subshift from a reduced aperiodic primitive constant-shape substitution ζ from a finite alphabet. If the set of matrices $M \in \vec{N}(X_\zeta, S, \mathbb{Z}^d)$ commuting with a power of the expansion matrix L_ζ is finite, then the quotient group $NC(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle$ is finite. A bound for $|NC(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle|$ is given by an explicit formula depending only on d, $|\mathcal{A}|$, $\|L_\zeta^{-1}\|$, $\|F_1^\zeta\|$, and $\sup_{N_M(X_\zeta, S, \mathbb{Z}^d) \neq \emptyset} \|M\|$.*

Proof. Let $\psi \in NC(X_\zeta, S, \mathbb{Z}^d)$, satisfying property (2) of Theorem 4.13. Following the proof of Proposition 4.8, ψ acts as a permutation of the ζ-invariant orbits. Since the set of matrices $M \in \vec{N}(X_\zeta, S, \mathbb{Z}^d)$ commuting with a power of L_ζ is finite, there exists $n > 0$ such that ψ^n is an automorphism of X_ζ. By Proposition 4.9 we have that ψ^n has finite order, which implies that ψ has finite order. The bound for $|NC(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle|$ is given by Theorem 4.13. ∎

Remark 4.18. Note that, if L_ζ is a diagonal matrix and all of its entries are different, the set of matrices $M \in \mathrm{GL}(d, \mathbb{Z})$ commuting with a power of L_ζ is finite.

5. Precisions on bijective constant-shape substitutions

Bijective substitutions are of great interest because of their mixed dynamic spectrum. They are never almost 1-to-1 extensions of their maximal equicontinuous factor. Bijective substitutions were studied before in [21] for block substitutions, where it was proved that the substitutive subshift is measurable-theoretic isomorphic to a skew product of one-dimensional odometers. Also, [8] studied the normalizer group of bijective block substitutions. We extend the study by describing the normalizer group for general constant-shape substitutions. To do this, we relate the linear representation group with different types of supports of the substitution and nondiagonal expansion matrices.

We start with a characterization of the automorphism group of substitutive subshifts from aperiodic bijective primitive constant-shape substitutions. Then we describe the non-deterministic directions of a substitutive subshift, by the supporting hyperplanes to $\mathrm{conv}(F_n^\zeta)$ (Theorem 5.2). We further emphasize when the convex hull of the digit tile $\mathrm{conv}(T_\zeta)$ is a polytope, because this implies strong geometrical restrictions on the supporting hyperplanes. Conversely, we provide a checkable combinatorial condition to ensure a vector to be nondeterministic (Corollary 5.13). Finally, we deduce dynamical consequences for $(X_\zeta, S, \mathbb{Z}^d)$ on the normalizer group. For instance, the normalizer group is virtually generated by the shift action (Theorem 5.17) and we provide restrictions on the linear representation group (Proposition 5.15). It follows that all the results of the previous sections may apply in this case.

5.1. The automorphism group of substitutive subshifts from bijective constant-shape substitutions

Since bijective substitutions are reduced, Proposition 4.9 implies that the automorphism group of the substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$ is virtually $\mathbb{Z}^d$. In fact, we have a more rigid result in the bijective case as shown in the following proposition.

Proposition 5.1. *Let ζ_1, ζ_2 be two aperiodic bijective primitive constant-shape substitutions with the same expansion matrix L and support F_1. Then, any factor map $\psi : X_{\zeta_1} \to X_{\zeta_2}$ satisfying property (2) in Theorem 4.1 is induced by a letter-to-letter map. In particular, the automorphism group $\mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d)$ of a substitutive subshift given by an aperiodic bijective primitive constant-shape substitution is isomorphic to the direct product of $\mathbb{Z}^d$, generated by the shift action, with a finite group given by a permutation of the letters in the alphabet $\mathcal{A}$.*

Proof. Let $\psi \in \mathrm{Fac}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$ satisfying property (2) in Theorem 4.1, i.e., there exists $p \in F_1$ such that $S^p \psi \zeta_1 = \zeta_2 \psi$. Suppose that $p \neq 0$. Let $n > 0$ be large enough such that the set $F_n^{\circ C} = \{f \in F_n : f + C \subseteq F_n\}$ is nonempty, where C is the set defined in Lemma 4.6. Then, for any $x \in X_{\zeta_1}$, the coordinate x_0 determines the pattern $\zeta_1^n(x)|_{F_n}$. Hence the pattern $(S^{p_n} \psi \zeta_1^n)|_{p_n + (F_n)^{\circ C}}$ is also completely determined by the letter in x_0, where $p_n = \sum_{i=0}^{n-1} L^i(p)$. Set $m \in \mathbb{Z}^d$ such that $(p_n + (F_n)^{\circ C}) \cap (L^n(m) + F_n) \neq \emptyset$.

Since $S^{p_n}\psi\zeta_1^n$ is equal to $\zeta_2^n\psi$ and ζ_2 is bijective, the coordinate x_0 determines $\psi(x)_m$. This implies that $S^{-m}\psi$ is a factor map induced by a letter-to-letter map. Set $\phi = S^{-m}\psi$, we get that

$$S^{p_n+m-L^n(m)}\phi\zeta_1^n = \zeta_2^n\phi,$$

and, by bijectivity, the coordinate x_0 determines two coordinates of ψ, unless for any $n \in \mathbb{N}$ large enough $p_n + m$ is in $L_\zeta^n(\mathbb{Z}^d)$, i.e., for any n large enough, there exists $r_n \in \mathbb{Z}^d$ such that $p_n + m = L^n(r_n)$. Note that $p + r_n = L(r_{n+1})$, which implies that

$$\|r_{n+1}\| \le \|L^{-1}\|(\|r_n\| + \|p\|),$$

so $(r_n)_{n>0}$ is a bounded sequence. Hence, there exist $n > 0$ and $N > 1$ such that $r_{n+N} = r_n$, which implies that $p_{N-1} \in (L^N - \mathrm{id})(\mathbb{Z}^d)$. This is not possible by Remark 4.2.

If $p_n + m \notin L_\zeta^n(\mathbb{Z}^d)$, then x_0 determines two coordinates n_1, n_2 of $\psi(x)$, so determines the coordinates 0 and $n_2 - n_1$ of $\psi_1(x) = S^{-n_1}\psi(x)$. Hence ψ_1 is also induced via a letter-to-letter map. Note that, the map $\Psi_1 : \mathcal{A} \to \mathcal{B}$ inducing ψ_1 is bijective. If not, there are two fixed points x, y with $\Psi_1(x_0) = \Psi_2(y_0)$ and $x_0 \ne y_0$ generate two points with the same image, which is a contradiction. It follows that x_0 determines $x_{n_2-n_1}$, and then $x_{k(n_2-n_1)}$ for all $k \in \mathbb{Z}$, so x has a nontrivial period, which is a contradiction. Finally, we conclude by Corollary 4.10. ∎

5.2. Nondeterministic directions of substitutive subshifts from extremally permutative constant-shape substitutions

In this section, we give a characterization of the nondeterministic directions (defined in Section 2.6) of a substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$, in the case where ζ is *extremally permutative*. A starting remark is that, for each $n > 0$, the set of directions $\mathbb{S}^{d-1}$ is stratified by the opposite normal fan $\mathcal{N}(\mathrm{conv}(F_n^\zeta))$ (see Section 2.1.2). Our description of the nondeterministic directions is given in terms of union of these fans.

Following the notion of left and right permutative morphisms (see [4]), we say that a constant-shape substitution ζ is *extremally permutative* if the restriction p_f of ζ in f is bijective for all $f \in \mathrm{Ext}(\mathrm{conv}(F_1^\zeta))$. Since $\mathrm{Ext}(\mathrm{conv}(A + B)) \subseteq \mathrm{Ext}(\mathrm{conv}(A)) + \mathrm{Ext}(\mathrm{conv}(B))$, a substitution is extremally permutative if and only if for any $n > 0$ and $f \in \mathrm{Ext}(\mathrm{conv}(F_n^\zeta))$ the restriction p_f of ζ^n in f is bijective. Since $(F_n^\zeta)_{n>0}$ is a Følner sequence, there exists $n > 0$ such that $\mathrm{conv}(F_n^\zeta)$ is a nondegenerate polytope, so up to considering a power of ζ, we may assume that $\mathrm{conv}(F_1^\zeta)$ is a nondegenerate polytope. Using the recognizability property of substitutions and some basic results in convex geometry, we prove the following result.

Theorem 5.2. *Let ζ be an aperiodic extremally permutative primitive constant-shape substitution. Then the set of nondeterministic directions $\mathrm{ND}(X_\zeta, S, \mathbb{Z}^d)$ of its substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$ is the intersection of $\mathbb{S}^{d-1}$ with a nonempty union of limits of nested sequences of opposite normal cones of the form $\hat{N}_{G_n}(\mathrm{conv}(F_n^\zeta))$, where G_n is a face of $\mathrm{conv}(F_n^\zeta)$, for some integer $n > 0$.*

This theorem gives topological constraints on the set of nondeterministic directions. Actually, we will see that the convex hull of any digit tile is a polytope when $L_\xi = \lambda \mathrm{id}_{\mathbb{R}^d}$, i.e., it has a finite number of extreme points (Theorem 5.6). In this case, by Theorem 5.2, the set of nondeterministic directions $\mathrm{ND}(X_\xi, S, \mathbb{Z}^d)$ is a finite union of closed intervals (eventually degenerated). More explicitly, in the two-dimensional case, we obtain the following corollary, showing in particular that it cannot be a Cantor set.

Corollary 5.3. *In the two-dimensional case, under the hypothesis of Theorem 5.2, either the set* $\mathrm{ND}(X_\xi, S, \mathbb{Z}^2)$ *has nonempty interior, or it has at most 2 accumulation points.*

Proof. Assume that the set of nondeterministic directions $\mathrm{ND}(X_\xi, S, \mathbb{Z}^2)$ has empty interior. By Theorem 5.2, the elements of $\mathrm{ND}(X_\xi, S, \mathbb{Z}^2)$ are limits of normal vectors to edges of $\mathrm{conv}(F_n^\xi)$ for some $n > 0$. In [43] it was proved that such vectors are normalized vectors of the form $(L_\xi^*)^{-k} \boldsymbol{u}_k$, with $\boldsymbol{u}_k \in \mathbb{S}^1$ being a normal vector to an edge of $\mathrm{conv}(F_1^\xi)$ for some $k > 0$. Hence, their accumulation points are accumulation points of orbits of the projective action L_ξ^* on the circle $\mathbb{S}^1$. A standard analysis of this action (you can check [31, Theorem 3]) provides that the cardinality of the accumulation points is at most 2, when a power of one of the L_ξ-eigenvalues is a real number. Otherwise, the projective orbits of L_ξ^* are dense in the circle. Since $\mathrm{ND}(X_\xi, S, \mathbb{Z}^2)$ is closed, it is the whole circle, which is a contradiction. ∎

Proof of Theorem 5.2. Let $\boldsymbol{v}$ be a nondeterministic direction for $(X_\xi, S, \mathbb{Z}^d)$, and $x_1 \neq x_2 \in X_\xi$ such that $x_1|_{H_{\boldsymbol{v}}} = x_2|_{H_{\boldsymbol{v}}}$. Consider the set $D = \{\boldsymbol{n} \in \mathbb{Z}^d : x_1(\boldsymbol{n}) \neq x_2(\boldsymbol{n})\}$. Since $D \subseteq \mathbb{R}^d \setminus H_{\boldsymbol{v}}$, its convex hull is also contained in $\mathbb{R}^d \setminus H_{\boldsymbol{v}}$. We have two possibilities:

(1) The convex hull $\mathrm{conv}(D)$ has at least one extreme point. By Theorem 2.3 all the extreme points of $\mathrm{conv}(D)$ belong to D. Now, if $\boldsymbol{v} \in \mathbb{Z}^d$ is an extremal ray of $\mathrm{conv}(D)$, then for any extremal point $\boldsymbol{n} \in D$, the map $\mathrm{dist}(\boldsymbol{n} + t\boldsymbol{v}, H_{\boldsymbol{v}})$ must be increasing (if not, there exists $t^* > 0$ with $\boldsymbol{n} + t^*\boldsymbol{v} \in H_{\boldsymbol{v}}$, which is a contradiction). Hence, the distance map to H restricted to $\mathrm{conv}(D)$ is minimized in the extreme points of $\mathrm{conv}(D)$. Since the extreme points of $\mathrm{conv}(D)$ are in D, we can use the shift action in x_1, x_2 and assume that $x_1(\boldsymbol{0}) \neq x_2(\boldsymbol{0})$.

(2) If $\mathrm{conv}(D)$ does not have extreme points, then it contains a line. In this case, the hyperplane $\partial H_{\boldsymbol{v}}$ must be parallel to this line. Using similar arguments, we can assume that $x_1(\boldsymbol{0}) \neq x_2(\boldsymbol{0})$.

So we can assume that $\boldsymbol{0}$ is in a face $\boldsymbol{F}_0$ of smallest dimension of $\overline{\mathrm{conv}(D)}$ and then $\boldsymbol{v} \in \hat{N}_{\boldsymbol{F}_0}(\overline{\mathrm{conv}(D)})$. In fact, any element in $\hat{N}_{\boldsymbol{F}_0}(\overline{\mathrm{conv}(D)}) \cap \mathbb{S}^{d-1}$ is a nondeterministic direction for $(X_\xi, S, \mathbb{Z}^d)$.

Now, for any $k > 0$ consider $R^{(k)} > 0$ as the recognizability radius for ζ^k given by Proposition 3.7 and $R = 4R^{(k)}$. Since x_1 and x_2 coincide in an arbitrarily large ball, they have the same image under the maximal equicontinuous factor, hence $\pi_n(x_1) = \pi_n(x_2) \in F_n^\xi$ for any $n > 0$. By Lemma 3.12, there exist $n > 0$ and two words $\mathtt{w}_1^{(n)}, \mathtt{w}_2^{(n)} \in \mathcal{L}_{\overline{K}_\xi}(X_\xi)$ such that $x_i|_{B(\boldsymbol{0},R) \cap \mathbb{Z}^d} \sqsubseteq \zeta^n(\mathtt{w}_i^{(n)})$, for $i \in \{1, 2\}$. By the Pigeonhole principle, there exist

an infinite set $E \subseteq \mathbb{N}$, two patterns $\mathtt{w}_1, \mathtt{w}_2 \in \mathcal{L}_{\overline{K}_\zeta}(X_\zeta)$ and $\boldsymbol{k}_1, \boldsymbol{k}_2 \in \overline{K}_\zeta$ such that for all $n \in E$, $x_i|_{B(\mathbf{0},R) \cap \mathbb{Z}^d} = \zeta^n(\mathtt{w}_i)|_{L_\zeta^n(\boldsymbol{k}_i) + \pi_n(x_1) + B(\mathbf{0},R) \cap \mathbb{Z}^d}$, $i \in \{1,2\}$. The recognizability property implies the origin is in the boundary of $\mathrm{conv}(F_k^\zeta - \pi_k(x_1))$. Letting k to infinity, for all $n > 0$, the origin is in the boundary of $\mathrm{conv}(F_n^\zeta - \pi_n(x_1))$. If $\boldsymbol{G}_n$ is the face of smallest dimension containing $\pi_n(x_1)$ in $\mathrm{conv}(F_n^\zeta)$, then $\widehat{N}_{\boldsymbol{F}_0}(\overline{\mathrm{conv}(D)})$ is included in $\widehat{N}_{\boldsymbol{G}_n}(\mathrm{conv}(F_n^\zeta))$, for all $n > 0$.

We will show now the converse. We separate the proof in two cases.

Suppose first that $\mathrm{conv}(D)$ is closed. Let $\boldsymbol{F}_1$ be a face of $\mathrm{conv}(D)$ containing $\mathbf{0}$ of codimension 1. By Theorem 2.3 we have that $\boldsymbol{F}_1 = \mathrm{conv}(\boldsymbol{F}_1 \cap D)$. Fix $\boldsymbol{t} \in \boldsymbol{F}_1 \cap D$ different from $\mathbf{0}$. Since F_n^ζ is a fundamental domain of $L_\zeta^n(\mathbb{Z}^d)$, there are $\boldsymbol{h}_n \in F_n^\zeta$ and $\boldsymbol{z}_n(\boldsymbol{t}) \in \mathbb{Z}^d$ such that $\boldsymbol{t} = \boldsymbol{h}_n(\boldsymbol{t}) - \pi_n(x_1) + L_\zeta^n(\boldsymbol{z}_n(\boldsymbol{t}))$ and $\boldsymbol{t}$, being in the boundary of $\mathrm{conv}(D)$, lies in the boundary of $\mathrm{conv}(F_n^\zeta - \pi_n(x_1) + L_\zeta^n(\boldsymbol{z}_n(\boldsymbol{t})))$. Since this last set is a translated one of $\mathrm{conv}(F_n^\zeta - \pi_n(x_1))$ and they are both subsets of $\mathrm{conv}(D)$, a basic geometrical argument ensures that for any $n > 0$, $\boldsymbol{h}_n(\boldsymbol{t})$ and $\pi_n(x_1)$ are in the same face of $\mathrm{conv}(F_n^\zeta)$. The same arguments imply that $\boldsymbol{h}_n(\boldsymbol{t}_1), \boldsymbol{h}_n(\boldsymbol{t}_2)$ are in the same face for any $\boldsymbol{t}_1, \boldsymbol{t}_2 \in \boldsymbol{F}_1 \cap D$. Furthermore, $\boldsymbol{h}_n(\boldsymbol{t})$ and $\pi_n(x_1)$ are different for any $n \in E$ large enough. Indeed, assume the converse, taking $R > \|\boldsymbol{t}\|$, we have that $x_1|_{\boldsymbol{t}} = \zeta^n(\mathtt{w}_i)_{L_\zeta^n(\boldsymbol{k}_3) + \pi_n(x_1)}$, for some $\boldsymbol{k}_3 \in \overline{K}_\zeta$ for infinitely many $n \in E$. Since

$$\|\boldsymbol{t}\| = \|L_\zeta^n(\boldsymbol{k}_1 - \boldsymbol{k}_3)\|, \quad \text{for infinitely many } n \in E,$$

we have that necessarily $\boldsymbol{k}_1 = \boldsymbol{k}_3$, which is a contradiction.

Consider the face $\boldsymbol{H}_n$ of $\mathrm{conv}(F_n^\zeta)$ of smallest dimension generated by $\{\boldsymbol{h}_n(\boldsymbol{t})\}_{\boldsymbol{t} \in \boldsymbol{F}_1 \cap D}$. Notice that $\widehat{N}_{\boldsymbol{F}_1}(\mathrm{conv}(D)) \subseteq \bigcap_{n>0} \widehat{N}_{\boldsymbol{H}_n}(\mathrm{conv}(F_n^\zeta))$. We will prove that $\widehat{N}_{\boldsymbol{F}_1}(\mathrm{conv}(D)) = \bigcap_{n>0} \widehat{N}_{\boldsymbol{H}_n}(\mathrm{conv}(F_n^\zeta))$. By construction of x_1, x_2, the set $\{\boldsymbol{z}_n(\boldsymbol{t})\}_{\boldsymbol{t} \in \boldsymbol{F}_1 \cap D}$ is bounded (for all n large enough it belong to $\overline{K}_\zeta - \overline{K}_\zeta$), so there exists $\boldsymbol{t} \in \boldsymbol{F}_1$ and $\varepsilon > 0$ small enough such that for all $\boldsymbol{t}' \in B(\boldsymbol{t}, \varepsilon) \cap \boldsymbol{F}_1$ we have that $\boldsymbol{z}_n(\boldsymbol{t}') = \boldsymbol{z}_n(\boldsymbol{t})$ for all $n \in E$ large enough. Hence $\boldsymbol{H}_n$ is a face of codimension 1. An argument of dimensions ensures that $\widehat{N}_{\boldsymbol{F}_1}(\mathrm{conv}(D)) = \bigcap_{n \in E} \widehat{N}_{\boldsymbol{H}_n}(\mathrm{conv}(F_n)^\zeta)$.

Suppose now that $\mathrm{conv}(D)$ is not closed. Let $\boldsymbol{F}_1$ be a face of $\overline{\mathrm{conv}(D)}$ of codimension 1 containing $\mathbf{0}$ and $\boldsymbol{w} \in \widehat{N}_{\boldsymbol{F}_1}(\overline{\mathrm{conv}(D)})$. We will find a sequence of faces $\boldsymbol{H}_n$ of $\mathrm{conv}(F_n^\zeta)$ such that $\widehat{N}_{\boldsymbol{F}_1}(\overline{\mathrm{conv}(D)}) = \bigcap_{n>0} \widehat{N}_{\boldsymbol{H}_n}(\mathrm{conv}(F_n^\zeta))$. By definition we have that

$$\forall \boldsymbol{t} \in \boldsymbol{F}_1, \langle \boldsymbol{w}, \boldsymbol{t} \rangle = \inf_{\boldsymbol{n} \in D} \langle \boldsymbol{w}, \boldsymbol{n} \rangle = 0. \tag{5}$$

Now, set $\boldsymbol{t} \in \boldsymbol{F}_1$ and consider a sequence $(\boldsymbol{t}^m)_{m>0} \subseteq \mathrm{conv}(D)$ converging to $\boldsymbol{t}$. By Carathéodory's theorem for any $m > 0$, we can write $\boldsymbol{t}^m = \sum_{i=0}^d t_i^m \boldsymbol{f}_i^m$, with $t_i \geq 0$, $\sum_{i=0}^d t_i^m = 1$ and $\boldsymbol{f}_i^m \in D$. Then (5) implies that for all i, $t_i^m \langle \boldsymbol{w}, \boldsymbol{f}_i^m \rangle \xrightarrow[m \to \infty]{} 0$. The only difficulty to get the result, concerns the indices i such that $\liminf_{m \to \infty} t_i^m > 0$. For such i, we have that $\langle \boldsymbol{w}, \boldsymbol{f}_i^m \rangle \xrightarrow[m \to \infty]{} 0$. Using the recognizability property, for all $n \in E$ we write

$$\boldsymbol{f}_i^m = \boldsymbol{h}(m,i,n) - \pi_n(x_1) + L_\zeta^n(\boldsymbol{z}(m,i,n)),$$

with $\boldsymbol{h}(m, i, n) \in F_n^\zeta$ and $\boldsymbol{z}(m, i, n) \in \mathbb{Z}^d$. Since $\langle \boldsymbol{w}, \boldsymbol{h}(m, i, n) - \pi_n(x_1)\rangle \geq 0$ and $\langle \boldsymbol{w}, L_\zeta^n(\boldsymbol{z}(m, i, n))\rangle \geq 0$, we have that for all $n > 0$

$$\langle \boldsymbol{w}, \boldsymbol{h}(m, i, n) - \pi_n(x_1)\rangle \xrightarrow[m \to \infty]{} 0 \quad \wedge \quad \langle \boldsymbol{w}, L_\zeta^n(\boldsymbol{z}(m, i, n))\rangle \xrightarrow[m \to \infty]{} 0.$$

Since F_n^ζ is finite, we conclude that for all $n \in E$, there exists $m(n)$ such that for all $m \geq m(n)$,

$$\langle \boldsymbol{w}, \boldsymbol{h}(m, i, n)\rangle = \langle \boldsymbol{w}, \pi_n(x_1)\rangle = 0. \tag{6}$$

The same argument as the former one when $\mathrm{conv}(D)$ is closed, gives that $\boldsymbol{h}(m, i, n) \neq \pi_n(x_1)$ and if $i \neq j$, $\boldsymbol{h}(m, i, n) \neq \boldsymbol{h}(m, j, n)$ for all n large enough.

Now, for any $n > 0$, we define $\boldsymbol{H}_n$ as the face of $\mathrm{conv}(F_n^\zeta)$ of smallest dimension containing $\pi_n(x_1)$ and $\{\boldsymbol{h}(m, i, n): \boldsymbol{t} \in \boldsymbol{F}_1, 0 \leq i \leq d, m \geq m(n)$ with $\liminf_{m \to \infty} t_i^m > 0\}$. In particular, (6) shows that $\hat{N}_{\boldsymbol{F}_1}(\overline{\mathrm{conv}(D)}) \subseteq \bigcap_{n > 0} \hat{N}_{\boldsymbol{H}_n}(\mathrm{conv}(F_n^\zeta))$. We claim

$$\bigcap_{n > 0} \hat{N}_{\boldsymbol{H}_n}(\mathrm{conv}(F_n^\zeta)) = \hat{N}_{\boldsymbol{F}_1}(\overline{\mathrm{conv}(D)}).$$

First, note that taking subsequences if its necessary, for all $n \in E$ we get the following limits

$$\lim_{m \to \infty} \sum_{i=1}^d t_i^m \boldsymbol{h}(m, i, n) = \boldsymbol{h}_n(\boldsymbol{t}) \quad \wedge \quad \lim_{m \to \infty} \sum_{i=1}^d t_i^m \boldsymbol{z}(m, i, n) = \boldsymbol{z}_n(\boldsymbol{t}).$$

Hence, for all $n \in E$, $\boldsymbol{h}_n(\boldsymbol{t})$ is in $\boldsymbol{H}_n$. Also, $\boldsymbol{z}_n(\boldsymbol{t})$ is in $\mathrm{conv}(\overline{K}_\zeta - \overline{K}_\zeta)$ for all $n \in E$ large enough. A geometric argument shows that there exist $\boldsymbol{t} \in \boldsymbol{F}_1$ and $\varepsilon > 0$ small enough such that for all $\boldsymbol{t}' \in \boldsymbol{F}_1 \cap B(\boldsymbol{t}, \varepsilon)$, $\boldsymbol{z}_n(\boldsymbol{t}') = \boldsymbol{z}_n(\boldsymbol{t})$. So $\boldsymbol{H}_n$ is a face of codimension 1 for all $n \in E$ large enough. We then conclude that $\bigcap_{n \in E} \hat{N}_{\boldsymbol{H}_n}(\mathrm{conv}(F_n^\zeta)) = \hat{N}_{\boldsymbol{F}_1}(\overline{\mathrm{conv}(D)})$.

Consequently, the extremal rays of $\hat{N}_{\boldsymbol{F}_0}(\overline{\mathrm{conv}(D)})$ are equal to sets of the form $\bigcap_{n > 0} \hat{N}_{\boldsymbol{H}_n}(\mathrm{conv}(F_n^\zeta))$, with $\boldsymbol{H}_n$ being faces, eventually, of codimension 1 of $\mathrm{conv}(F_n^\zeta)$ containing $\pi_n(x_1)$. ∎

Then, to determine the nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$, we need to study the supporting hyperplanes to $\mathrm{conv}(F_n^\zeta)$. To do this, we focus on the convex hull of the digit tile of the substitution. In general, this convex hull is not a polytope, i.e., it can have infinitely many extreme points, even if the expansion matrix is diagonal, as we see in Example 5.4.

Example 5.4 (A digit tile, with a nonpolytope convex hull). Consider $L = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ and $F_1 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 2), (1, -2)\}$ (see Figure 13).
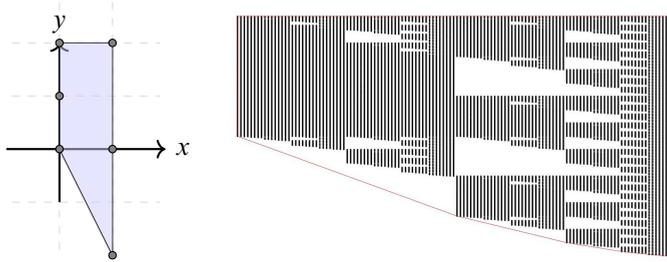
Figure 13. The fundamental domain and an approximation of the digit tile of Example 5.4.

A direct computation shows that, for any $n > 0$, the set of extreme points of $\text{conv}(F_n)$ is the set $\{(0, 0), (0, 3^n - 1), (2^n - 1, 3^n - 1)\} \cup \{(2^n - 2^k, 3^k - 3^n): 0 \leq k \leq n - 1\}$, which implies that

$$\text{Ext}(\text{conv}(T(L, F_1))) = \{(0, 0), (0, 1), (1, 1), (1, -1)\} \cup \{(1 - 2^{-k}, -1 + 3^{-k}): k \geq 0\}.$$

5.3. The polytope case

Here, we focus in the case when the convex hull of the digit tile is a polytope. We present some known results about this set that we will use in the rest of this article.

Definition 5.5. We say that a substitution ζ is a *polytope substitution* if it is extremally permutative, and the convex hull of the digit tile $T_\zeta = T(L_\zeta, F_1^\zeta)$ is a polytope.

From now on, we only consider polytope substitutions. This geometrical hypothesis implies several algebraic restrictions on the expansion matrix L_ζ (Proposition 5.10) and some dynamical consequences for the substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$ (Theorem 5.17).

We recall here some results characterizing the polytope case in terms of the extreme points of $\text{conv}(F_n^\zeta)$ [30], and the inward unit normal vectors of the $(d - 1)$-dimensional faces of $\text{conv}(T_\zeta)$ [43].

Theorem 5.6. *Let T be the digit tile for an expansion matrix $L \in M_d(\mathbb{R})$ and a fundamental domain $F_1 \subseteq \mathbb{R}^d$. The following statements are equivalent:*

(1) *The convex hull of the digit tile $T(L, F_1)$ is a polytope.*

(2) *The inward unit normal vectors of the $(d - 1)$-dimensional faces of $\text{conv}(F_1)$ are eigenvectors of $(L^*)^k$ for some k [43, Theorem 4.2].*

(3) *The cardinality of $\text{Ext}(\text{conv}(F_n))$ and $\text{Ext}(\text{conv}(F_{n+1}))$ are the same for some $n > 0$. In such a case, for any $m > n$, $|\text{Ext}(\text{conv}(F_m))| = |\text{Ext}(\text{conv}(F_n))|$, and then $|\text{Ext}(\text{conv}(T(L, F_1)))| = |\text{Ext}(\text{conv}(F_n))|$ [30, Theorem 2.2].*

Remark 5.7. In the case $L = \lambda \text{id}_{\mathbb{R}^d}$, with $\lambda > 1$, a direct computation shows that the statements (2) and (3) of Theorem 5.6 are satisfied without taking any power of L.

A big family for the polytope case is when a power of the expansion matrix L is an integer multiple of the identity, because for any fundamental domain F of $L(\mathbb{Z}^d)$, the convex hull of the digit tile generated by L and F is a polytope. In particular, all the convex hulls of the digit tiles of the examples in Figure 4 are polytopes. Nevertheless, it is not the only case where Theorem 5.6 can be applied.

Example 5.8.   (1) (Example of a non-self-similar matrix with a polytope digit tile) Consider $L = \left(\begin{smallmatrix} 2 & 0 \\ -1 & 3 \end{smallmatrix}\right)$ and $F_1 = \{(0,0),(1,2),(-2,-1),(-2,-3),(1,0),(-1,-1)\}$ (see Figure 14).

We have that $L^* = \left(\begin{smallmatrix} 2 & -1 \\ 0 & 3 \end{smallmatrix}\right)$ with eigenvectors equal to $\{(-1,1),(1,0)\}$. A direct computation shows that the set of extreme points of $\mathrm{conv}(F_n)$ is equal to

$$\{(2^n - 1, 2^n - (3^n + 1)/2), (2^n - 1, (2^{n+1} + 3^n - 3)/2),$$
$$(-2(2^n - 1), (3^n + 3 - 2^{n+2})/2), (-2(2^n - 1), (5 - 3^n - 2^{n+2})/2)\},$$

so the set of extreme points of $\mathrm{conv}(T(L, F_1))$ is

$$\{(1, 1/2), (1, 3/2), (-2, -3/2), (-2, -5/2)\}.$$
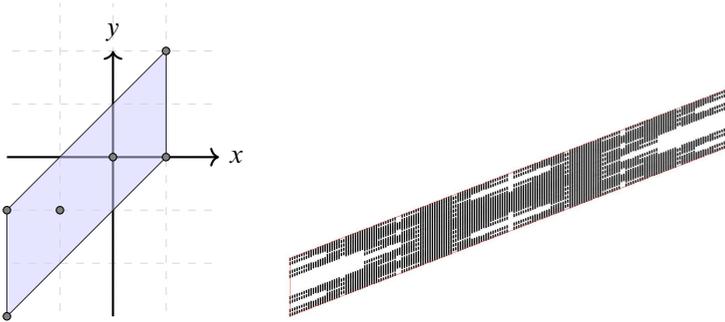
Figure 14. The fundamental domain and an approximation of the digit tile of a non-self-similar matrix.

(2) As an example where the statement (3) in Theorem 5.6 to be applied is not necessarily satisfied in $n = 1$, consider $L = \left(\begin{smallmatrix} -2 & 0 \\ 0 & -2 \end{smallmatrix}\right)$ and $F_1 = \{(0,0),(1,0),(0,1),(-1,-1)\}$. We have that

$$F_2 = \{(-1,-3),(0,-2),(1,-2),(-3,-1),(-1,-1),(0,-1),(-2,0),(-1,0),$$
$$(0,0),(1,0),(-2,1),(0,1),(1,1),(2,2),(3,2),(2,3)\},$$

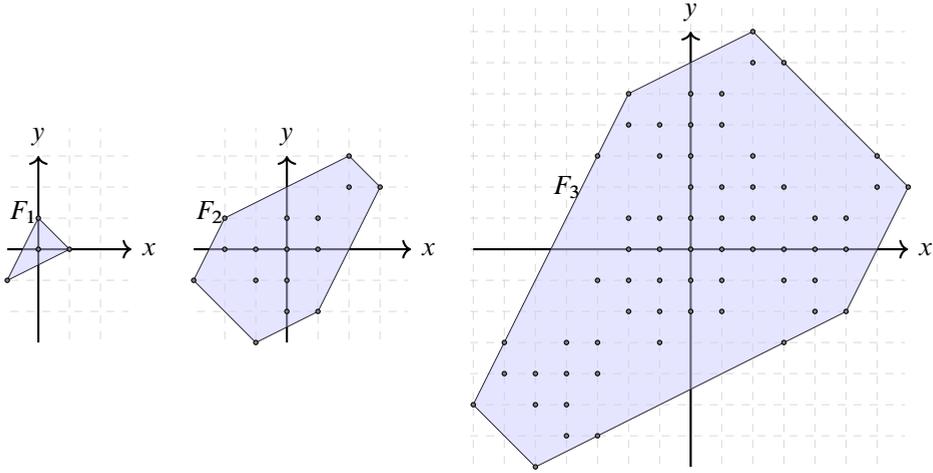so $\mathrm{conv}(F_2)$ has 3 extreme points, while $\mathrm{conv}(F_1)$ has 6 extreme points as shown in Figure 15.

Figure 15. The sets F_1, F_2 and F_3. Note that conv(F_2) and conv(F_3) have 6 extreme points.

In [30], the following result was proved about the extreme points of conv($T(L, F_1)$) as well as the extreme points of conv(F_m) for any $m > n$, where n is such that

$$|\text{Ext}(\text{conv}(F_n))| = |\text{Ext}(\text{conv}(F_{n+1}))|.$$

Proposition 5.9 ([30, Theorem 4.8]). *If* $|\text{Ext}(\text{conv}(F_n))| = |\text{Ext}(\text{conv}(F_{n+1}))|$, *then all the extreme points of* conv($T(L, F_1)$) *are of the form* $\sum_{j>0} L^{-(n+1)j} (\sum_{i=0}^{n} L^i(f_i))$, *with* $\sum_{i=0}^{n} L^i(f_i)$ *being an extreme point of* conv(F_{n+1}).

This implies that conv($T(L, F_1)$) is equal to $(L^m - \text{id})^{-1} \text{conv}(F_m)$ for all $m > n$. Now, assume that we are under the condition $|\text{Ext}(\text{conv}(F_1))| = |\text{Ext}(\text{conv}(T(L, F_1)))|$ and for all $n > 0$, conv($T(L, F_1)$) $= (L^n - \text{id})^{-1} \text{conv}(F_n)$. Let $\boldsymbol{u}$ be an inward unit normal vector of a $(d - 1)$-dimensional face of conv($T(L, F_1)$). For each $n > 0$, $((L^n - \text{id})^*)^{-1}\boldsymbol{u}$ is an inward normal vector of a $(d - 1)$-dimensional face of F_n. By Theorem 5.6 (1), there exists $k > 0$ such that $((L - \text{id})^*)^{-1}\boldsymbol{u}$ is an eigenvector of $(L^*)^k$. Hence by commutation, $\boldsymbol{u}$ is an eigenvector of $(L^*)^k$. Since conv($T(L, F_1)$) is a polytope, we can take $n > 0$ large enough such that any of the inward unit normal vectors of conv(F_1) is an eigenvector of the same power $(L^*)^n$. Hence, by the same arguments, up to considering a power of L, we may assume that all of the inward unit normal vectors of the $(d - 1)$-dimensional faces of conv(F_1) are eigenvectors of L^*. This is equivalent to the hyperplane $\partial H[\boldsymbol{u}] = \{\boldsymbol{t} \in \mathbb{R}^d : \langle \boldsymbol{t}, \boldsymbol{u} \rangle = 0\}$ (the vector space of an affine hull of a face of conv(F_1)) generated by $\boldsymbol{u}$, being preserved by L, i.e., $L\partial H[\boldsymbol{u}] = \partial H[\boldsymbol{u}]$. This implies that the normal fan $\mathcal{N}(\text{conv}(F_n))$ is the same for all $n > 0$, and it is equal to the one of conv($T(L, F_1)$).

Since for some $n > 0$, conv(F_n) is nondegenerate (by the Følner condition), it has d linearly independent inward normal vectors (that have integer coordinates with no com-

mon divisor), which are eigenvectors of L^*. The polytope condition implies then the following algebraic restrictions on the expansion matrix L. The proof is left to the reader.

Proposition 5.10. *If* $|\mathrm{Ext}(\mathrm{conv}(F_1^\zeta))| = |\mathrm{Ext}(\mathrm{conv}(T(L, F_1)))|$, *then the eigenvalues of* L *are integer numbers.*

Moreover, if $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ *are linearly dependent inward unit normal vectors of* $(d-1)$-*dimensional faces of* $\mathrm{conv}(T(L, F_1))$, *then* L *restricted to the vector space generated by these vectors acts as an integer multiple of the identity.*

In particular, in the two-dimensional case, if the digit tile has 3 or at least 5 edges, then it follows that the expansion matrix is an integer multiple of the identity.

Finally, up to taking an appropriate power of a substitution, we may assume the following hypotheses.

(PC 1) The expansion matrix L is diagonalizable, with positive integer eigenvalues.

(PC 2) The convex set $\mathrm{conv}(F_1)$ is nondegenerate and

$$|\mathrm{Ext}(\mathrm{conv}(F_1))| = |\mathrm{Ext}(\mathrm{conv}(T(L, F_1)))|.$$

(PC 3) Any inward unit normal vector of a $(d-1)$-dimensional face of $\mathrm{conv}(F_1)$ is an eigenvector of L^*.

(PC 4) The set K given by Proposition 2.10 is equal to $(\mathrm{id} - L)^{-1}(F_1) \cap \mathbb{Z}^d$, i.e., for any $\boldsymbol{k} \in K$, there exists $\boldsymbol{f} \in F_1$ such that $\boldsymbol{k} = L(\boldsymbol{k}) + \boldsymbol{f}$.

5.4. Dynamical properties of substitutive subshifts from polytope substitutions

As we saw in the previous subsection, under the hypotheses (PC 1), (PC 2), (PC 3) and (PC 4), the normal fan is the same for the convex hull of the supports of ζ^n, for any $n > 0$, and for the convex hull of the digit tile, so we have the following interpretation of Theorem 5.2 in the polytope case.

Corollary 5.11 (Nondeterministic directions in the polytope case). *Let* ζ *be an aperiodic primitive polytope substitution. The set of nondeterministic directions* $\mathrm{ND}(X_\zeta, S, \mathbb{Z}^d)$ *is the intersection of* $\mathbb{S}^{d-1}$ *with a nonempty union of opposite normal cones of the form* $\widehat{N}_{\boldsymbol{G}}(\mathrm{conv}(T_\zeta))$, *where* $\boldsymbol{G}$ *is a face of* $\mathrm{conv}(T_\zeta)$.

Hence, in the two-dimensional case, the former corollary implies strong restrictions on the set of nondeterministic directions. For instance, the number of its connected components is bounded by the number of edges of $\mathrm{conv}(T_\zeta)$.

Now, as shown in the proof of Theorem 5.2, to establish which opposite normal vectors of $\mathrm{conv}(F_1^\zeta)$ appears as nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$, we study the convex sets $\mathrm{conv}(L_\zeta^n(\boldsymbol{k}) + F_n^\zeta)$ generated by the points $\boldsymbol{k}$ in K_ζ, which depend on the combinatorics of the substitution. We say that a subset $W \subseteq K_\zeta$ is a *set of differences* if there exist two patterns $\mathtt{w}_1, \mathtt{w}_2 \in \mathcal{L}_{K_\zeta}(X_\zeta)$ such that $\mathtt{w}_1(\boldsymbol{k})$ is equal to $\mathtt{w}_2(\boldsymbol{k})$ if and only if $\boldsymbol{k}$ is in $K_\zeta \setminus W$.

The next lemma gives a sufficient condition to ensure that a vector $\boldsymbol{v}$ is a nondeterministic direction for $(X_\xi, S, \mathbb{Z}^d)$, seen as the converse of Theorem 5.2 in the polytope case. As in Lemma 3.12, we consider a set $C \Subset \mathbb{Z}^d$ such that for all $n > 0$, $C + F_n^\xi + F_n^\xi \subseteq L_\xi^n(C) + F_n^\xi$ and $\overline{K}_\xi = K_\xi + C$.

Lemma 5.12. *Let $W \subseteq K_\xi$ be a set of differences, $\boldsymbol{k} \in W$, $n > 0$, a point $\boldsymbol{f} \in \partial \operatorname{conv}(L_\xi^n(\boldsymbol{k}) + F_n^\xi)$ and $\boldsymbol{v} \in \mathbb{S}^{d-1}$ be such that $\boldsymbol{f} + \partial H_{\boldsymbol{v}}$ supports $\operatorname{conv}(L_\xi^n(\boldsymbol{k}) + F_n^\xi)$ at $\boldsymbol{f}$. Suppose that $\boldsymbol{f}$ satisfies the following conditions:*

  (H1) $\boldsymbol{f} + \overline{K}_\xi \subseteq L_\xi^n(K_\xi) + F_n^\xi$,

  (H2) $\boldsymbol{f} + (\overline{K}_\xi \cap H_{\boldsymbol{v}}) \subseteq L_\xi^n(K_\xi \setminus W) + F_n^\xi$.

Then $\boldsymbol{v}$ is nondeterministic for $(X_\xi, S, \mathbb{Z}^d)$.

Proof. Let $\mathtt{w}_1, \mathtt{w}_2$ be two patterns such that $\mathtt{w}_1(\boldsymbol{k}') = \mathtt{w}_2(\boldsymbol{k}')$ if and only if $\boldsymbol{k}' \in K_\xi \setminus W$. Note that condition (H1) is equivalent to for all $m > 0$, $L_\xi^m(\boldsymbol{f}) + L_\xi^m(\overline{K}_\xi) + F_m^\xi \subseteq L_\xi^{n+m}(K_\xi) + F_{n+m}^\xi$. Since $\overline{K}_\xi = K_\xi + C$, Remark 2.13 (2) implies that for all $m > 0$,

$$L_\xi^m(\boldsymbol{f}) + F_m^\xi + (L_\xi^m(K_\xi) + F_m^\xi) \subseteq L_\xi^{n+m}(K_\xi) + F_{n+m}^\xi. \tag{7}$$

If $\boldsymbol{f}$ is an extreme point of $\operatorname{conv}(L_\xi^n(\boldsymbol{k}) + F_n^\xi)$, there exists $\boldsymbol{g} \in \operatorname{Ext}(\operatorname{conv}(F_1^\xi))$ such that $\boldsymbol{f} = L_\xi^n(\boldsymbol{k}) + \sum_{i=0}^{n-1} L_\xi^i(\boldsymbol{g})$. If $\boldsymbol{f}$ is in the relative interior of a k-dimensional face of $\operatorname{conv}(L_\xi^n(\boldsymbol{k}) + F_n^\xi)$, $(1 \leq k \leq d-1)$, we consider $\boldsymbol{g} \in \operatorname{Ext}(\operatorname{conv}(F_1^\xi))$ such that $L_\xi^n(\boldsymbol{k}) + \sum_{i=0}^{n-1} L_\xi^i(\boldsymbol{g})$ and $\boldsymbol{f}$ are in the same k-dimensional face of $\operatorname{conv}(L_\xi^n(\boldsymbol{k}) + F_n^\xi)$ as shown in Figure 16.

Now, condition (H2) is equivalent to for all $m > 0$,

$$L_\xi^m(\boldsymbol{f} + (\overline{K}_\xi \cap H_{\boldsymbol{v}})) + F_m \subseteq L_\xi^{n+m}(K_\xi \setminus W) + F_{n+m}^\xi.$$

We will prove that for all $m > 0$

$$L_\xi^m(\boldsymbol{f}) + \sum_{i=0}^{m-1} L_\xi^i(\boldsymbol{g}) + ((L_\xi^m(K_\xi) + F_m^\xi) \cap H_{\boldsymbol{v}}) \subseteq L_\xi^m(\boldsymbol{f}) + (\overline{K}_\xi \cap H_{\boldsymbol{v}}) + F_m^\xi. \tag{8}$$
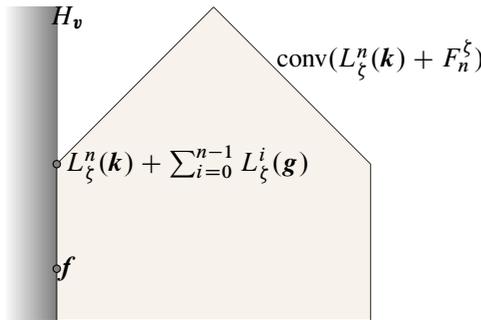
Figure 16. The hyperplane $\partial H_{\boldsymbol{v}}$ supports $\operatorname{conv}(L_\xi^n(\boldsymbol{k}) + F_n^\xi)$ at $\boldsymbol{f}$ and $L_\xi^n(\boldsymbol{k}) + \sum_{i=0}^{n-1} L_\xi^i(\boldsymbol{g})$.

We have that

$$\langle v, g \rangle = \min_{i \in F_1^\zeta} \langle v, i \rangle. \tag{9}$$

Fix $n \in L_\zeta^m(f) + \sum_{i=0}^{m-1} L_\zeta^i(g) + ((L_\zeta^m(K_\zeta) + F_m^\zeta) \cap H_v)$. There exist $k_1 \in K_\zeta$, $j \in F_m^\zeta$ and $h \in H_v$ such that

$$n = L_\zeta^m(f) + \sum_{i=0}^{m-1} L_\zeta^i(g) + L_\zeta^m(k_1) + j = L_\zeta^m(f) + \sum_{i=0}^{m-1} L_\zeta^i(g) + h.$$

On the other hand, by definition of $\overline{K}_\zeta$, there exist $c \in \overline{K}_\zeta$ and $l \in F_m^\zeta$ such that

$$\sum_{i=0}^{m-1} L_\zeta^i(g) + L_\zeta^m(k_1) + j = L_\zeta^m(c) + l.$$

To prove (8), it is enough to show that $c \in H_v$. Indeed, writing $l = \sum_{i=0}^{m-1} L_\zeta^i(l^{(i)})$, with $l^{(i)} \in F_1^\zeta$ for $0 \le i \le m - 1$, we have that $c = L_\zeta^{-m}(\sum_{i=0}^{m-1} L_\zeta^i(g - l^{(i)}) + h)$. So, we get that

$$\langle v, c \rangle = \left\langle v, L_\zeta^{-m}\left(\sum_{i=0}^{m-1} L_\zeta^i(g - l^{(i)}) + h \right) \right\rangle$$

$$= \frac{1}{\lambda^m}\left(\sum_{i=0}^{m-1} \langle v, L_\zeta^i(g - l^{(i)}) \rangle \right) + \frac{1}{\lambda^m} \langle v, h \rangle$$

$$= \frac{1}{\lambda^m}\left(\sum_{i=0}^{m-1} \lambda^i \langle v, g - l^{(i)} \rangle \right) + \frac{1}{\lambda^m} \langle v, h \rangle.$$

Since $h \in H_v$, we have $\langle v, h \rangle \le 0$, and by (9) we have that $\langle v, g - l^{(i)} \rangle \le 0$, for all $1 \le i \le m - 1$. Since $\lambda > 0$, we conclude that $\langle v, c \rangle \le 0$, which implies that $c \in H_v$.

By (7) and Proposition 2.10, the iterations of the substitution on the patterns $\mathtt{w}_1$, $\mathtt{w}_2$ leads to two points $x_1 \ne x_2 \in X_\zeta$ such that, x_i is in $[\zeta^n(\mathtt{w}_1)]_{-L_\zeta^n(f) - \sum_{i=0}^{n-1} L_\zeta^i(g)}$, for $i \in \{1, 2\}$. Finally, (8) implies that $x_1|_{H_v} = x_2|_{H_v}$. The fact that f is in the set of differences of $\mathtt{w}_1$ and $\mathtt{w}_2$ ensures that $x_1(\mathbf{0}) \ne x_2(\mathbf{0})$, so $x_1 \ne x_2$. We then conclude that v is nondeterministic for $(X_\zeta, S, \mathbb{Z}^d)$. ∎

As described in Theorem 5.2, depending on to which faces of $\mathrm{conv}(L_\zeta^n(k) + F_n^\zeta)$ the point f satisfying condition (H1) belongs, we may have more nondeterministic directions, obtaining the following corollary.

Corollary 5.13. *Let $W \subseteq K_\zeta$ be a set of differences, $k \in W$, $n > 0$, and a point $f \in \partial \mathrm{conv}(L_\zeta^n(k) + F_n^\zeta) \cap \partial \mathrm{conv}(L_\zeta^n(W) + F_n^\zeta)$ satisfying condition (H1) of Lemma 5.12. Then any v in $\widehat{N}_F(\mathrm{conv}(L_\zeta^n(W) + F_n^\zeta)) \cap \mathbb{S}^{d-1}$ (with F being the face of smallest dimension where f belongs to $\mathrm{conv}(L_\zeta^n(W) + F_n^\zeta)$) is nondeterministic for $(X_\zeta, S, \mathbb{Z}^d)$.*

In addition to this result, the proof of Theorem 5.2 provides that, for some $n > 0$, the opposite normal cone $\hat{N}_{\boldsymbol{F}}(\mathrm{conv}(L_\zeta^n(W) + F_n^\zeta))$ is equal to an opposite normal cone $\hat{N}_{\boldsymbol{G}}(\mathrm{conv}(F_n^\zeta))$, where $\boldsymbol{G}$ is a face of $\mathrm{conv}(F_n^\zeta)$.

Example 5.14 (Different behaviors for the nondeterministic directions). In the following we present two examples with different behaviors given by Lemma 5.12.

(1) Consider the 2D-Thue–Morse substitution with $L_{\zeta_{TM}} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, $F_1^{\zeta_{TM}} = [\![0,1]\!]^2$, given by

$$\zeta_{TM}: \quad 0 \mapsto \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \quad 1 \mapsto \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}.$$

The following is a pattern of ζ_{TM}:

$$
\begin{array}{c}
0110100110010110 \\
1001011001101001 \\
0110100110010110 \\
1001011001101001 \\
1001011001101001 \\
0110100110010110.
\end{array}
$$

In this case, the set $K_{\zeta_{TM}}$ is equal to $[\![-1,0]\!]^2$. We have that

$$\mathcal{L}_{K_{\zeta_{TM}}}(X_{TM}) = \left\{ \begin{smallmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{smallmatrix} \right\}.$$

The sets of differences for the 2D-Thue–Morse substitution are $\{\{(0,-1),(0,0)\}, \{(-1,0),(0,0)\}, \{(-1,0),(0,-1)\}, \{(-1,-1),(0,0)\}, \{(-1,-1),(-1,0)\}\}$. By Lemma 5.12 it can be proved that $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ are the only nondeterministic directions for $(X_{\zeta_{TM}}, S, \mathbb{Z}^2)$.

(2) Consider the substitution of the table tiling [38], with $L_{\zeta_t} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, $F_1^{\zeta_t} = [\![0,1]\!]^2$, given by ζ_t

$$0 \mapsto \begin{matrix} 3 & 0 \\ 1 & 0 \end{matrix}, \quad 1 \mapsto \begin{matrix} 1 & 1 \\ 0 & 2 \end{matrix}, \quad 2 \mapsto \begin{matrix} 2 & 3 \\ 2 & 1 \end{matrix}, \quad 3 \mapsto \begin{matrix} 0 & 2 \\ 3 & 3 \end{matrix}.$$

The following is a pattern of ζ_t:

$$
\begin{array}{c}
3023302302021111 \\
1021102133330202 \\
0230230211110230 \\
3310213302023310 \\
1130231130231130 \\
0210210210210210.
\end{array}
$$

The set K_{ζ_t} is equal to $[\![-1,0]\!]^2$ and the sets of differences is equal to $2^{K_{\zeta_t}} \setminus \{\emptyset, K_{\zeta_t}, \{(-1,-1),(0,0)\}, \{(0,-1),(-1,0)\}\}$. By Lemma 5.12, it can be proved that the set of nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^2)$ is the whole circle $\mathbb{S}^1$.

Now, we proceed to determine the normalizer semigroup $N(X_\zeta, S, \mathbb{Z}^d)$ of substitutive subshifts given by polytope substitutions. Set $M \in \vec{N}(X_\zeta, S, \mathbb{Z}^d)$. By Proposition 2.9, if $\boldsymbol{v}$ is a nondeterministic direction for $(X_\zeta, S, \mathbb{Z}^d)$, then $M^*\boldsymbol{v}/\|M^*\boldsymbol{v}\|$ is also a nondeterministic direction for $(X_\zeta, S, \mathbb{Z}^d)$. Moreover, Theorem 5.2 ensures that the matrix M acts on the opposite normal cones of $\mathrm{conv}(T_\zeta)$, that appeared as nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$. In particular, the matrix M^* permutes the hyperplanes defined by the $(d-1)$-dimensional faces of $\mathrm{conv}(T_\zeta)$ whose unit opposite normal cones are nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$. If there are d linearly independent nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$, we get the following result about the linear representation semigroup.

Proposition 5.15. *Let ζ be an aperiodic primitive polytope substitution. If the set of nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$ contains d linearly independent vectors, then the following hold:*

(1) *Any homomorphism $\phi \in N(X_\zeta, S, \mathbb{Z}^d)$ is invertible.*

(2) *The linear representation semigroup $\vec{N}(X_\zeta, S, \mathbb{Z}^d)$ is a finite group, and it is isomorphic to a subgroup of $\mathrm{GL}(d, \mathbb{Z}/3\mathbb{Z})$.*

(3) *The norm of any matrix M in the linear representation group $\vec{N}(X_\zeta, S, \mathbb{Z}^d)$ is bounded by an explicit formula only depending on the convex hull of the digit tile T_ζ.*

Proof. By assumption and Corollary 5.11 there are d linearly independent inward unit normal vectors to the $(d-1)$-dimensional faces of $\mathrm{conv}(T_\zeta)$ that are nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$. Let $n \geq d$ be the number of unit normal vectors to the $(d-1)$-dimensional faces of $\mathrm{conv}(T_\zeta)$ that are nondeterministic directions. Any matrix M in the linear representation group of $(X_\zeta, S, \mathbb{Z}^d)$ permutes the hyperplanes defined by these $(d-1)$-dimensional faces of $\mathrm{conv}(T_\zeta)$. By condition (PC 3), the normal lines (generated by the inward vectors) of these hyperplanes are invariant by a power of the expansion matrix L_ζ^*. Hence M^* permutes n eigenspaces $\{\mathbb{Q}\boldsymbol{v}_1, \ldots, \mathbb{Q}\boldsymbol{v}_n\}$ of some power of L_ζ^*. Moreover, we can assume that these vectors have integer coordinates not having common divisors except ± 1. Note that each vector is unique up to a sign and does not depend on M. Since M is in $\mathrm{GL}(d, \mathbb{Z})$, M^* is also in $\mathrm{GL}(d, \mathbb{Z})$. This implies that M^* sends vectors with integer coordinates with no common divisor to vectors with the same property. Therefore, for all $1 \leq i \leq n$, $(M^*)^{2n!}\boldsymbol{v}_i = \pm\boldsymbol{v}_i$. Since the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ contains d linearly independent vectors, we get that $(M^*)^{2n!}$ is the identity matrix, which implies that M has finite order. By Lemma 4.12 any homomorphism of $(X_\zeta, S, \mathbb{Z}^d)$ is invertible. We recall that a subgroup of $\mathrm{GL}(d, \mathbb{Z})$ is finite if and only if any element in the subgroup has finite order. We then conclude that the linear representation group $\vec{N}(X_\zeta, S, \mathbb{Z}^d)$ is finite, and by Minkowski's theorem, we have that $\vec{N}(X_\zeta, S, \mathbb{Z}^d) \leq \mathrm{GL}(d, \mathbb{Z}/3\mathbb{Z})$.

Finally, note that $\|M\| \leq \|P\| \cdot \|Q_M\| \cdot \|P^{-1}\|$, where $\|P\|, \|P^{-1}\|$ and

$$\sup_{M \in \vec{N}(X_\zeta, S, \mathbb{Z}^d)} \|Q_M\| < \infty$$

only depend on the convex hull of the digit tile T_ζ.   ∎

Remark 5.16. In particular, it follows from the proof that if $n = d$, each matrix Q_M is a permutation matrix, so $\vec{N}(X_\zeta, S, \mathbb{Z}^d)$ is conjugate to a subgroup of the hyperoctaedral group W_d. These recover results in [8] for block substitutions. By the realization result [8, Theorem 35] these results obtained are optimal.

The following theorem summarizes all the properties satisfied for aperiodic primitive reduced polytope substitutions.

Theorem 5.17. *Let ζ be an aperiodic reduced primitive polytope substitution. Then,*

(1) *the system $(X_\zeta, S, \mathbb{Z}^d)$ is coalescent, and also any homomorphism in $N(X_\zeta, S, \mathbb{Z}^d)$ is invertible.*

If there are d linearly independent vectors that are nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$, we have that

(2) *the normalizer is virtually generated by the shift action;*

(3) *the linear representation group $\vec{N}(X_\zeta, S, \mathbb{Z}^d)$ acts as a permutation group on the set $\{N_G(\mathrm{conv}(T_\zeta)) : N_G(\mathrm{conv}(T_\zeta)) \subseteq \mathrm{ND}(X_\zeta, S, \mathbb{Z}^d), G \text{ a face of } \mathrm{conv}(T_\zeta)\}$. In particular, if $\mathrm{ND}(X_\zeta, S, \mathbb{Z}^d) = \mathbb{S}^{d-1}$, then the linear representation group $\vec{N}(X_\zeta, S, \mathbb{Z}^d)$ is isomorphic to a subgroup of the automorphism group of the normal fan of $\mathrm{conv}(T_\zeta)$.*

Proof. Statement (1) is true by Proposition 4.8 and Lemma 4.12.

Now, by the third isomorphism theorem we have that

$$N(X_\zeta, S, \mathbb{Z}^d)/\mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d) \cong \left(N(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle\right)/\left(\mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle\right).$$

Then, Proposition 5.15 gives that the quotient group $N(X_\zeta, S, \mathbb{Z}^d)/\mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d) = \vec{N}(X_\zeta, S, \mathbb{Z}^d)$ is finite and Proposition 4.9 implies that $\mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle$ is also finite. We conclude that $N(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle$ is a finite group.

Finally, statement (3) is true by Proposition 5.15.   ∎

The hypothesis of having d linearly independent vectors as nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$ is decidable by an algorithm, studying all the sets of differences and then applying Lemma 5.12. Until now we did not find an aperiodic d-dimensional reduced primitive polytope substitution with less than d linearly independent nondeterministic directions leaving the following question.

Question 5.18. *Does there exist an aperiodic primitive reduced polytope substitution ζ with less than d linearly independent nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$?*

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