

THE QUARK PROPAGATOR FROM THE DYSON-SCHWINGER EQUATIONS:

I. THE CHIRAL SOLUTION

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Abstract

Within the framework of the Dyson-Schwinger equations in the axial gauge, we study the effect that non-perturbative glue has on the quark propagator. We show that Ward-Takahashi identities, combined with the requirement of matching perturbative QCD at high momentum transfer, guarantee the multiplicative renormalisability of the answer. Technically, the matching with perturbation theory is accomplished by the introduction of a transverse part to the quark-gluon vertex. We show that this transverse vertex is crucial for chiral symmetry breaking, and that massless solutions exist below a critical value of α_S . Using the gluon propagator that we previously obtained, we obtain small corrections to the quark propagator, which keeps a pole at the origin in the chiral phase.

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1. Introduction

The Dyson-Schwinger (DS) equations of motion are one of the main tools for the investigation of non-perturbative effects. These equations are particularly suited to the study of the interface between perturbative and nonperturbative regimes, as they allow us to compute the evolution of Green functions when one enters the infrared region, and hence suggest modifications to the usual Feynman rules.

In a previous paper, we have studied [1] the DS equations in the quenched approximation, *i.e.* for pure gauge QCD, and found that, as was first suggested by Cornwall [2], some solutions for the gluon propagator are flatter than a pole in the infrared region. In order to avoid the question of ghost propagators, we worked in the axial gauge. Other groups [3, 4, 5] have also found that flatter solutions exist, in covariant gauges, although it is at present difficult to relate our results to theirs. The absence of a pole in the gluon propagator at $k^2 = 0$ is natural if one assumes that gluons do not propagate to infinity, *i.e.* these solutions should describe *confined* gluons. Furthermore, as was pointed out by Landshoff and Nachtmann [6], the existence of such solutions is highly desirable in phenomenological applications, and gives us a practical way to extend usual perturbative estimates to the strongly-interacting sector of the theory. The use of these solutions has already met with some success in diffractive calculations [7].

The DS equations constitute an infinite tower of integral relations between n-point functions. Hence, by themselves, they cannot be solved. For instance, if we consider the DS equation for the quark two-point function (the propagator), it involves a three-point function (the quark-gluon vertex) as well as the gluon two-point function (the propagator). One can imagine solving the pure-gauge DS equations for the gluon propagator as a first approximation, hence we shall assume here that the gluon propagator is known. However, resorting to higher-order DS equations to calculate the vertex will only bring in higher-order n-point functions, which are also a priori unknown. Hence one needs to use another input.

The simplest assumption is to take the perturbative γ_μ vertex, which is the so-called “rainbow” or “ladder” approximation. One can be however more sophisticated, and follow the observation of Baker, Ball and Zachariasen (BBZ) [8], namely that one needs to choose a vertex that will obey the Ward-Takahashi-Slavnov-Taylor identities. These constitute another nonperturbative statement of field theory, and hence must be valid in general.

In our study of the gluon propagator [1], we chose for the three-gluon vertex the simplest function that would obey the Ward-Slavnov-Taylor identities. We showed that besides the original $1/k^4$ solution, there exists another solution, behaving like $1/k^{0.2}$ near $k^2 = 0$. To obtain a full picture of nonperturbative effects in QCD, one then needs to calculate the quark propagator corresponding to that gluon propagator. As we shall see, the problem of multiple solutions arises again, as for sufficiently small α_S , a chiral solution exists together with a massive one. We shall show that one can constrain the spectral density of the gluon propagator and the value of the coupling so that only the massive solution survives.

As the equation for the quarks is much simpler than that for the gluons, we investigate in more detail the effect of the ansatz for the quark-gluon vertex on the quark propagator. Namely, the minimal vertex, used by BBZ for gluons and by Ball and Zachariasen (BZ) for quarks [12], leads to the breakdown of multiplicative renormalisability. We demonstrate that, following the method of Curtis and Pennington (CP) [9], it is possible to recover it by choosing a specific form for the quark-gluon vertex, which agrees with perturbation theory in the ultraviolet region, and which generalizes the B(B)Z ansatz. We also show that massive solutions exist only for the CP vertex. Simpler ansätze lead to an inconsistent ultraviolet behaviour, unless the propagator is massless. We also explain under which conditions massless solutions will exist, even for the CP vertex. We proceed to solve the DS equation in the massless phase, and show that nonperturbative effects do not remove the pole of the quark propagator in the chiral phase of the QCD vacuum, suggesting that confined quarks can exist only after chiral symmetry breaking.

This paper is set out as follows. In Section 2, we review our results for the gluon propagator, and outline the formalism that lead to these. We point out the problems linked with the BBZ choice of a three-gluon vertex. In Section 3, we discuss the DS equation for the quark propagator in the axial gauge, and the approximations made when imposing the Ward-Takahashi identity. In Section 4, we consider the part of the propagator that preserves chiral symmetry. We show that the problem can be reduced to a one-dimensional integral equation, which we then renormalise. We show explicitly that we recover multiplicative renormalisability. In Section 5, we give a criterion for the existence of massless solutions, and then proceed to the numerical solution of the equation in the massless case, for the gluon propagator that we previously derived.

2. The Gluon Propagator

The fundamental observation [8] is that one can use the Ward-Slavnov-Taylor (WST) identities to obtain a closed equation from the DS equation, if one neglects quark loops and works with gluons only. In the axial gauge ($n \cdot A = 0$), where we do not have to worry about ghost degrees of freedom, the DS equation relates the propagator to the three- and four-point vertices.

The three-point vertex, $\Gamma_\mu^{(3)}$ can be split into a part $\Gamma_T^{(3)}{}_\mu$ transverse to the external gluon momentum k_μ , $\Gamma_T^{(3)} \cdot k = 0$, and a longitudinal part, $\Gamma_L^{(3)\mu}$, with $\Gamma_L^{(3)} \cdot k \neq 0$. As the vertex is a third rank tensor, depending on the three vectors entering the vertex, this split is not uniquely defined. The WST identities relate $\Gamma_L^{(3)}$ to the propagator, and these identities can be “solved”, hence producing an ansatz for the longitudinal part [8, 12]. It is this ansatz that we *define* as being the longitudinal part of the vertex in the following.

One can then obtain a closed equation for the propagator if one makes two assumptions. First, the full propagator is supposed to have the same spin and gauge structure as the free propagator:

$$D_{\mu\nu}(k^2) = -D_{\mu\nu}^0(k, n) \frac{\mathcal{Z}(k^2)}{k^2} \quad (2.1)$$

where

$$D_{\mu\nu}^0(k, n) = \delta_{\mu\nu} - \frac{k_\mu n_\nu + n_\mu k_\nu}{n \cdot k} + \frac{n \cdot n \, k_\mu k_\nu}{(n \cdot k)^2} \quad (2.2)$$

This spin structure projects out the four-point vertex contribution to the equation. If one further assumes that the three-point vertex is dominated by its longitudinal part $\Gamma_L^{(3)}$, which is itself known from the propagator via the WST identities, one obtains an equation which involves only the gluon propagator.

This equation still needs to be renormalised. One can show without approximation [8] that a propagator with the spin structure (2.1) has to be singular as $k^2 \rightarrow 0$. Thus the inverse propagator should vanish at the origin. This allows one to subtract the quadratic divergences, thus renormalising the gluon mass to zero and cancelling the tadpole graphs. One is then left with logarithmic divergences, which can be dealt with through wave function renormalisation at a renormalisation point μ_g , by defining

$$\mathcal{Z}(k^2) = \mathcal{Z}(\mu_g^2) \mathcal{Z}_R(k^2) \quad (2.3)$$

However, the resulting definition of the renormalised coupling constant α_g is slightly different from the usual one:

$$\alpha_g(\mu_g) = \frac{\alpha_b \mathcal{Z}(\mu_g^2)}{1 + \alpha_b \mathcal{Z}(\mu_g^2) [\int_b^\infty dk^2 \Psi(k^2, \mathcal{Z})]_{\mathcal{Z}=\mathcal{Z}_R}} \quad (2.4)$$

where α_b is the bare coupling and Ψ a function linear in \mathcal{Z} , the explicit form of which can be found in ref. [11]. It has to be noted at this point that Eq. (2.4) leads to two related problems. First of all, the propagator asymptotically behaves like $1/\log(k^2)^{11/16}$ as $k^2 \rightarrow \infty$ [8], which is not what is obtained in perturbation theory. Furthermore, one obviously loses multiplicative renormalisability. This has the nasty consequence that, if the same truncation is applied to the quark propagator, the renormalised couplings are not equal anymore, because the functions Ψ are different for quarks and gluons.

The solution to the resulting equation, found in ref. [8], has a very singular behaviour proportional to $1/k^4$ as the momentum squared goes to zero. This was advertised as a signal for a confining potential. It has to be pointed out that in order to renormalise the equation in the presence of a $1/k^4$ singularity, one needs to perform additional subtractions, and hence one obtains such solutions only if one assumes they exist. In ref. [1], we exploited the fact that the integral equation obtained in ref. [8] is non-linear and can in general admit several solutions. We solved the original equation, without additional subtractions, and obtained a gluon propagator which, although singular as $k^2 \rightarrow 0$ as required in the axial gauge, has a cut singularity, flatter than a pole. This describes a soft gluon which is confined rather than confining and can conveniently be used in models of the pomeron such as that suggested by Landshoff and Nachtmann [6]. This propagator may be written

in the axial gauge ($n.A = 0$) as

$$\frac{Z_R(k^2)}{k^2} = \frac{\mu_g^{-2}}{a_1 \left(\frac{k^2}{\mu_g^2}\right)^{b_1} + a_2 \left(\frac{k^2}{\mu_g^2}\right)^{b_2} + c \ln \left[d \left(\frac{k^2}{\mu_g^2}\right) + e \right]} \quad (2.5)$$

The dimensionless constants a_1, b_1, a_2, b_2, c, d and e , are given in the Table. The mass scale μ_g is the value of momentum at which $Z_R(k^2)$ takes the value 1, so that the propagator matches the value of the free gluon propagator. It is not obtained from the DS equation, which is scale invariant, but rather from a fit to the total cross-section for proton-proton scattering calculated using this propagator. An optimum value of 0.8 GeV was found for μ_g . It is worth pointing out that most of the ideas discussed here can be applied to other theories, living at other mass scales, such as technicolour, and that one would only need to adjust μ_g to do so.

Table: The constants used in Eq. (2.5)

a_1	0.88	c	0.59
a_2	-0.95	d	2.1
b_1	0.22	e	4.1
b_2	0.86	$\alpha_g(\mu_g)$	1.4

Note that in the case of a multiplicatively renormalisable theory, the actual value of α_S should not matter. In principle, one could always rescale α_S by varying μ according to

$$\begin{aligned} \alpha_S(\mu') Z_R(\mu'^2, q^2) &= \alpha_S(\mu) Z_R(\mu^2, q^2) \\ Z_R(\mu^2, q^2) &= Z_R(\mu^2, \mu'^2) Z_R(\mu'^2, q^2) \end{aligned} \quad (2.6)$$

where the first argument of Z refers to the renormalisation scale: $Z(\mu^2, \mu^2) = 1$. However, our gluonic solution does not obey these equations because of (2.4), and one obtains a solution only for a given value¹. This value of $\alpha_g(\mu_g)$ enters phenomenological calculations, which then determine μ_g .

Hence the assumption of a given spin structure (2.1) and the neglect of the transverse part of the three gluon vertex leads to a DS equation which admits two solutions, and which breaks multiplicative renormalisability. In order to explain the discrepancy of the BBZ result with more recent ones, the assumed spin structure of the gluon propagator has been put into question [5]. However, there is no doubt that such solutions exist. Further solutions can of course exist, given that the equation is highly nonlinear. The neglect of the transverse part of the vertex, on the other hand, may have important consequences, which we shall explore in this paper for the quark propagator.

¹Note that there was a mistake of a factor 2 in our definition of α_g in Refs. [1]

3. The Dyson-Schwinger equation for quarks

Following their work with Baker on the gluon propagator [8], Ball and Zachariasen have considered the DS equation for the quark propagator [12]. Choosing again a purely longitudinal quark-gluon vertex, they found that the quark propagator corresponding to a $1/k^4$ gluon was suppressed, and became constant near the origin. Furthermore, they showed that a chiral solution was always possible, and argued that a solution could be found which would break chiral symmetry. We shall see that, in the case of a less singular gluon propagator, only the chiral solution is possible if one neglects the transverse part of the vertex. However, implementing the improvements proposed by Curtis and Pennington [9], we shall show that multiplicative renormalisability (and hence the transverse part of the vertex) leads to an equation that allows chiral symmetry breaking, and even suggests a range of parameters (describing the coupling and the gluon propagator) which would lead to a unique, massive, solution.

3.1. The Ball-Zachariasen equation

We define the propagator for a quark with momentum q as:

$$S(q) = F(q^2)\gamma \cdot q + G(q^2) \quad (3.1)$$

so that $F(q^2)$ represents the chiral-symmetry conserving part and $G(q^2)$ represents the chiral-symmetry breaking part. We shall also use the equivalent notation:

$$S(q) = \frac{\mathcal{F}(q^2) [\gamma \cdot q + \Sigma(q^2)]}{q^2 - \Sigma^2(q^2)} \quad (3.2)$$

The free propagator is therefore obtained by setting $F(q^2)$ to $1/q^2$ and $G(q^2)$ to zero (neglecting all current masses). In the axial gauge Eq. (3.1) is not the most general form for the quark propagator. First of all, the functions F and G can depend on $n \cdot q$ as well as q^2 . As in the case of the gluon propagator [8, 1], we seek solutions for which these functions are independent of $n \cdot q$. Furthermore, the gauge dependence can also arise through extra spin structures, proportional to $\gamma \cdot n$ and $n \cdot q$, which although absent for the free propagator can in general occur for the dressed propagator. As pointed out in ref. [12]), these extra terms drop out of the equations for F and G if one specializes to a gauge vector orthogonal to q . This gauge choice $n \cdot q = 0$ is the only one that makes the algebra sufficiently tractable that reliable solutions to the DS equations can be obtained. The question of the dependence of the quark propagator on n , although important, is difficult to address in the axial gauge, and beyond the scope of this paper.

The DS equation for $S(q)$, sketched in Fig. 1, in Euclidean space is given by

$$1 = \gamma \cdot q S_b(q) - C_F \alpha_b \int \frac{d^4 k}{4\pi^3 k^2} \mathcal{Z}_b(k^2) \gamma^\mu D_{\mu\nu}(k, n) S_b(q - k) \Gamma^\nu(q - k, q) S_b(q) \quad (3.3)$$

Figure 1: A pictorial representation of Eq. (3.3). The hatched circles represent the exact two-point functions, and the cross-hatched circle the exact three-point function.

where the subscript b on α , \mathcal{Z} and S indicate that these are bare quantities which will have to be renormalised. $C_F = 4/3$ is the quark Casimir invariant.

In general it is not possible to solve this equation because it involves the unknown quark-gluon vertex function $\Gamma^\nu(q-k, q)$. This vertex function is related to a four-point Green function via a DS equation. Thus we get an infinite tower of coupled integral equations. On the other hand, in the axial gauge, this quark-gluon vertex function connecting a quark of momentum $p \equiv q - k$ to one of momentum q through the absorption of a gluon of momentum k , obeys the Ward-Takahashi (WT) identity sketched in Fig. 2:

$$k^\nu \Gamma_\nu(p, q) = S(q)^{-1} - S(p)^{-1} \quad (3.4)$$

Figure 2: A representation of the Ward-Takahashi identity of Eq. (3.4) using the same convention as for Fig. 1.

One can then solve this equation to determine Γ_L^μ , the part of the vertex longitudinal to k . Following ref. [12], we obtain:

$$\begin{aligned} -S(q)\Gamma_\mu^L(p, q)S(p) &= \frac{1}{2} [F_b(p) + F_b(q)] \gamma^\mu + \frac{1}{2} [F_b(p) - F_b(q)] \frac{2\gamma \cdot q \gamma^\mu \gamma \cdot p}{p^2 - q^2} \\ &+ \frac{1}{2} [F_b(p) - F_b(q)] \frac{q^2 + p^2}{p^2 - q^2} \gamma^\mu \\ &+ [G_b(p) - G_b(q)] \frac{\gamma^\mu \gamma \cdot p + \gamma \cdot q \gamma^\mu}{p^2 - q^2} \end{aligned} \quad (3.5)$$

The simplest approach consists, as suggested in ref. [13], in neglecting the transverse part of the vertex: one then assumes that the integral equation is dominated by the part (3.5) which is determined from the above WT identity.² The resulting integral equations for the two functions $F_b(q^2)$ and $G_b(q^2)$ separate and we have

$$\begin{aligned} 1 &= q^2 F_b(q^2) - C_F \alpha_b \int \frac{d^4 k}{4\pi^3 k^2} \mathcal{Z}_b(k^2) D_{\mu\nu}(k, n) \left\{ \frac{F_b(q^2) + F_b(p^2)}{2} \delta_{\mu\nu} \right. \\ &+ \left. \frac{1}{(q^2 - p^2)} [F_b(q^2) - F_b(p^2)] \left(\frac{1}{2} k^2 \delta_{\mu\nu} + q_\mu p_\nu + p_\mu q_\nu \right) \right\} \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} 0 &= G_b(q^2) - C_F \alpha_b \int \frac{d^4 k}{4\pi^3 k^2} \mathcal{Z}_b(k^2) D_{\mu\nu}(k, n) \\ &\left\{ \frac{1}{(q^2 - p^2)} [G_b(q^2) - G_b(p^2)] (q \cdot k \delta_{\mu\nu} + q_\mu p_\nu + p_\mu q_\nu) \right\} \end{aligned} \quad (3.7)$$

²In other gauges the Ward identity Eq. (3.4) must be replaced by the full Slavnov-Taylor identities [14] which involve the interactions of Faddeev-Popov ghosts. Nevertheless it was argued in ref. [15] that the contributions from Faddeev-Popov ghosts are small and so the Ward identity of Eq. (3.4) was imposed on the integral equation formulated in a covariant gauge.

where again we have introduced the subscript b on F and G to indicate that these quantities are to be renormalised. (There is no renormalisation for Σ since this function is zero for the free propagator, i.e. in the absence of an explicit current mass there is no parameter in the QCD Lagrangian to renormalise, hence the ratio F/G does not get renormalised).

3.2. Multiplicative renormalisability and the transverse part of the vertex

As was observed in [12], Eq. (3.6, 3.7) are not multiplicatively renormalisable. In other words, if we renormalise the quark and gluon wavefunctions by imposing $F_b(q^2) = F_b(\mu_f^2) F_R(q^2)$, $G_b(q^2) = F_b(\mu_f^2) G_R(q^2)$ and $\mathcal{Z}_b(q^2) = \mathcal{Z}_b(\mu_g^2) \mathcal{Z}_R(q^2)$, additional terms need to be introduced in the definition of the renormalised coupling, as in Eq. (2.4). These terms are not the same in the quark and in the gluon case, so one loses not only multiplicative renormalisability, but also the universality of the QCD coupling constant!

These problems come from the ultraviolet region, and can be traced back to the neglect of the transverse part of the vertex in the solution of the WT identities: the vertex (3.5) does not match the perturbative one at high momentum transfer. Hence, it is necessary to postulate a transverse part that will restore multiplicative renormalisability. Curtis and Pennington have shown [9] that this goal can be achieved in QED by considering the perturbative limit of the vertex. Their argument can be trivially extended to QCD in the axial gauge: the one-loop corrections to the propagator are identical to those of QED up to the quark Casimir invariant C_F , and the vertex corrections are also the same: because the vertex and the wavefunction renormalisation constants $Z_1 = Z_2$ are equal in this gauge, and because Z_2 is a function of C_F only, the diagrams involving the three-gluon vertex, which depend on $C_F - C_A/2$, with C_A the adjoint Casimir invariant, have to cancel. Hence the argument is totally similar to that of ref. [9] and goes as follows.

In general, the quark propagator has the perturbative limit

$$\lim_{q^2 \rightarrow \infty} \mathcal{F}(q^2) = 1 + \frac{\alpha_S \xi}{4\pi} \ln \frac{q^2}{\Lambda^2} \quad (3.8)$$

with ξ the anomalous dimension. The one-loop vertex can be shown [9] to tend to

$$\begin{aligned} \lim_{q^2/p^2 \rightarrow \infty} \Gamma_\mu^{pert}(p, q) &= \gamma_\mu \left[1 - \frac{\alpha_S \xi}{4\pi} \ln \left(\frac{q^2}{\Lambda^2} \right) \right] \\ &- \frac{\alpha_S}{4\pi} \frac{\ln \left(\frac{q^2}{\Lambda^2} \right)}{q^2} (\gamma \cdot p \gamma_\mu \gamma \cdot p - q_\mu \gamma \cdot p + \xi q_\mu \gamma \cdot p) \end{aligned} \quad (3.9)$$

One can compare this answer with the vertex (3.5) in the same limit, to conclude that the transverse part must give:

$$\lim_{q^2/p^2 \rightarrow \infty} \Gamma_\mu^T(p, q) = \frac{\alpha_S \xi \ln \left(\frac{q^2}{\Lambda^2} \right)}{4\pi q^2} (-q_\mu \gamma \cdot p + \gamma \cdot q \gamma_\mu \gamma \cdot p) \quad (3.10)$$

This tensor is indeed transverse to $q \approx k$. One can then extend the tensor structure of this vertex so that it becomes transverse to k for any value of p and q . We find that the simplest extension is:

$$S(p)\Gamma_\mu^T(p, q)S(q) = (\mathcal{F}(q) - \mathcal{F}(p)) \frac{\gamma_\mu(q^2 - p^2) - (q_\mu + p_\mu)(\gamma \cdot q - \gamma \cdot p)}{\mathcal{D}} \quad (3.11)$$

where \mathcal{D} is an expression symmetric in p and q , and behaving as q^4 in the large- q^2 limit. \mathcal{D} must not introduce any singularity, must be symmetric in q and p , and must satisfy $\Gamma_\mu^T(p, p) = 0$. In general,

$$\mathcal{D} = (q^2 + p^2)^2 \eta \left(\frac{q \cdot p}{(q^2 + p^2)}, \frac{q^2 p^2}{(q^2 + p^2)^2} \right) \quad (3.12)$$

with η a regular function such that $\eta(0, 0) = 1$.

It is worth noting that the tensor structure of the transverse vertex (3.11) can be generalised. Indeed, Ball and Chiu [10] have given a set of 8 independent tensor structures spanning the space of regular transverse vertices. Only three of these have the correct helicity structure to contribute to the DS equation in the chiral limit, and are (in the notation of Ball and Chiu):

$$\begin{aligned} T_2^\mu &= [p^\mu(k \cdot q) - q^\mu(p \cdot k)]\gamma \cdot (p + q) \\ T_3^\mu &= k^2 \gamma^\mu - k^\mu \gamma \cdot k \\ T_6^\mu &= \gamma^\mu(q^2 - p^2) - (q + p)^\mu \gamma \cdot (q - p) \end{aligned} \quad (3.13)$$

The transverse vertex given by Eq. (3.11) corresponds to:

$$\Gamma^T(p, q) = \frac{1}{\mathcal{D}} \left(\frac{1}{\mathcal{F}(q)} - \frac{1}{\mathcal{F}(p)} \right) \left(\frac{1}{2}(p^2 + q^2)T_6 + \frac{1}{2}(p^2 - q^2)T_3 + \dots \right) \quad (3.14)$$

where the ellipses refer to terms that vanish in the DS equation. This differs from the structure of ref. [9], where the transverse vertex was chosen to be proportional to T_6 . This can be understood from the fact that the expression of the longitudinal part of the vertex used here differs from theirs, by a transverse tensor proportional to T_3 , in such a way that we get the same high- q^2 or p^2 leading term for the total vertex. The addition of such extra subleading terms is always possible, and we shall investigate the effect of these through the variation of function \mathcal{D} in Eq. (3.11).

The transverse vertex (3.11), together with the longitudinal one (3.5), can then be included in Eq. (3.3) to obtain:

$$\begin{aligned} 1 &= p^2 F_b(p) + C_F \alpha_b \int \frac{d^4 k}{4\pi^3 k^2} D_{\mu\nu}^b \mathcal{Z}_b(k^2) \\ &\left\{ -\frac{1}{2} [F_b(p) + F_b(q)] \delta_{\mu\nu} - \frac{F_b(p) - F_b(q)}{p^2 - q^2} \left(p_\nu q_\mu + p_\mu q_\nu + \frac{1}{2}(p - q)\delta_{\mu\nu} \right) \right. \\ &\left. + \frac{(p^2 F_b(p) - q^2 F_b(q)) (\delta_{\mu\nu}(q^2 - p^2) + p_\mu p_\nu - q_\mu q_\nu + p_\mu q_\nu - p_\nu q_\mu)}{\mathcal{D}} \right\} \end{aligned} \quad (3.15)$$

4. Solving the equation

4.1. The angular integral

The angular integral in Eqs. (3.6, 3.7, 3.15) cannot be performed analytically without prior knowledge of the function F , since this occurs with the argument $(q - k)^2$. The approximation, first proposed by Schoenmaker [11] in the gluon case, consists in replacing a function $f((q - k)^2)$ by $f(q^2 + k^2)$. This is clearly valid in the regions $k^2 \gg q^2$ and $k^2 \ll q^2$. One then needs to find a function f which is sufficiently slowly varying that the error generated by the integration over the whole range of k^2 be negligible.

The first choice would be $f = F$. In this case, the contribution of the longitudinal part of the quark-gluon vertex to the DS equation vanishes. Hence the BZ equations (3.6, 3.7) admit only the trivial solution $F(q^2) = 1/q^2$, $G = 0$. If the transverse part of the vertex is added, the resulting equation (3.15) leads to a propagator close to the perturbative one, with a pole at $k^2 = 0$. Neither solution is a slowly varying function, hence the approximation is not justified in this case.

In the following, we shall use $f = \mathcal{F}$. Our solutions show that this assumption is reasonable so that the above ansatz is justified *a posteriori*. The only region where this approximation may have led to substantial errors would be for small $q^2 \approx k^2$, but then the integrand is suppressed because (as can be seen from Eq. (2.5)) the function $\mathcal{Z}_R(k^2)$ vanishes. This approximation keeps the contributions both from the longitudinal and from the transverse part of the vertex, hence we shall obtain nontrivial solutions both for Eqs. (3.6, 3.7) and (3.15).

With this approximation the angular part of the integration over momentum k in Eqs. (3.6, 3.7, 3.15) may be performed, and both equations can be recast into the following form if $G = 0$:

$$1 = q^2 F_b(q^2) - \frac{C_F \alpha_b}{4\pi} \int dk^2 \left[\mathcal{Z}(k^2) \Delta_1(k^2/q^2) F_b(q^2) + \mathcal{Z}(k^2) \Delta_2(k^2/q^2) F_b(k^2 + q^2) \right] \quad (4.1)$$

The kernels Δ_1 and Δ_2 are given, both in the case of Eqs. (3.6, 3.7) and in the case of Eq. (3.15) in the Appendix. This is a Fredholm equation of the second kind which must now be solved numerically. Before we can do this, however, we must consider the question of the renormalisations required to absorb the ultraviolet divergences.

4.2. Renormalisation

The renormalisation procedure is similar whether one neglects the transverse part of the vertex or not. The renormalisation constant $\mathcal{Z}_b(\mu_g^2)$ has been introduced in Eq. (2.3). As mentioned in Section 2 we choose this constant such that the renormalised function,

$\mathcal{Z}_R(q^2)$ takes the value 1 at $q^2 = \mu_g^2$, with μ_g taken to be 0.8 GeV. We likewise introduce a renormalisation constant $\mathcal{F}_b(\mu_f)$ such that

$$F_R(q^2) = \frac{F_b(q^2)}{\mathcal{F}_b(\mu_f)}$$

and

$$G_R(q^2) = \frac{G_b(q^2)}{\mathcal{F}_b(\mu_f)}$$

are ultraviolet finite and we choose it so that the quantity $\mathcal{F}_R(q^2)$ takes the value 1 at $q^2 = \mu_f^2$. Eq. (4.1) then becomes finite if one rewrites it in terms of the coupling

$$\alpha_f = \frac{\alpha_b \mathcal{Z}(\mu_f)}{1 - \alpha_b \mathcal{Z}(\mu_f) C_F / 4\pi \int dk^2 \mathcal{Z}(k^2) \Delta_1(k^2, \mu_f^2)} \quad (4.2)$$

and the renormalised equation (4.1) becomes:

$$\begin{aligned} 1 &= \left(1 - \frac{C_F \alpha_f}{4\pi} \int dk^2 \mathcal{Z}_R(k^2) \left[\Delta_1(k^2, q^2) - \Delta_1(k^2, \mu_f^2) \right] \right) F_R(q^2) \\ &- \frac{C_F \alpha_f}{4\pi} \int dk^2 \mathcal{Z}_R(k^2) \left(\Delta_2(k^2, q^2) F_R(k^2 + q^2) - \Delta_2(k^2, \mu_f^2) F_R(k^2 + \mu_f^2) \right) \end{aligned} \quad (4.3)$$

1. Renormalised BZ equation

In the case of the BZ kernels, the UV divergence of the equation is concentrated in the Δ_1 term, which has a log singularity as $k^2 \rightarrow \infty$. This means that the coupling (4.2) cannot be related to the usual renormalised QCD coupling

$$\alpha_S(\mu_f) = Z_b(\mu_f) \alpha_b \quad (4.4)$$

Hence one loses the universality of the QCD coupling.

Furthermore, the equation for the chirality breaking term G is identical to Eq. (4.3) with the replacement $F \rightarrow G$, $1 \rightarrow 0$. The leading behaviour at high q^2 comes from the remnant of the UV term. If we assume, in agreement with the renormalisation group, that $\mathcal{Z}_R(q^2) \sim 1/[q^2 \log(q^2)]$ as $q^2 \rightarrow \infty$, the leading terms of the equation for the G term are:

$$0 \approx G_R(q^2) - \frac{C_F \alpha_f}{4\pi} \log(\log(q^2)) G_R(q^2) \quad (4.5)$$

which has only $G = 0$ as a consistent solution. Hence, unless the gluon propagator is highly singular, the BZ equation does not lead to chiral symmetry breaking.

If one overlooks these problems, and goes ahead to solve the equation, one obtains a function $F_R(q^2)$ which is very close to the perturbative $1/q^2$. The propagator in this case keeps a pole, the residue of which is slightly bigger than the perturbative one.

As we shall see, the chiral solution will continue to possess these properties, even after we get a consistent equation for the quark propagator by the introduction of the transverse part of the vertex.

2. A multiplicatively renormalised equation

The main effect of the inclusion of the transverse vertex is to shift the divergence from the integral of Δ_1 to that of $\Delta_2 F$ in Eq. (4.3). Indeed, the integral over k^2 of Δ_2 in Eq. (4.1) now diverges logarithmically since $\Delta_2(k^2, q^2)$ behaves like $1/k^2$ as $k^2 \rightarrow \infty$, whereas the integral of the Δ_1 term is finite.

This fact means that the renormalised coupling (4.2) is related to the usual one (4.4) by a finite renormalisation:

$$\alpha_f = \frac{\alpha_S(\mu_f)}{1 - \alpha_S(\mu_f) C_F / 4\pi \int dk^2 \mathcal{Z}(k^2) \Delta_1(k^2, \mu_f^2)} \quad (4.6)$$

Hence we can now write our equation in terms of the true QCD coupling:

$$\begin{aligned} \frac{1}{\mathcal{F}_b(\mu_f)} &= q^2 F_R(q^2) \\ &- \frac{C_F \alpha_S(\mu_f)}{4\pi} \int dk^2 \left[\mathcal{Z}_R(k^2) \Delta_1(k^2, q^2) F_R(q^2) \right. \\ &\left. + \mathcal{Z}_R(k^2) \Delta_2(k^2, q^2) F_R(k^2 + q^2) \right] \end{aligned} \quad (4.7)$$

Setting $q^2 = \mu_f^2$ in the above, we obtain:

$$\frac{1}{\mathcal{F}_b(\mu_f)} = 1 - \frac{C_F \alpha_S(\mu_f)}{4\pi} \int dk^2 \left[\mathcal{Z}_R(k^2) \frac{\Delta_1(k^2, \mu_f^2)}{\mu_f^2} + \mathcal{Z}_R(k^2) \Delta_2(k^2, \mu_f^2) F_R(k^2 + \mu_f^2) \right] \quad (4.8)$$

We can then equate both expressions for $1/\mathcal{F}_b(\mu_f)$ to obtain:

$$\begin{aligned} 1 &= q^2 F_R(q^2) \\ &- \frac{C_F \alpha_S(\mu_f)}{4\pi} \int dk^2 \left[\mathcal{Z}_R(k^2) \Delta_1(k^2, q^2) F_R(q^2) - \mathcal{Z}_R(k^2) \frac{\Delta_1(k^2, \mu_f^2)}{\mu_f^2} \right] \\ &- \frac{C_F \alpha_S(\mu_f)}{4\pi} \int dk^2 \mathcal{Z}_R(k^2) \left[\Delta_2(k^2, q^2) F_R(k^2 + q^2) - \Delta_2(k^2, \mu_f^2) F_R(k^2 + \mu_f^2) \right] \end{aligned} \quad (4.9)$$

As can be seen the large k^2 behaviour of Δ_2 cancels out and the integral is now ultraviolet convergent.

5. The chiral solution

5.1. Asymptotic behaviour

The first test of the consistency of our results is a direct comparison of equation (4.9) with perturbation theory, *i.e.* for $q^2 \rightarrow \infty$. As we have explained, before subtractions, the

equation is ultraviolet divergent because of the terms proportional to Δ_2 . After subtraction, the leading $\log q^2$ comes from those terms in Δ_2 that behave like $1/k^2$. The equation then becomes, for $q^2 \rightarrow \infty$:

$$1 \approx q^2 F_R(q^2) - \frac{C_F \alpha_S}{4\pi} \int_{\mu_f^2}^{q^2} \frac{3dk^2}{2k^2} F_R(k^2) [\mathcal{Z}_R(k^2)] \quad (5.1)$$

Writing $q^2 F_R(q^2) \approx 1 + \xi \log(q^2)$, $Z_R \approx 1$, one then gets a consistent solution to order $\xi \sim \alpha_S$, provided that

$$\xi = \frac{3C_F \alpha_S}{8\pi} \quad (5.2)$$

in agreement with one-loop results (in axial gauge).

Furthermore, the equation also agrees with RG-improved perturbation theory. Assuming that for large q^2 one has $Z_R(q^2) \approx \log(q^2)^{-1}$, one then gets a consistent asymptotic behaviour $q^2 F_R(q^2) \sim C + \log(q^2)^{-\xi}$, with C a constant, and ξ still given by Eq. (5.2). Hence we see that Eq. (4.9) encompasses our perturbative knowledge of propagators.

Finally, as we shall now explain, it suggests that the $G = 0$ solution cannot be valid for arbitrary values of $\alpha_S(\mu_f)$.

5.2. Critical value of α_S and chiral symmetry breaking

By shifting the k^2 integration in the terms proportional to Δ_2 , one can recast the equation in the following form:

$$\phi(q^2) \mathcal{F}_R(q^2) = \phi(\mu_f^2) + \int dk^2 \mathcal{K}(q^2, k^2) \mathcal{F}_R(k^2) \quad (5.3)$$

with:

$$\begin{aligned} \phi(q^2) &= 1 - \frac{C_F \alpha_S}{4\pi} \int dk^2 \left[\mathcal{Z}_R(k^2) \frac{\Delta_1(k^2, q^2)}{q^2} \right] \\ \mathcal{K}(q^2, k^2) &= \frac{C_F \alpha_S}{4\pi} \frac{1}{k^2} \left[\mathcal{Z}_R(k^2 - q^2) \Delta_2(k^2, q^2) - \mathcal{Z}_R(k^2 - \mu_f^2) \Delta_2(k^2, \mu_f^2) \right] \end{aligned} \quad (5.4)$$

In the case where $\phi(q^2) \neq 0$ for all q^2 , the equation can be reduced to a Fredholm equation of the second kind. Its kernel \mathcal{K} is integrable and bounded, so that there is a unique solution. We obtain it by discretising Eq. (5.3) and inverting the matrix equation thus obtained, which is equivalent to Fredholm's solution [17]. We then get a smoother solution by introducing the obtained points as the input of an iterative method, where we use the left-hand side of Eq. (5.3) as the output, and the left-hand side as the input, which is the Liouville-Neumann method [18]. This converges nicely as long as $\phi(q^2) \neq 0$ for all q^2 .

Figure 3: Massless quark solutions to the Dyson-Schwinger equations exist in the shaded region, the boundary of which is the value of α_S for which the equation becomes singular for some value of q^2 , given the mass squared σ which enters the Källén-Lehmann representation of the gluon propagator. The thick curve shows the values of α_S at which the effective quark coupling (4.2) becomes infinite. The curves are for $\eta = 1$.

When $\phi(q^2)$ has a zero, we have a Fredholm equation of the third kind. In that case, both the Fredholm solution, and the Liouville-Neumann iterations fail. This is because, effectively, the Fredholm solution involves integrals of $\int dq^2 K(q^2, k^2)/\phi(q^2)$, which are ill-defined, whereas the Liouville-Neumann solution involves a series with terms of the form $\int dk^2 K(q^2, k^2)\phi(k^2)/\phi(q^2)$, which clearly diverges near the zero of ϕ .

In fact it is clear that the solution dramatically changes. When $\phi(\mu_f^2) = 0$, the solution \mathcal{F} can at best be determined up to a constant as the equation becomes homogeneous. Furthermore, it is not in general possible to remove the pole that the zero of ϕ introduces in \mathcal{F} , hence the propagator develops an imaginary part, which is not allowed in the t -channel. $\phi(q^2)$ is 1 at $\alpha_S = 0$ and steadily decreases until it reaches a zero value (Note that $\Delta_1(k^2, q^2) \rightarrow 0$ as $q^2 \rightarrow 0$, hence ϕ is finite for all q^2). ϕ becomes zero at small q^2 first, and once $\phi(0)$ has crossed zero, then there will be a zero of $\phi(q^2)$ at some nonzero q^2 for larger values of α_S . Therefore there is a critical value of α_S past which ϕ is not positive definite.

Physically, one can view the divergence of the kernel as the divergence of the effective coupling $\alpha_f(q^2)$ of Eq. (4.2) at some value of q^2 . This divergence suggests that the chiral equation stops having physically relevant solutions, and hence that chirality needs to be broken past a certain critical value of the coupling. One should really speak of the value of $\alpha_S(\mu)\mathcal{Z}_R(\mu)$, as one can always change the value of α_S according to Eq. (2.6). This interpretation is reinforced when one realizes that the singularity of the kernel is entirely due to the fact that we have neglected the terms proportional to G . If these are reintroduced, the singularity will disappear, and one will keep a solution. Hence, the singularity is the place at which the quarks develop a mass.

We can in fact study this for a general gluon propagator, and hence not limit ourselves to the solution found in ref. [1]. We simply need to assume that the gluon propagator has a Källén-Lehmann representation:

$$\frac{\mathcal{Z}(q^2)}{q^2} = \int d\sigma \frac{\rho(\sigma)}{q^2 + \sigma} \quad (5.5)$$

$\phi(q^2)$ can then be written, using the condition $\mathcal{Z}(\mu_g) = 1$:

$$\begin{aligned} \phi(q^2) &= \int d\sigma \rho(\sigma) \Phi(\sigma, q^2) \\ \text{with } \Phi(\sigma, q^2) &= \left\{ \frac{\mu_g^2}{\mu_g^2 + \sigma} - \frac{C_F \alpha_S}{4\pi} \int dk^2 \Delta_1(k^2, q^2) \frac{1}{k^2 + \sigma} \right\} \end{aligned} \quad (5.6)$$

We show in Fig. 3 the region in the $(\alpha_S, \sigma/\mu_g^2)$ plane in which with $\Phi(\sigma, q^2)$ is positive

Figure 4: The solutions for massless quark propagators that come from the nonperturbative gluon propagator of ref. [1], for $\alpha_s(mf) = 0.2$ (plain), 0.6 (dashed), 1.0 (dot-dashed) and 1.4 (dashed).

for all q^2 . This means that if $\rho(\sigma)$ has support in an interval $[\sigma_0, \sigma_1]$, then there will be massless solutions if $\alpha_S \leq \alpha_S^{critical}(\sigma_1)$.

We have plotted Fig. 3 for the function η of Eq. (3.12) equal to one. It is easy to see that a critical α_S will exist for a broad range of η and that its value can be calculated from Fig. 3. Indeed, as shown in the Appendix, for $\eta\left(0, \frac{q^2 p^2}{(q^2 + p^2)^2}\right)$, we can write $\Delta_1 = \eta\Delta_1^T + \Delta_1^L$. Unless η is such that the sign of

$$I(\eta) = \int dk^2 \frac{[\Delta_1^L(k^2/q^2) + \eta\Delta_1^T(k^2/q^2)]}{(k^2 + \sigma)} \quad (5.7)$$

changes, there will exist a critical α_S . Its value will be $\alpha_c(\eta) = \alpha_c(\eta = 1)I(\eta = 1)/I(\eta)$. We therefore conclude that there exists a wide range of choices for the transverse vertex which give rise to chiral symmetry breaking.

It is of course true that even for Φ negative in part of $[\sigma_0, \sigma_1]$ it is possible to get $\phi > 0$, hence the condition is sufficient only: in the shaded region, there will be massless solutions. It is interesting to note that large- σ modes in the Källén-Lehmann density of the gluon imply chiral symmetry breaking in the quark sector. Only for a specific gluon propagator can one find the exact value of α_S beyond which these solutions do not exist anymore. This is what we are going to do in the next section.

5.3. Chiral solution for a specific gluon propagator

As explained in Section 2, the gluon propagator (2.5) that we found in ref. [1] has the problems linked with the breakdown of multiplicative renormalisability. The procedure used here to obtain a consistent ansatz for the transverse part of the vertex can be extended to the gluon case. However, one may hope that the effect on gluons will be less dramatic than that on quarks. Indeed, we found that imposing the true asymptotic behaviour as $q^2 \rightarrow \infty$ does not appreciably change the behaviour of the propagator at moderate q^2 (compare the solution of the first paper of ref. 1 with Eq. (2.5)).

Hence we shall use that solution in the quark equation as an example of what the chiral solution looks like. We shall assume that both equations are renormalised at the same point, and that the two couplings have the same value. As we already explained, one should be able to derive a gluon propagator for any value of $\alpha_S(\mu)$ as it can always be changed according to Eq. (2.5). As our gluonic equation did not respect multiplicative renormalisability, we were not able to do so, and got an optimum $\alpha_S \approx 1.4$. To see what the effect of the coupling on the quark propagator is, we shall vary α_S independently of \mathcal{Z} , although the two are really correlated.

We show the result of this exercise in Fig. 4. As the value of the coupling grows, the quark propagator is enhanced near the origin. As α_S grows further, the propagator

Figure 5: Same as Fig. 4, but for with a transverse vertex multiplied by $\frac{(q^2+p^2)^2}{q^4+p^4}$.

experiences oscillations, until the chiral solution is lost. It is interesting to note that the value of α_S that we obtained for the gluon propagator is very close to the critical value beyond which the chiral solution disappears. We also see that the criterion of Section 5.2 can in practice work backwards, i.e. it not only predicts when there will be chiral solutions, but also when these will disappear.

In order to investigate the sensitivity of the solution to our choice of the function \mathcal{D} in Eq. (3.11), we consider two possible choices for that function. In Fig. 5, we show the behaviour of our solution for for $\mathcal{D} = q^4 + p^4$, instead of $\mathcal{D} = (q^2 + p^2)^2$ (as in Fig. 3). As expected, the two solutions have the same high- q^2 behaviour and are equal at $q^2 = \mu_g^2$. We see that the low-momentum behaviours vary by a modest amount, and that the solutions only differ in the region $q^2 \approx \mu_f^2$, and only when the integrand oscillates a lot. This is where we do not trust our approximation for the angular integrals and hence the result is stable where our approximations hold.

Figure 6: The dashed curve shows the value of α_c for various functions η (see text), and the plain curve shows the value of the intercept of the propagator for $\alpha_S = 0.75 \alpha_c$.

Finally, Fig. 6 illustrates the insensitivity of our results to the choice of transverse vertex. We consider the functions $\eta = [(q^4 + p^4)/(q^2 + p^2)^2]^n$ and plot the results in terms of n . We show the variation of the critical value of α_S , which changes by a factor 2 when n changes by a factor 100. We also show the value of the intercept at the origin of the propagator calculated at $\alpha_S = 0.75\alpha_c$, and which hardly changes with n . This clearly illustrates that the results we have obtained hold for a wide class of transverse vertices.

6. Conclusion

We have shown that the transverse part of the vertex plays an essential role in the quark DS equation in the axial gauge. It restores multiplicative renormalisability, allows chiral symmetry breaking and provides solutions which match with perturbation theory at large q^2 . Hence we have obtained an equation that possesses all the properties that are required in QCD.

The solution that we obtain in the chiral phase has a pole at $q^2 = 0$ despite the fact that we expect quarks to be confined. This contrasts with the results of a similar analysis carried out in ref. [15] for the gluon propagator of ref. [8] in which it was shown that for the *confining* propagator of ref. [8] the quark propagator does not have a singular behaviour at $q^2 = 0$ and can therefore be considered to be confined. The present formalism can be used to solve for the chirality breaking solution, as we shall explain in a future publication [16]. It is an interesting question to see whether these solutions are confined, and hence whether

confinement and chiral symmetry breaking are related.

It has not so far been possible to obtain a solution to the (modified) DS equations for the gluon propagator for time-like momenta. This is because we expect that in this regime the propagator develops an imaginary part with cuts corresponding to the thresholds for glueball production etc. and consequently such a solution involves coupled non-linear equations for the real and imaginary parts of $\mathcal{Z}_R(k^2)$. Therefore the corresponding solution for the quark propagator we have obtained is only valid for quarks propagating with space-like momenta (in the t-channel). It is tempting to try to analytically continue the solution obtained into time-like momenta by fitting the solution obtained to known analytic functions and possibly then using the DS equation for the quark propagator to extract information about the gluon propagator with time-like momenta. Unfortunately this has not been possible. We have obtained several very accurate fits (errors nowhere worse than 2%) to the curve shown in Fig. 4, using different parametrisations inside various analytic functions. Whereas these all fit the quark propagator remarkably well in the fit region they give wildly different projected behaviours for the quark propagator for time-like momenta. We therefore have to accept for the moment that the important problem of describing the exchange of soft gluons and quarks in the s-channel remains unsolved.

As we explained earlier, the equation for the gluon propagator also will have to be modified to incorporate a transverse vertex and recover multiplicative renormalisability. Furthermore, as has been pointed out in ref. [19] the DS equations for the quark and gluon propagators are coupled and after having obtained the solution for the quark propagator one must check the assumption that one may neglect quark loops in the equation for the gluon propagator. In ref. [19] it was shown that for the gluon propagator of ref. [8] this was not the case and the gluon propagator had to be modified accordingly. A similar analysis should also be carried out for the gluon and quark propagators discussed here.

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7. Appendix: Kernels

The kernels entering Eq. (4.1) are given by the following expressions, where we define $\rho = k^2/q^2$.

For the BZ equation (3.6):

$$\Delta_1(\rho) = \frac{\rho}{4} - \theta(\rho - 4) \sqrt{1 - \frac{4}{\rho}} \left(\frac{\rho}{4} + \frac{1}{2} \right) \quad (7.1)$$

and

$$\begin{aligned} \Delta_2(\rho) = & \theta(1 - \rho) \frac{\rho(\rho + 1)}{4} + \theta(\rho - 1) \left(\frac{3}{2} \left(1 + \frac{1}{\rho} \right) - \frac{\rho(\rho + 1)}{4} \right) \\ & + \theta(\rho - 4)(1 + \rho) \sqrt{1 - \frac{1}{\rho}} \left(\frac{\rho}{4} + \frac{1}{2} \right) \end{aligned} \quad (7.2)$$

For Eq. (3.15), and for η a function of $\frac{q^2 p^2}{(q^2 + p^2)^2} \approx \frac{1 + \rho}{(2 + \rho)^2}$ only:

$$\begin{aligned} \Delta_1(\rho) = & \frac{\eta}{4\rho} \left[-\rho - 8 + \frac{\rho^2 + 16 + 2\rho}{\sqrt{\rho^2 + 4}} \right] \\ & + \frac{1}{4} \left[\frac{-\rho^2 + 2\rho + 8}{\sqrt{\rho^2 - 4\rho}} + \rho \right] \\ & + \theta(4 - \rho) \frac{1}{4} \frac{\rho^2 - 2\rho - 8}{\sqrt{\rho^2 - 4\rho}} \end{aligned} \quad (7.3)$$

$$\begin{aligned} \Delta_2(\rho) = & \frac{\eta}{4\rho} \left\{ 9\rho + \rho^2 + 8 - \frac{1}{\sqrt{\rho^2 + 4}} [\rho^3 + \rho^2 + 18\rho + 16] \right\} \\ & + \theta(\rho - 1) \frac{3}{2\rho} \\ & + \theta(1 - \rho) \frac{-3 + \rho^2 + \rho}{2} \\ & + \theta(4 - \rho) \frac{-\rho^3 + \rho^2 + 10\rho + 8}{4\sqrt{\rho^2 - 4\rho}} \\ & + \frac{6 - \rho^2 - \rho}{4} + \frac{\rho^3 - \rho^2 - 10\rho - 8}{4\sqrt{\rho^2 - 4\rho}} \end{aligned} \quad (7.4)$$

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THE QUARK PROPAGATOR FROM THE DYSON-SCHWINGER EQUATIONS:

I. THE CHIRAL SOLUTION

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Abstract

Within the framework of the Dyson-Schwinger equations in the axial gauge, we study the effect that non-perturbative glue has on the quark propagator. We show that Ward-Takahashi identities, combined with the requirement of matching perturbative QCD at high momentum transfer, guarantee the multiplicative renormalisability of the answer. Technically, the matching with perturbation theory is accomplished by the introduction of a transverse part to the quark-gluon vertex. We show that this transverse vertex is crucial for chiral symmetry breaking, and that massless solutions exist below a critical value of α_S . Using the gluon propagator that we previously obtained, we obtain small corrections to the quark propagator, which keeps a pole at the origin in the chiral phase.

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1. Introduction

The Dyson-Schwinger (DS) equations of motion are one of the main tools for the investigation of non-perturbative effects. These equations are particularly suited to the study of the interface between perturbative and nonperturbative regimes, as they allow us to compute the evolution of Green functions when one enters the infrared region, and hence suggest modifications to the usual Feynman rules.

In a previous paper, we have studied [1] the DS equations in the quenched approximation, *i.e.* for pure gauge QCD, and found that, as was first suggested by Cornwall [2], some solutions for the gluon propagator are flatter than a pole in the infrared region. In order to avoid the question of ghost propagators, we worked in the axial gauge. Other groups [3, 4, 5] have also found that flatter solutions exist, in covariant gauges, although it is at present difficult to relate our results to theirs. The absence of a pole in the gluon propagator at $k^2 = 0$ is natural if one assumes that gluons do not propagate to infinity, *i.e.* these solutions should describe *confined* gluons. Furthermore, as was pointed out by Landshoff and Nachtmann [6], the existence of such solutions is highly desirable in phenomenological applications, and gives us a practical way to extend usual perturbative estimates to the strongly-interacting sector of the theory. The use of these solutions has already met with some success in diffractive calculations [7].

The DS equations constitute an infinite tower of integral relations between n-point functions. Hence, by themselves, they cannot be solved. For instance, if we consider the DS equation for the quark two-point function (the propagator), it involves a three-point function (the quark-gluon vertex) as well as the gluon two-point function (the propagator). One can imagine solving the pure-gauge DS equations for the gluon propagator as a first approximation, hence we shall assume here that the gluon propagator is known. However, resorting to higher-order DS equations to calculate the vertex will only bring in higher-order n-point functions, which are also a priori unknown. Hence one needs to use another input.

The simplest assumption is to take the perturbative γ_μ vertex, which is the so-called “rainbow” or “ladder” approximation. One can be however more sophisticated, and follow the observation of Baker, Ball and Zachariasen (BBZ) [8], namely that one needs to choose a vertex that will obey the Ward-Takahashi-Slavnov-Taylor identities. These constitute another nonperturbative statement of field theory, and hence must be valid in general.

In our study of the gluon propagator [1], we chose for the three-gluon vertex the simplest function that would obey the Ward-Slavnov-Taylor identities. We showed that besides the original $1/k^4$ solution, there exists another solution, behaving like $1/k^{0.2}$ near $k^2 = 0$. To obtain a full picture of nonperturbative effects in QCD, one then needs to calculate the quark propagator corresponding to that gluon propagator. As we shall see, the problem of multiple solutions arises again, as for sufficiently small α_S , a chiral solution exists together with a massive one. We shall show that one can constrain the spectral density of the gluon propagator and the value of the coupling so that only the massive solution survives.

As the equation for the quarks is much simpler than that for the gluons, we investigate in more detail the effect of the ansatz for the quark-gluon vertex on the quark propagator. Namely, the minimal vertex, used by BBZ for gluons and by Ball and Zachariasen (BZ) for quarks [12], leads to the breakdown of multiplicative renormalisability. We demonstrate that, following the method of Curtis and Pennington (CP) [9], it is possible to recover it by choosing a specific form for the quark-gluon vertex, which agrees with perturbation theory in the ultraviolet region, and which generalizes the B(B)Z ansatz. We also show that massive solutions exist only for the CP vertex. Simpler ansätze lead to an inconsistent ultraviolet behaviour, unless the propagator is massless. We also explain under which conditions massless solutions will exist, even for the CP vertex. We proceed to solve the DS equation in the massless phase, and show that nonperturbative effects do not remove the pole of the quark propagator in the chiral phase of the QCD vacuum, suggesting that confined quarks can exist only after chiral symmetry breaking.

This paper is set out as follows. In Section 2, we review our results for the gluon propagator, and outline the formalism that lead to these. We point out the problems linked with the BBZ choice of a three-gluon vertex. In Section 3, we discuss the DS equation for the quark propagator in the axial gauge, and the approximations made when imposing the Ward-Takahashi identity. In Section 4, we consider the part of the propagator that preserves chiral symmetry. We show that the problem can be reduced to a one-dimensional integral equation, which we then renormalise. We show explicitly that we recover multiplicative renormalisability. In Section 5, we give a criterion for the existence of massless solutions, and then proceed to the numerical solution of the equation in the massless case, for the gluon propagator that we previously derived.

2. The Gluon Propagator

The fundamental observation [8] is that one can use the Ward-Slavnov-Taylor (WST) identities to obtain a closed equation from the DS equation, if one neglects quark loops and works with gluons only. In the axial gauge ($n \cdot A = 0$), where we do not have to worry about ghost degrees of freedom, the DS equation relates the propagator to the three- and four-point vertices.

The three-point vertex, $\Gamma_\mu^{(3)}$ can be split into a part $\Gamma_T^{(3)}{}_\mu$ transverse to the external gluon momentum k_μ , $\Gamma_T^{(3)} \cdot k = 0$, and a longitudinal part, $\Gamma_L^{(3)\mu}$, with $\Gamma_L^{(3)} \cdot k \neq 0$. As the vertex is a third rank tensor, depending on the three vectors entering the vertex, this split is not uniquely defined. The WST identities relate $\Gamma_L^{(3)}$ to the propagator, and these identities can be “solved”, hence producing an ansatz for the longitudinal part [8, 12]. It is this ansatz that we *define* as being the longitudinal part of the vertex in the following.

One can then obtain a closed equation for the propagator if one makes two assumptions. First, the full propagator is supposed to have the same spin and gauge structure as the free propagator:

$$D_{\mu\nu}(k^2) = -D_{\mu\nu}^0(k, n) \frac{\mathcal{Z}(k^2)}{k^2} \quad (2.1)$$

where

$$D_{\mu\nu}^0(k, n) = \delta_{\mu\nu} - \frac{k_\mu n_\nu + n_\mu k_\nu}{n \cdot k} + \frac{n \cdot n k_\mu k_\nu}{(n \cdot k)^2} \quad (2.2)$$

This spin structure projects out the four-point vertex contribution to the equation. If one further assumes that the three-point vertex is dominated by its longitudinal part $\Gamma_L^{(3)}$, which is itself known from the propagator via the WST identities, one obtains an equation which involves only the gluon propagator.

This equation still needs to be renormalised. One can show without approximation [8] that a propagator with the spin structure (2.1) has to be singular as $k^2 \rightarrow 0$. Thus the inverse propagator should vanish at the origin. This allows one to subtract the quadratic divergences, thus renormalising the gluon mass to zero and cancelling the tadpole graphs. One is then left with logarithmic divergences, which can be dealt with through wave function renormalisation at a renormalisation point μ_g , by defining

$$\mathcal{Z}(k^2) = \mathcal{Z}(\mu_g^2) \mathcal{Z}_R(k^2) \quad (2.3)$$

However, the resulting definition of the renormalised coupling constant α_g is slightly different from the usual one:

$$\alpha_g(\mu_g) = \frac{\alpha_b \mathcal{Z}(\mu_g^2)}{1 + \alpha_b \mathcal{Z}(\mu_g^2) [\int_b^\infty dk^2 \Psi(k^2, \mathcal{Z})]_{\mathcal{Z}=\mathcal{Z}_R}} \quad (2.4)$$

where α_b is the bare coupling and Ψ a function linear in \mathcal{Z} , the explicit form of which can be found in ref. [11]. It has to be noted at this point that Eq. (2.4) leads to two related problems. First of all, the propagator asymptotically behaves like $1/\log(k^2)^{11/16}$ as $k^2 \rightarrow \infty$ [8], which is not what is obtained in perturbation theory. Furthermore, one obviously loses multiplicative renormalisability. This has the nasty consequence that, if the same truncation is applied to the quark propagator, the renormalised couplings are not equal anymore, because the functions Ψ are different for quarks and gluons.

The solution to the resulting equation, found in ref. [8], has a very singular behaviour proportional to $1/k^4$ as the momentum squared goes to zero. This was advertised as a signal for a confining potential. It has to be pointed out that in order to renormalise the equation in the presence of a $1/k^4$ singularity, one needs to perform additional subtractions, and hence one obtains such solutions only if one assumes they exist. In ref. [1], we exploited the fact that the integral equation obtained in ref. [8] is non-linear and can in general admit several solutions. We solved the original equation, without additional subtractions, and obtained a gluon propagator which, although singular as $k^2 \rightarrow 0$ as required in the axial gauge, has a cut singularity, flatter than a pole. This describes a soft gluon which is confined rather than confining and can conveniently be used in models of the pomeron such as that suggested by Landshoff and Nachtmann [6]. This propagator may be written

in the axial gauge ($n \cdot A = 0$) as

$$\frac{Z_R(k^2)}{k^2} = \frac{\mu_g^{-2}}{a_1 \left(\frac{k^2}{\mu_g^2}\right)^{b_1} + a_2 \left(\frac{k^2}{\mu_g^2}\right)^{b_2} + c \ln \left[d \left(\frac{k^2}{\mu_g^2}\right) + e \right]} \quad (2.5)$$

The dimensionless constants a_1, b_1, a_2, b_2, c, d and e , are given in the Table. The mass scale μ_g is the value of momentum at which $Z_R(k^2)$ takes the value 1, so that the propagator matches the value of the free gluon propagator. It is not obtained from the DS equation, which is scale invariant, but rather from a fit to the total cross-section for proton-proton scattering calculated using this propagator. An optimum value of 0.8 GeV was found for μ_g . It is worth pointing out that most of the ideas discussed here can be applied to other theories, living at other mass scales, such as technicolour, and that one would only need to adjust μ_g to do so.

Table: The constants used in Eq. (2.5)

a_1	0.88	c	0.59
a_2	-0.95	d	2.1
b_1	0.22	e	4.1
b_2	0.86	$\alpha_g(\mu_g)$	1.4

Note that in the case of a multiplicatively renormalisable theory, the actual value of α_S should not matter. In principle, one could always rescale α_S by varying μ according to

$$\begin{aligned} \alpha_S(\mu') \mathcal{Z}_R(\mu'^2, q^2) &= \alpha_S(\mu) \mathcal{Z}_R(\mu^2, q^2) \\ \mathcal{Z}_R(\mu^2, q^2) &= \mathcal{Z}_R(\mu^2, \mu'^2) \mathcal{Z}_R(\mu'^2, q^2) \end{aligned} \quad (2.6)$$

where the first argument of \mathcal{Z} refers to the renormalisation scale: $\mathcal{Z}(\mu^2, \mu^2) = 1$. However, our gluonic solution does not obey these equations because of (2.4), and one obtains a solution only for a given value¹. This value of $\alpha_g(\mu_g)$ enters phenomenological calculations, which then determine μ_g .

Hence the assumption of a given spin structure (2.1) and the neglect of the transverse part of the three gluon vertex leads to a DS equation which admits two solutions, and which breaks multiplicative renormalisability. In order to explain the discrepancy of the BBZ result with more recent ones, the assumed spin structure of the gluon propagator has been put into question [5]. However, there is no doubt that such solutions exist. Further solutions can of course exist, given that the equation is highly nonlinear. The neglect of the transverse part of the vertex, on the other hand, may have important consequences, which we shall explore in this paper for the quark propagator.

¹Note that there was a mistake of a factor 2 in our definition of α_g in Refs. [1]

3. The Dyson-Schwinger equation for quarks

Following their work with Baker on the gluon propagator [8], Ball and Zachariasen have considered the DS equation for the quark propagator [12]. Choosing again a purely longitudinal quark-gluon vertex, they found that the quark propagator corresponding to a $1/k^4$ gluon was suppressed, and became constant near the origin. Furthermore, they showed that a chiral solution was always possible, and argued that a solution could be found which would break chiral symmetry. We shall see that, in the case of a less singular gluon propagator, only the chiral solution is possible if one neglects the transverse part of the vertex. However, implementing the improvements proposed by Curtis and Pennington [9], we shall show that multiplicative renormalisability (and hence the transverse part of the vertex) leads to an equation that allows chiral symmetry breaking, and even suggests a range of parameters (describing the coupling and the gluon propagator) which would lead to a unique, massive, solution.

3.1. The Ball-Zachariasen equation

We define the propagator for a quark with momentum q as:

$$S(q) = F(q^2)\gamma \cdot q + G(q^2) \quad (3.1)$$

so that $F(q^2)$ represents the chiral-symmetry conserving part and $G(q^2)$ represents the chiral-symmetry breaking part. We shall also use the equivalent notation:

$$S(q) = \frac{\mathcal{F}(q^2)[\gamma \cdot q + \Sigma(q^2)]}{q^2 - \Sigma^2(q^2)} \quad (3.2)$$

The free propagator is therefore obtained by setting $F(q^2)$ to $1/q^2$ and $G(q^2)$ to zero (neglecting all current masses). In the axial gauge Eq. (3.1) is not the most general form for the quark propagator. First of all, the functions F and G can depend on $n \cdot q$ as well as q^2 . As in the case of the gluon propagator [8, 1], we seek solutions for which these functions are independent of $n \cdot q$. Furthermore, the gauge dependence can also arise through extra spin structures, proportional to $\gamma \cdot n$ and $n \cdot q$, which although absent for the free propagator can in general occur for the dressed propagator. As pointed out in ref. [12]), these extra terms drop out of the equations for F and G if one specializes to a gauge vector orthogonal to q . This gauge choice $n \cdot q = 0$ is the only one that makes the algebra sufficiently tractable that reliable solutions to the DS equations can be obtained. The question of the dependence of the quark propagator on n , although important, is difficult to address in the axial gauge, and beyond the scope of this paper.

The DS equation for $S(q)$, sketched in Fig. 1, in Euclidean space is given by

$$1 = \gamma \cdot q S_b(q) - C_F \alpha_b \int \frac{d^4 k}{4\pi^3 k^2} \mathcal{Z}_b(k^2) \gamma^\mu D_{\mu\nu}(k, n) S_b(q - k) \Gamma^\nu(q - k, q) S_b(q) \quad (3.3)$$

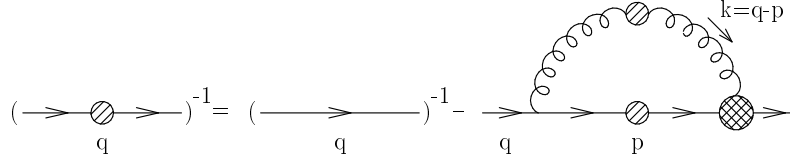


Figure 1: A pictorial representation of Eq. (3.3). The hatched circles represent the exact two-point functions, and the cross-hatched circle the exact three-point function.

where the subscript b on α , \mathcal{Z} and S indicate that these are bare quantities which will have to be renormalised. $C_F = 4/3$ is the quark Casimir invariant.

In general it is not possible to solve this equation because it involves the unknown quark-gluon vertex function $\Gamma^\nu(q-k, q)$. This vertex function is related to a four-point Green function via a DS equation. Thus we get an infinite tower of coupled integral equations. On the other hand, in the axial gauge, this quark-gluon vertex function connecting a quark of momentum $p \equiv q - k$ to one of momentum q through the absorption of a gluon of momentum k , obeys the Ward-Takahashi (WT) identity sketched in Fig. 2:

$$k^\nu \Gamma_\nu(p, q) = S(q)^{-1} - S(p)^{-1} \quad (3.4)$$

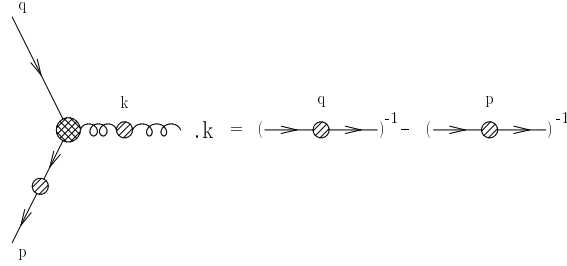


Figure 2: A representation of the Ward-Takahashi identity of Eq. (3.4) using the same convention as for Fig. 1.

One can then solve this equation to determine Γ_L^μ , the part of the vertex longitudinal to k . Following ref. [12], we obtain:

$$\begin{aligned} -S(q)\Gamma_\mu^L(p, q)S(p) &= \frac{1}{2}[F_b(p) + F_b(q)]\gamma^\mu + \frac{1}{2}[F_b(p) - F_b(q)]\frac{2\gamma \cdot q \gamma^\mu \gamma \cdot p}{p^2 - q^2} \\ &+ \frac{1}{2}[F_b(p) - F_b(q)]\frac{q^2 + p^2}{p^2 - q^2}\gamma^\mu \\ &+ [G_b(p) - G_b(q)]\frac{\gamma^\mu \gamma \cdot p + \gamma \cdot q \gamma^\mu}{p^2 - q^2} \end{aligned} \quad (3.5)$$

The simplest approach consists, as suggested in ref. [13], in neglecting the transverse part of the vertex: one then assumes that the integral equation is dominated by the part (3.5)

which is determined from the above WT identity.² The resulting integral equations for the two functions $F_b(q^2)$ and $G_b(q^2)$ separate and we have

$$\begin{aligned}
1 &= q^2 F_b(q^2) - C_F \alpha_b \int \frac{d^4 k}{4\pi^3 k^2} \mathcal{Z}_b(k^2) D_{\mu\nu}(k, n) \left\{ \frac{F_b(q^2) + F_b(p^2)}{2} \delta_{\mu\nu} \right. \\
&+ \left. \frac{1}{(q^2 - p^2)} [F_b(q^2) - F_b(p^2)] \left(\frac{1}{2} k^2 \delta_{\mu\nu} + q_\mu p_\nu + p_\mu q_\nu \right) \right\}
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
0 &= G_b(q^2) - C_F \alpha_b \int \frac{d^4 k}{4\pi^3 k^2} \mathcal{Z}_b(k^2) D_{\mu\nu}(k, n) \\
&\left\{ \frac{1}{(q^2 - p^2)} [G_b(q^2) - G_b(p^2)] (q \cdot k \delta_{\mu\nu} + q_\mu p_\nu + p_\mu q_\nu) \right\}
\end{aligned} \tag{3.7}$$

where again we have introduced the subscript b on F and G to indicate that these quantities are to be renormalised. (There is no renormalisation for Σ since this function is zero for the free propagator, i.e. in the absence of an explicit current mass there is no parameter in the QCD Lagrangian to renormalise, hence the ratio F/G does not get renormalised).

3.2. Multiplicative renormalisability and the transverse part of the vertex

As was observed in [12], Eq. (3.6, 3.7) are not multiplicatively renormalisable. In other words, if we renormalise the quark and gluon wavefunctions by imposing $F_b(q^2) = F_b(\mu_f^2) F_R(q^2)$, $G_b(q^2) = F_b(\mu_f^2) G_R(q^2)$ and $\mathcal{Z}_b(q^2) = \mathcal{Z}_b(\mu_g^2) \mathcal{Z}_R(q^2)$, additional terms need to be introduced in the definition of the renormalised coupling, as in Eq. (2.4). These terms are not the same in the quark and in the gluon case, so one loses not only multiplicative renormalisability, but also the universality of the QCD coupling constant!

These problems come from the ultraviolet region, and can be traced back to the neglect of the transverse part of the vertex in the solution of the WT identities: the vertex (3.5) does not match the perturbative one at high momentum transfer. Hence, it is necessary to postulate a transverse part that will restore multiplicative renormalisability. Curtis and Pennington have shown [9] that this goal can be achieved in QED by considering the perturbative limit of the vertex. Their argument can be trivially extended to QCD in the axial gauge: the one-loop corrections to the propagator are identical to those of QED up to the quark Casimir invariant C_F , and the vertex corrections are also the same: because the vertex and the wavefunction renormalisation constants $Z_1 = Z_2$ are equal in this gauge, and because Z_2 is a function of C_F only, the diagrams involving the three-gluon vertex,

²In other gauges the Ward identity Eq. (3.4) must be replaced by the full Slavnov-Taylor identities [14] which involve the interactions of Faddeev-Popov ghosts. Nevertheless it was argued in ref. [15] that the contributions from Faddeev-Popov ghosts are small and so the Ward identity of Eq. (3.4) was imposed on the integral equation formulated in a covariant gauge.

which depend on $C_F - C_A/2$, with C_A the adjoint Casimir invariant, have to cancel. Hence the argument is totally similar to that of ref. [9] and goes as follows.

In general, the quark propagator has the perturbative limit

$$\lim_{q^2 \rightarrow \infty} \mathcal{F}(q^2) = 1 + \frac{\alpha_S \xi}{4\pi} \ln \frac{q^2}{\Lambda^2} \quad (3.8)$$

with ξ the anomalous dimension. The one-loop vertex can be shown [9] to tend to

$$\begin{aligned} \lim_{q^2/p^2 \rightarrow \infty} \Gamma_\mu^{pert}(p, q) &= \gamma_\mu \left[1 - \frac{\alpha_S \xi}{4\pi} \ln \left(\frac{q^2}{\Lambda^2} \right) \right] \\ &- \frac{\alpha_S \ln \left(\frac{q^2}{\Lambda^2} \right)}{4\pi} \frac{1}{q^2} (\gamma \cdot p \gamma_\mu \gamma \cdot p - q_\mu \gamma \cdot p + \xi q_\mu \gamma \cdot p) \end{aligned} \quad (3.9)$$

One can compare this answer with the vertex (3.5) in the same limit, to conclude that the transverse part must give:

$$\lim_{q^2/p^2 \rightarrow \infty} \Gamma_\mu^T(p, q) = \frac{\alpha_S \xi \ln \left(\frac{q^2}{\Lambda^2} \right)}{4\pi q^2} (-q_\mu \gamma \cdot p + \gamma \cdot q \gamma_\mu \gamma \cdot p) \quad (3.10)$$

This tensor is indeed transverse to $q \approx k$. One can then extend the tensor structure of this vertex so that it becomes transverse to k for any value of p and q . We find that the simplest extension is:

$$S(p) \Gamma_\mu^T(p, q) S(q) = (\mathcal{F}(q) - \mathcal{F}(p)) \frac{\gamma_\mu (q^2 - p^2) - (q_\mu + p_\mu) (\gamma \cdot q - \gamma \cdot p)}{\mathcal{D}} \quad (3.11)$$

where \mathcal{D} is an expression symmetric in p and q , and behaving as q^4 in the large- q^2 limit. \mathcal{D} must not introduce any singularity, must be symmetric in q and p , and must satisfy $\Gamma_\mu^T(p, p) = 0$. In general,

$$\mathcal{D} = (q^2 + p^2)^2 \eta \left(\frac{q \cdot p}{(q^2 + p^2)}, \frac{q^2 p^2}{(q^2 + p^2)^2} \right) \quad (3.12)$$

with η a regular function such that $\eta(0, 0) = 1$.

It is worth noting that the tensor structure of the transverse vertex (3.11) can be generalised. Indeed, Ball and Chiu [10] have given a set of 8 independent tensor structures spanning the space of regular transverse vertices. Only three of these have the correct helicity structure to contribute to the DS equation in the chiral limit, and are (in the notation of Ball and Chiu):

$$\begin{aligned} T_2^\mu &= [p^\mu (k \cdot q) - q^\mu (p \cdot k)] \gamma \cdot (p + q) \\ T_3^\mu &= k^2 \gamma^\mu - k^\mu \gamma \cdot k \\ T_6^\mu &= \gamma^\mu (q^2 - p^2) - (q + p)^\mu \gamma \cdot (q - p) \end{aligned} \quad (3.13)$$

The transverse vertex given by Eq. (3.11) corresponds to:

$$\Gamma^T(p, q) = \frac{1}{\mathcal{D}} \left(\frac{1}{\mathcal{F}(q)} - \frac{1}{\mathcal{F}(p)} \right) \left(\frac{1}{2}(p^2 + q^2)T_6 + \frac{1}{2}(p^2 - q^2)T_3 + \dots \right) \quad (3.14)$$

where the ellipses refer to terms that vanish in the DS equation. This differs from the structure of ref. [9], where the transverse vertex was chosen to be proportional to T_6 . This can be understood from the fact that the expression of the longitudinal part of the vertex used here differs from theirs, by a transverse tensor proportional to T_3 , in such a way that we get the same high- q^2 or p^2 leading term for the total vertex. The addition of such extra subleading terms is always possible, and we shall investigate the effect of these through the variation of function \mathcal{D} in Eq. (3.11).

The transverse vertex (3.11), together with the longitudinal one (3.5), can then be included in Eq. (3.3) to obtain:

$$\begin{aligned} 1 &= p^2 F_b(p) + C_F \alpha_b \int \frac{d^4 k}{4\pi^3 k^2} D_{\mu\nu}^b \mathcal{Z}_b(k^2) \\ &\left\{ -\frac{1}{2} [F_b(p) + F_b(q)] \delta_{\mu\nu} - \frac{F_b(p) - F_b(q)}{p^2 - q^2} \left(p_\nu q_\mu + p_\mu q_\nu + \frac{1}{2}(p - q) \delta_{\mu\nu} \right) \right. \\ &\left. + \frac{(p^2 F_b(p) - q^2 F_b(q)) (\delta_{\mu\nu} (q^2 - p^2) + p_\mu p_\nu - q_\mu q_\nu + p_\mu q_\nu - p_\nu q_\mu)}{\mathcal{D}} \right\} \quad (3.15) \end{aligned}$$

4. Solving the equation

4.1. The angular integral

The angular integral in Eqs. (3.6, 3.7, 3.15) cannot be performed analytically without prior knowledge of the function F , since this occurs with the argument $(q - k)^2$. The approximation, first proposed by Schoenmaker [11] in the gluon case, consists in replacing a function $f((q - k)^2)$ by $f(q^2 + k^2)$. This is clearly valid in the regions $k^2 \gg q^2$ and $k^2 \ll q^2$. One then needs to find a function f which is sufficiently slowly varying that the error generated by the integration over the whole range of k^2 be negligible.

The first choice would be $f = F$. In this case, the contribution of the longitudinal part of the quark-gluon vertex to the DS equation vanishes. Hence the BZ equations (3.6, 3.7) admit only the trivial solution $F(q^2) = 1/q^2$, $G = 0$. If the transverse part of the vertex is added, the resulting equation (3.15) leads to a propagator close to the perturbative one, with a pole at $k^2 = 0$. Neither solution is a slowly varying function, hence the approximation is not justified in this case.

In the following, we shall use $f = \mathcal{F}$. Our solutions show that this assumption is reasonable so that the above ansatz is justified *a posteriori*. The only region where this approximation may have led to substantial errors would be for small $q^2 \approx k^2$, but then the integrand is suppressed because (as can be seen from Eq. (2.5)) the function $\mathcal{Z}_R(k^2)$

vanishes. This approximation keeps the contributions both from the longitudinal and from the transverse part of the vertex, hence we shall obtain nontrivial solutions both for Eqs. (3.6, 3.7) and (3.15).

With this approximation the angular part of the integration over momentum k in Eqs. (3.6, 3.7, 3.15) may be performed, and both equations can be recast into the following form if $G = 0$:

$$1 = q^2 F_b(q^2) - \frac{C_F \alpha_b}{4\pi} \int dk^2 \left[\mathcal{Z}(k^2) \Delta_1(k^2/q^2) F_b(q^2) + \mathcal{Z}(k^2) \Delta_2(k^2/q^2) F_b(k^2 + q^2) \right] \quad (4.1)$$

The kernels Δ_1 and Δ_2 are given, both in the case of Eqs. (3.6, 3.7) and in the case of Eq. (3.15) in the Appendix. This is a Fredholm equation of the second kind which must now be solved numerically. Before we can do this, however, we must consider the question of the renormalisations required to absorb the ultraviolet divergences.

4.2. Renormalisation

The renormalisation procedure is similar whether one neglects the transverse part of the vertex or not. The renormalisation constant $\mathcal{Z}_b(\mu_g^2)$ has been introduced in Eq. (2.3). As mentioned in Section 2 we choose this constant such that the renormalised function, $\mathcal{Z}_R(q^2)$ takes the value 1 at $q^2 = \mu_g^2$, with μ_g taken to be 0.8 GeV. We likewise introduce a renormalisation constant $\mathcal{F}_b(\mu_f)$ such that

$$F_R(q^2) = \frac{F_b(q^2)}{\mathcal{F}_b(\mu_f)}$$

and

$$G_R(q^2) = \frac{G_b(q^2)}{\mathcal{F}_b(\mu_f)}$$

are ultraviolet finite and we choose it so that the quantity $\mathcal{F}_R(q^2)$ takes the value 1 at $q^2 = \mu_f^2$. Eq. (4.1) then becomes finite if one rewrites it in terms of the coupling

$$\alpha_f = \frac{\alpha_b \mathcal{Z}(\mu_f)}{1 - \alpha_b \mathcal{Z}(\mu_f) C_F / 4\pi \int dk^2 \mathcal{Z}(k^2) \Delta_1(k^2, \mu_f^2)} \quad (4.2)$$

and the renormalised equation (4.1) becomes:

$$1 = \left(1 - \frac{C_F \alpha_f}{4\pi} \int dk^2 \mathcal{Z}_R(k^2) \left[\Delta_1(k^2, q^2) - \Delta_1(k^2, \mu_f^2) \right] \right) F_R(q^2) - \frac{C_F \alpha_f}{4\pi} \int dk^2 \mathcal{Z}_R(k^2) \left(\Delta_2(k^2, q^2) F_R(k^2 + q^2) - \Delta_2(k^2, \mu_f^2) F_R(k^2 + \mu_f^2) \right) \quad (4.3)$$

1. Renormalised BZ equation

In the case of the BZ kernels, the UV divergence of the equation is concentrated in the Δ_1 term, which has a log singularity as $k^2 \rightarrow \infty$. This means that the coupling (4.2) cannot be related to the usual renormalised QCD coupling

$$\alpha_S(\mu_f) = Z_b(\mu_f)\alpha_b \quad (4.4)$$

Hence one loses the universality of the QCD coupling.

Furthermore, the equation for the chirality breaking term G is identical to Eq. (4.3) with the replacement $F \rightarrow G$, $1 \rightarrow 0$. The leading behaviour at high q^2 comes from the remnant of the UV term. If we assume, in agreement with the renormalisation group, that $\mathcal{Z}_R(q^2) \sim 1/[q^2 \log(q^2)]$ as $q^2 \rightarrow \infty$, the leading terms of the equation for the G term are:

$$0 \approx G_R(q^2) - \frac{C_F \alpha_f}{4\pi} \log(\log(q^2)) G_R(q^2) \quad (4.5)$$

which has only $G = 0$ as a consistent solution. Hence, unless the gluon propagator is highly singular, the BZ equation does not lead to chiral symmetry breaking.

If one overlooks these problems, and goes ahead to solve the equation, one obtains a function $F_R(q^2)$ which is very close to the perturbative $1/q^2$. The propagator in this case keeps a pole, the residue of which is slightly bigger than the perturbative one.

As we shall see, the chiral solution will continue to possess these properties, even after we get a consistent equation for the quark propagator by the introduction of the transverse part of the vertex.

2. A multiplicatively renormalised equation

The main effect of the inclusion of the transverse vertex is to shift the divergence from the integral of Δ_1 to that of $\Delta_2 F$ in Eq. (4.3). Indeed, the integral over k^2 of Δ_2 in Eq. (4.1) now diverges logarithmically since $\Delta_2(k^2, q^2)$ behaves like $1/k^2$ as $k^2 \rightarrow \infty$, whereas the integral of the Δ_1 term is finite.

This fact means that the renormalised coupling (4.2) is related to the usual one (4.4) by a finite renormalisation:

$$\alpha_f = \frac{\alpha_S(\mu_f)}{1 - \alpha_S(\mu_f) C_F / 4\pi \int dk^2 \mathcal{Z}(k^2) \Delta_1(k^2, \mu_f^2)} \quad (4.6)$$

Hence we can now write our equation in terms of the true QCD coupling:

$$\begin{aligned} \frac{1}{\mathcal{F}_b(\mu_f)} &= q^2 F_R(q^2) \\ &- \frac{C_F \alpha_S(\mu_f)}{4\pi} \int dk^2 \left[\mathcal{Z}_R(k^2) \Delta_1(k^2, q^2) F_R(q^2) \right. \\ &+ \left. \mathcal{Z}_R(k^2) \Delta_2(k^2, q^2) F_R(k^2 + q^2) \right] \end{aligned} \quad (4.7)$$

Setting $q^2 = \mu_f^2$ in the above, we obtain:

$$\frac{1}{\mathcal{F}_b(\mu_f)} = 1 - \frac{C_F \alpha_S(\mu_f)}{4\pi} \int dk^2 \left[\mathcal{Z}_R(k^2) \frac{\Delta_1(k^2, \mu_f^2)}{\mu_f^2} + \mathcal{Z}_R(k^2) \Delta_2(k^2, \mu_f^2) F_R(k^2 + \mu_f^2) \right] \quad (4.8)$$

We can then equate both expressions for $1/\mathcal{F}_b(\mu_f)$ to obtain:

$$\begin{aligned} 1 &= q^2 F_R(q^2) \\ &- \frac{C_F \alpha_S(\mu_f)}{4\pi} \int dk^2 \left[\mathcal{Z}_R(k^2) \Delta_1(k^2, q^2) F_R(q^2) - \mathcal{Z}_R(k^2) \frac{\Delta_1(k^2, \mu_f^2)}{\mu_f^2} \right] \\ &- \frac{C_F \alpha_S(\mu_f)}{4\pi} \int dk^2 \mathcal{Z}_R(k^2) \left[\Delta_2(k^2, q^2) F_R(k^2 + q^2) - \Delta_2(k^2, \mu_f^2) F_R(k^2 + \mu_f^2) \right] \end{aligned} \quad (4.9)$$

As can be seen the large k^2 behaviour of Δ_2 cancels out and the integral is now ultraviolet convergent.

5. The chiral solution

5.1. Asymptotic behaviour

The first test of the consistency of our results is a direct comparison of equation (4.9) with perturbation theory, *i.e.* for $q^2 \rightarrow \infty$. As we have explained, before subtractions, the equation is ultraviolet divergent because of the terms proportional to Δ_2 . After subtraction, the leading $\log q^2$ comes from those terms in Δ_2 that behave like $1/k^2$. The equation then becomes, for $q^2 \rightarrow \infty$:

$$1 \approx q^2 F_R(q^2) - \frac{C_F \alpha_S}{4\pi} \int_{\mu_f^2}^{q^2} \frac{3dk^2}{2k^2} F_R(k^2) \left[\mathcal{Z}_R(k^2) \right] \quad (5.1)$$

Writing $q^2 F_R(q^2) \approx 1 + \xi \log(q^2)$, $Z_R \approx 1$, one then gets a consistent solution to order $\xi \sim \alpha_S$, provided that

$$\xi = \frac{3C_F \alpha_S}{8\pi} \quad (5.2)$$

in agreement with one-loop results (in axial gauge).

Furthermore, the equation also agrees with RG-improved perturbation theory. Assuming that for large q^2 one has $Z_R(q^2) \approx \log(q^2)^{-1}$, one then gets a consistent asymptotic behaviour $q^2 F_R(q^2) \sim C + \log(q^2)^{-\xi}$, with C a constant, and ξ still given by Eq. (5.2). Hence we see that Eq. (4.9) encompasses our perturbative knowledge of propagators.

Finally, as we shall now explain, it suggests that the $G = 0$ solution cannot be valid for arbitrary values of $\alpha_S(\mu_f)$.

5.2. Critical value of α_S and chiral symmetry breaking

By shifting the k^2 integration in the terms proportional to Δ_2 , one can recast the equation in the following form:

$$\phi(q^2)\mathcal{F}_R(q^2) = \phi(\mu_f^2) + \int dk^2 \mathcal{K}(q^2, k^2)\mathcal{F}_R(k^2) \quad (5.3)$$

with:

$$\begin{aligned} \phi(q^2) &= 1 - \frac{C_F \alpha_S}{4\pi} \int dk^2 \left[\mathcal{Z}_R(k^2) \frac{\Delta_1(k^2, q^2)}{q^2} \right] \\ \mathcal{K}(q^2, k^2) &= \frac{C_F \alpha_S}{4\pi} \frac{1}{k^2} \left[\mathcal{Z}_R(k^2 - q^2) \Delta_2(k^2, q^2) - \mathcal{Z}_R(k^2 - \mu_f^2) \Delta_2(k^2, \mu_f^2) \right] \end{aligned} \quad (5.4)$$

In the case where $\phi(q^2) \neq 0$ for all q^2 , the equation can be reduced to a Fredholm equation of the second kind. Its kernel \mathcal{K} is integrable and bounded, so that there is a unique solution. We obtain it by discretising Eq. (5.3) and inverting the matrix equation thus obtained, which is equivalent to Fredholm's solution [17]. We then get a smoother solution by introducing the obtained points as the input of an iterative method, where we use the left-hand side of Eq. (5.3) as the output, and the left-hand side as the input, which is the Liouville-Neumann method [18]. This converges nicely as long as $\phi(q^2) \neq 0$ for all q^2 .

When $\phi(q^2)$ has a zero, we have a Fredholm equation of the third kind. In that case, both the Fredholm solution, and the Liouville-Neumann iterations fail. This is because, effectively, the Fredholm solution involves integrals of $\int dq^2 K(q^2, k^2)/\phi(q^2)$, which are ill-defined, whereas the Liouville-Neumann solution involves a series with terms of the form $\int dk^2 K(q^2, k^2)\phi(k^2)/\phi(q^2)$, which clearly diverges near the zero of ϕ .

In fact it is clear that the solution dramatically changes. When $\phi(\mu_f^2) = 0$, the solution \mathcal{F} can at best be determined up to a constant as the equation becomes homogeneous. Furthermore, it is not in general possible to remove the pole that the zero of ϕ introduces in \mathcal{F} , hence the propagator develops an imaginary part, which is not allowed in the t -channel. $\phi(q^2)$ is 1 at $\alpha_S = 0$ and steadily decreases until it reaches a zero value (Note that $\Delta_1(k^2, q^2) \rightarrow 0$ as $q^2 \rightarrow 0$, hence ϕ is finite for all q^2). ϕ becomes zero at small q^2 first, and once $\phi(0)$ has crossed zero, then there will be a zero of $\phi(q^2)$ at some nonzero q^2 for larger values of α_S . Therefore there is a critical value of α_S past which ϕ is not positive definite.

Physically, one can view the divergence of the kernel as the divergence of the effective coupling $\alpha_f(q^2)$ of Eq. (4.2) at some value of q^2 . This divergence suggests that the chiral equation stops having physically relevant solutions, and hence that chirality needs to be broken past a certain critical value of the coupling. One should really speak of the value of $\alpha_S(\mu)\mathcal{Z}_R(\mu)$, as one can always change the value of α_S according to Eq. (2.6). This interpretation is reinforced when one realizes that the singularity of the kernel is entirely due to the fact that we have neglected the terms proportional to G . If these are reintroduced,

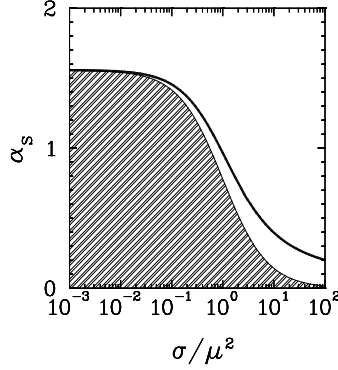


Figure 3: Massless quark solutions to the Dyson-Schwinger equations exist in the shaded region, the boundary of which is the value of α_S for which the equation becomes singular for some value of q^2 , given the mass squared σ which enters the Källén-Lehmann representation of the gluon propagator. The thick curve shows the values of α_S at which the effective quark coupling (4.2) becomes infinite. The curves are for $\eta = 1$.

the singularity will disappear, and one will keep a solution. Hence, the singularity is the place at which the quarks develop a mass.

We can in fact study this for a general gluon propagator, and hence not limit ourselves to the solution found in ref. [1]. We simply need to assume that the gluon propagator has a Källén-Lehmann representation:

$$\frac{\mathcal{Z}(q^2)}{q^2} = \int d\sigma \frac{\rho(\sigma)}{q^2 + \sigma} \quad (5.5)$$

$\phi(q^2)$ can then be written, using the condition $\mathcal{Z}(\mu_g) = 1$:

$$\begin{aligned} \phi(q^2) &= \int d\sigma \rho(\sigma) \Phi(\sigma, q^2) \\ \text{with } \Phi(\sigma, q^2) &= \left\{ \frac{\mu_g^2}{\mu_g^2 + \sigma} - \frac{C_F \alpha_S}{4\pi} \int dk^2 \Delta_1(k^2, q^2) \frac{1}{k^2 + \sigma} \right\} \end{aligned} \quad (5.6)$$

We show in Fig. 3 the region in the $(\alpha_S, \sigma/\mu_g^2)$ plane in which with $\Phi(\sigma, q^2)$ is positive for all q^2 . This means that if $\rho(\sigma)$ has support in an interval $[\sigma_0, \sigma_1]$, then there will be massless solutions if $\alpha_S \leq \alpha_S^{\text{critical}}(\sigma_1)$.

We have plotted Fig. 3 for the function η of Eq. (3.12) equal to one. It is easy to see that a critical α_S will exist for a broad range of η and that its value can be calculated from Fig. 3. Indeed, as shown in the Appendix, for $\eta \left(0, \frac{q^2 p^2}{(q^2 + p^2)^2}\right)$, we can write $\Delta_1 = \eta \Delta_1^T + \Delta_1^L$. Unless η is such that the sign of

$$I(\eta) = \int dk^2 \frac{[\Delta_1^L(k^2/q^2) + \eta \Delta_1^T(k^2/q^2)]}{(k^2 + \sigma)} \quad (5.7)$$

changes, there will exist a critical α_S . Its value will be $\alpha_c(\eta) = \alpha_c(\eta = 1)I(\eta = 1)/I(\eta)$. We therefore conclude that there exists a wide range of choices for the transverse vertex which give rise to chiral symmetry breaking.

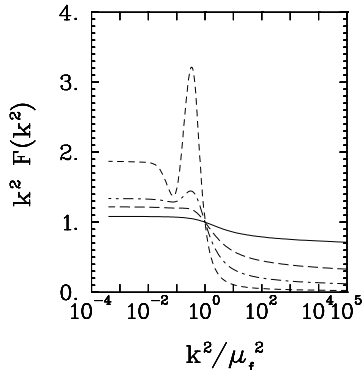


Figure 4: The solutions for massless quark propagators that come from the nonperturbative gluon propagator of ref. [1], for $\alpha_s(mf) = 0.2$ (plain), 0.6 (dashed), 1.0 (dot-dashed) and 1.4 (dashed).

It is of course true that even for Φ negative in part of $[\sigma_0, \sigma_1]$ it is possible to get $\phi > 0$, hence the condition is sufficient only: in the shaded region, there will be massless solutions. It is interesting to note that large- σ modes in the Källén-Lehmann density of the gluon imply chiral symmetry breaking in the quark sector. Only for a specific gluon propagator can one find the exact value of α_S beyond which these solutions do not exist anymore. This is what we are going to do in the next section.

5.3. Chiral solution for a specific gluon propagator

As explained in Section 2, the gluon propagator (2.5) that we found in ref. [1] has the problems linked with the breakdown of multiplicative renormalisability. The procedure used here to obtain a consistent ansatz for the transverse part of the vertex can be extended to the gluon case. However, one may hope that the effect on gluons will be less dramatic than that on quarks. Indeed, we found that imposing the true asymptotic behaviour as $q^2 \rightarrow \infty$ does not appreciably change the behaviour of the propagator at moderate q^2 (compare the solution of the first paper of ref. 1 with Eq. (2.5)).

Hence we shall use that solution in the quark equation as an example of what the chiral solution looks like. We shall assume that both equations are renormalised at the same point, and that the two couplings have the same value. As we already explained, one should be able to derive a gluon propagator for any value of $\alpha_S(\mu)$ as it can always be changed according to Eq. (2.5). As our gluonic equation did not respect multiplicative renormalisability, we were not able to do so, and got an optimum $\alpha_S \approx 1.4$. To see what the effect of the coupling on the quark propagator is, we shall vary α_S independently of \mathcal{Z} , although the two are really correlated.

We show the result of this exercise in Fig. 4. As the value of the coupling grows, the quark propagator is enhanced near the origin. As α_S grows further, the propagator experiences oscillations, until the chiral solution is lost. It is interesting to note that the value of α_S that we obtained for the gluon propagator is very close to the critical value

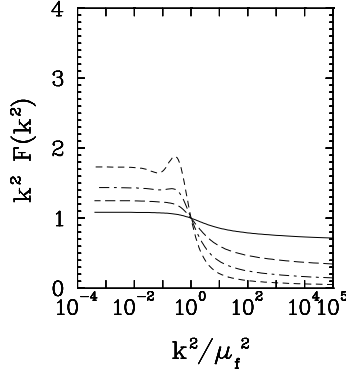


Figure 5: Same as Fig. 4, but for with a transverse vertex multiplied by $\frac{(q^2+p^2)^2}{q^4+p^4}$.

beyond which the chiral solution disappears. We also see that the criterion of Section 5.2 can in practice work backwards, i.e. it not only predicts when there will be chiral solutions, but also when these will disappear.

In order to investigate the sensitivity of the solution to our choice of the function \mathcal{D} in Eq. (3.11), we consider two possible choices for that function. In Fig. 5, we show the behaviour of our solution for for $\mathcal{D} = q^4 + p^4$, instead of $\mathcal{D} = (q^2 + p^2)^2$ (as in Fig. 3). As expected, the two solutions have the same high- q^2 behaviour and are equal at $q^2 = \mu_g^2$. We see that the low-momentum behaviours vary by a modest amount, and that the solutions only differ in the region $q^2 \approx \mu_f^2$, and only when the integrand oscillates a lot. This is where we do not trust our approximation for the angular integrals and hence the result is stable where our approximations hold.

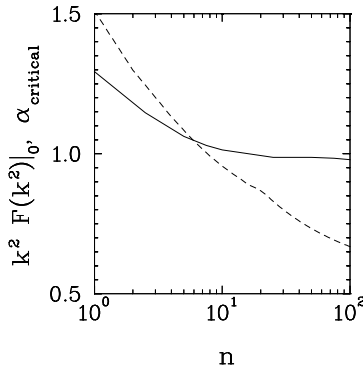


Figure 6: The dashed curve shows the value of α_c for various functions η (see text), and the plain curve shows the value of the intercept of the propagator for $\alpha_S = 0.75 \alpha_c$.

Finally, Fig. 6 illustrates the insensitivity of our results to the choice of transverse vertex. We consider the functions $\eta = [(q^4 + p^4)/(q^2 + p^2)^2]^n$ and plot the results in terms of n . We show the variation of the critical value of α_S , which changes by a factor 2 when n changes by a factor 100. We also show the value of the intercept at the origin of the propagator calculated at $\alpha_S = 0.75\alpha_c$, and which hardly changes with n . This clearly illustrates that

the results we have obtained hold for a wide class of transverse vertices.

6. Conclusion

We have shown that the transverse part of the vertex plays an essential role in the quark DS equation in the axial gauge. It restores multiplicative renormalisability, allows chiral symmetry breaking and provides solutions which match with perturbation theory at large q^2 . Hence we have obtained an equation that possesses all the properties that are required in QCD.

The solution that we obtain in the chiral phase has a pole at $q^2 = 0$ despite the fact that we expect quarks to be confined. This contrasts with the results of a similar analysis carried out in ref. [15] for the gluon propagator of ref. [8] in which it was shown that for the *confining* propagator of ref. [8] the quark propagator does not have a singular behaviour at $q^2 = 0$ and can therefore be considered to be confined. The present formalism can be used to solve for the chirality breaking solution, as we shall explain in a future publication [16]. It is an interesting question to see whether these solutions are confined, and hence whether confinement and chiral symmetry breaking are related.

It has not so far been possible to obtain a solution to the (modified) DS equations for the gluon propagator for time-like momenta. This is because we expect that in this regime the propagator develops an imaginary part with cuts corresponding to the thresholds for glueball production etc. and consequently such a solution involves coupled non-linear equations for the real and imaginary parts of $\mathcal{Z}_R(k^2)$. Therefore the corresponding solution for the quark propagator we have obtained is only valid for quarks propagating with space-like momenta (in the t-channel). It is tempting to try to analytically continue the solution obtained into time-like momenta by fitting the solution obtained to known analytic functions and possibly then using the DS equation for the quark propagator to extract information about the gluon propagator with time-like momenta. Unfortunately this has not been possible. We have obtained several very accurate fits (errors nowhere worse than 2%) to the curve shown in Fig. 4, using different parametrisations inside various analytic functions. Whereas these all fit the quark propagator remarkably well in the fit region they give wildly different projected behaviours for the quark propagator for time-like momenta. We therefore have to accept for the moment that the important problem of describing the exchange of soft gluons and quarks in the s-channel remains unsolved.

As we explained earlier, the equation for the gluon propagator also will have to be modified to incorporate a transverse vertex and recover multiplicative renormalisability. Furthermore, as has been pointed out in ref. [19] the DS equations for the quark and gluon propagators are coupled and after having obtained the solution for the quark propagator one must check the assumption that one may neglect quark loops in the equation for the gluon propagator. In ref. [19] it was shown that for the gluon propagator of ref. [8] this was not the case and the gluon propagator had to be modified accordingly. A similar analysis should also be carried out for the gluon and quark propagators discussed here.

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7. Appendix: Kernels

The kernels entering Eq. (4.1) are given by the following expressions, where we define $\rho = k^2/q^2$.

For the BZ equation (3.6):

$$\Delta_1(\rho) = \frac{\rho}{4} - \theta(\rho - 4) \sqrt{1 - \frac{4}{\rho}} \left(\frac{\rho}{4} + \frac{1}{2} \right) \quad (7.1)$$

and

$$\begin{aligned} \Delta_2(\rho) = & \theta(1 - \rho) \frac{\rho(\rho + 1)}{4} + \theta(\rho - 1) \left(\frac{3}{2} \left(1 + \frac{1}{\rho} \right) - \frac{\rho(\rho + 1)}{4} \right) \\ & + \theta(\rho - 4)(1 + \rho) \sqrt{1 - \frac{1}{\rho}} \left(\frac{\rho}{4} + \frac{1}{2} \right) \end{aligned} \quad (7.2)$$

For Eq. (3.15), and for η a function of $\frac{q^2 p^2}{(q^2 + p^2)^2} \approx \frac{1 + \rho}{(2 + \rho)^2}$ only:

$$\begin{aligned} \Delta_1(\rho) = & \frac{\eta}{4\rho} \left[-\rho - 8 + \frac{\rho^2 + 16 + 2\rho}{\sqrt{\rho^2 + 4}} \right] \\ & + \frac{1}{4} \left[\frac{-\rho^2 + 2\rho + 8}{\sqrt{\rho^2 - 4\rho}} + \rho \right] \\ & + \theta(4 - \rho) \frac{1}{4} \frac{\rho^2 - 2\rho - 8}{\sqrt{\rho^2 - 4\rho}} \end{aligned} \quad (7.3)$$

$$\begin{aligned} \Delta_2(\rho) = & \frac{\eta}{4\rho} \left\{ 9\rho + \rho^2 + 8 - \frac{1}{\sqrt{\rho^2 + 4}} [\rho^3 + \rho^2 + 18\rho + 16] \right\} \\ & + \theta(\rho - 1) \frac{3}{2\rho} \\ & + \theta(1 - \rho) \frac{-3 + \rho^2 + \rho}{2} \\ & + \theta(4 - \rho) \frac{-\rho^3 + \rho^2 + 10\rho + 8}{4\sqrt{\rho^2 - 4\rho}} \\ & + \frac{6 - \rho^2 - \rho}{4} + \frac{\rho^3 - \rho^2 - 10\rho - 8}{4\sqrt{\rho^2 - 4\rho}} \end{aligned} \quad (7.4)$$

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