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Conformal Invariance and Degrees of Freedom in the QCD String

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Abstract

We demonstrate that the Hagedorn-like growth of the number of observed meson states can be used to constrain the degrees of freedom of the underlying effective QCD string. We find that the temperature relevant for such string theories is not given by the usual Hagedorn value $T_H \approx 160$ MeV, but is considerably higher. This resolves an apparent conflict with the results from a static quark-potential analysis, and suggests that conformal invariance and modular invariance are indeed reflected in the hadronic spectrum. We also find that the $D_\perp = 2$ scalar string is in excellent agreement with data.

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It is by now a well-established fact that many aspects of hadronic physics can be successfully modelled by strings. Indeed, an effective "QCD string" theory would simultaneously explain the existence of Regge trajectories, linear confinement, the exponential rise in hadron-state densities, and s- and t-channel duality. In fact, it has recently been proposed that string modular invariance might even explain relative meson/baryon abundances [1].

Over the years many different string models have been proposed for describing the QCD color flux tube deemed responsible for quark confinement in mesons: early examples include the scalar (Nambu) string, the Ramond string, and the Neveu-Schwarz (NS) string, and more recent examples [2] include Polyakov's "rigid string", Green's "Dirichlet string", and the Polchinski-Strominger effective string. While all of these models are endowed with a certain number $D_\perp$ of bosonic degrees of freedom on the two-dimensional string worldsheet (corresponding to the vibrational and rotational string degrees of freedom in the external spacetime), the early models place additional, *purely internal* degrees of freedom on the string worldsheet. The later string models instead introduce effective interactions for the spacetime degrees of freedom which alter their short-distance behavior.

Conformal invariance nevertheless plays an important role in each case. In the early models, conformal invariance is exact (though classical), and indeed an infinite array of models of this type can be constructed by placing on the string worldsheet virtually any two-dimensional conformal field theory (CFT). The quantum excitations of the corresponding new worldsheet fields should then also produce new hadronic states. Conformal invariance in some of the more recent models, by contrast, is an *effective* symmetry, valid in the long-string limit. The effective CFT's which emerge in this limit are nevertheless equally responsible for reproducing the salient features of hadronic spectroscopy.

A comparison with data is therefore necessary in order to constrain the class of strings appropriate for modelling hadronic physics, or, more generally, to test the validity of the string approach by determining the extent to which conformal invariance is actually reflected in the observed hadron spectrum. In particular, we shall use data to constrain the worldsheet central charge c of the effective QCD string, since this parameter is a unique measure of the degrees of freedom in a general two-dimensional theory. Most previous efforts in this direction have employed the static quark potential method, and favor values $c \approx 2$. We propose using the well-known Hagedorn-like growth in the number of meson states as an independent method constraining not only the total central charge, but also its distribution between spacetime and internal degrees of freedom. While the value $T_H \approx 160$ MeV commonly taken for the Hagedorn temperature [3] is too low to agree with the above results (yielding $c \approx 7$), we will show that for comparisons with strings an alternative treatment is necessary. This will resolve the conflict between these two methods, and in so doing provide experimental evidence that the QCD string possesses conformal – and indeed modular – invariance. As a by-product, we will also find that the $D_\perp = 2$ scalar string is in excellent agreement with data.

Let us first review the basic ideas behind the static potential $V(R)$ between two quarks a distance R apart. Modelling confinement as a color flux tube, one can show [4] that this potential takes the exact form

$$\begin{aligned} V(R) &= \left[(\sigma R)^2 + {M_0}^2\right]^{1/2} \\ &\sim \sigma R + ({M_0}^2/2\sigma)\, R^{-1} + \mathcal{O}(R^{-2}) \ , \end{aligned} \tag{1}$$

where σ is the string tension of the flux tube and ${M_0}^2 \leq 0$ is a constant independent of R. While the first (linear) term in the large-R expansion of $V(R)$ represents the classical energy in the effective string, the second "pseudo-Coulomb" term is an attractive universal quantum correction (or Casimir energy) which arises due to transverse zero-point vibrations of the string. As such, this term is to be distinguished from the true attractive Coulomb term which has the same form and which arises at short distances from gluon exchange. The form of the exact string result (1) indicates that while σR plays the role of a string "momentum," the quantity M_0 appears as the string "rest mass" (or ground-state energy) [5]. The fact that ${M_0}^2 \leq 0$ (or equivalently that the long-distance pseudo-Coulomb term is attractive) implies that the ground state of the effective QCD string is tachyonic, yet this causes no inconsistency in the large-R limit [6].

If we assume the dynamics of the color flux tube to be modelled by a two-dimensional conformal field theory, then this ground state energy M_0 and the central charge c of the corresponding worldsheet theory are related by

$$\alpha' \, {M_0}^2 \;=\; h - c/24 \tag{2}$$

where $\alpha' \equiv (2\pi\sigma)^{-1}$ is the Regge slope, and h is the conformal dimension of that primary field in the worldsheet theory which produces the ground state. In most cases this primary field is merely the identity field with $h = 0$, so that the coefficient of the pseudo-Coulomb term directly yields the corresponding central charge. In all other cases, however, the coefficient of this term yields information concerning only the *difference* $h - c/24 \equiv -\tilde{c}/24$. This is dramatically illustrated in the Ramond string: here $h = c/24 = (D-2)/16$, whereupon $\tilde{c} = 0$, the ground state is massless, and the long-range pseudo-Coulomb term is absent.

By fitting the parameters of Eq. (1) to heavy-quark spectroscopic data and/or results from lattice QCD, many authors [7] have attempted to determine the ground-state energy ${M_0}^2$ and thereby the central charge of the QCD string. While the string tension σ is generally found to be in good agreement with the Regge-trajectory value $\alpha' \approx 0.85$ $(\text{GeV})^{-2}$, values of $\tilde{c}$ have been obtained throughout the range $0 < \tilde{c} \leq 4$, clustering near $\tilde{c} \approx 2$. Perhaps the largest source of error in these methods is the fact that they rely upon a full separation of the effects of the long-range Coulomb term from those of the true Coulomb interaction: while the latter are *a priori* unrelated to $\tilde{c}$, fits to spectroscopic data and/or lattice QCD results undoubtedly contain their contributions. Furthermore, as discussed, $\tilde{c}$ is not always a true measure of the degrees of freedom in the string worldsheet theory.

There exists, however, a different approach towards determining the central charge c, one which avoids all of the above difficulties and is complementary to that involving the static quark potential. In string theory (or more generally in any CFT), the number or degeneracy of states g_n at any excitation level n is given by the coefficients in a certain polynomial $\chi_h(x)$ called the *character* of the sector $[h]$ of the worldsheet CFT:

$$\chi_h(x) \;=\; x^{-\tilde{c}/24} \sum_{n=0}^{\infty} g_n \, x^n \;. \tag{3}$$

Since the spacetime mass M_n of a given string excitation level n is given by $\alpha' {M_n}^2 = n - \tilde{c}/24$, we see that the ground state energy ${M_0}^2$ in each sector $[h]$ is indeed given by Eq. (2). Thus, the static quark potential method, by fitting M_0, is essentially a test of the string $n \to 0$ limit. However, as has been well-known from the earliest days of the dual-resonance models, information can also be extracted from the *high*-energy limit ($n \to \infty$), for in this limit string theories predict an exponential rise in the degeneracy of states g_n with excitation number n:

$$\begin{aligned} g_n &\sim A\,[C^2(n-\tilde{c}/24)]^{-B}\, e^{C\sqrt{n-\tilde{c}/24}} \\ &= A\,(M/T_H)^{-2B}\, e^{M/T_H}\ . \end{aligned} \tag{4}$$

Here A, B, and C are constants, with $T_H \equiv (C\sqrt{\alpha'})^{-1}$. The form of these expressions demonstrates that T_H is the famous Hagedorn temperature [3], a critical temperature beyond which the partition functions of such theories (and indeed all of their thermodynamic quantities) cannot be defined. The exponent B also has profound physical consequences. Since the internal energy of such a hadronic system near T_H grows as $U(T) \sim (T_H - T)^{2B-3}$ [8], we see that T_H is a true maximum temperature if $B \leq 3/2$, and merely the site of a second-order QCD phase transition otherwise.

What makes these observations useful for our purposes, however, is the fact that T_H in Eq. (4) is directly related to the total central charge of the underlying CFT:

$$c \;=\; \frac{3}{2\pi^2}\left(\alpha'\, {T_H}^2\right)^{-1}\ . \tag{5}$$

Indeed, this result holds independently of the sector $[h]$ in which our string states are presumed to reside, yielding a value for the true central charge c rather than $\tilde{c}$ and providing a test independent of the static quark potential method. Furthermore, the exponent B can also be interpreted in terms of an underlying string theory, for B is universally related to $D_\perp$, the effective number of spacetime dimensions for transverse string oscillations (or equivalently the number of uncompactified bosonic degrees of freedom in the worldsheet CFT [9]):

$$B \;=\; \tfrac{1}{4}\,(3 + D_\perp)\ . \tag{6}$$

Thus, a fit of Eq. (4) to hadronic data yields information concerning not only the total number of degrees of freedom, but also their effective distribution between spacetime and internal excitations.

In practice, however, a number of subtleties arise which must be addressed before an adequate fit can be performed. First, by tabulating the experimentally measured masses M_i and widths Γ_i of observed mesons [10], we have calculated the density $\rho_{\rm exp}(M)$ of meson states, where $\rho_{\rm exp}(M) \equiv \sum_i W_i \mathcal{S}(M; M_i, \Gamma_i)$. Here $W_i \equiv \gamma_i(2I_i+1)(2J_i+1)$ is the number of states per resonance (with $\gamma_i = 1$ for charge self-conjugate states and $\gamma_i = 2$ otherwise), and $\mathcal{S}$ represents a statistical distribution function. The result is plotted in Fig. 1, with error bars determined by varying $\mathcal{S}$ between Breit-Wigner, Gaussian, and fixed-width ($\Gamma_i = 200$ MeV) distributions. These errors also include an estimate of the uncertainties resulting from the possible subtraction of quark masses. While this density clearly experiences the predicted exponential growth over much of the plotted mass range, we see that the rate of growth sharply diminishes beyond 1.7 GeV. This is attributable to experimental difficulties, for at higher energies it becomes harder to distinguish mesons from background. Indeed, a comparison with explicit quark-model calculations [11] shows 1.7 GeV to be the first energy where fewer mesons are observed than predicted. At the other extreme, the data below 0.3 GeV depends on pion contributions whose small masses result from chiral symmetries which string theory is not expected to model. Thus, we shall limit our attention to the experimental meson data for $0.3 \leq M \leq 1.7$ GeV.

This in turn requires a more sophisticated treatment of the theoretical string predictions than was sketched above. In particular, taking $\alpha' \approx 0.85$ (GeV)$^{-2}$, we see that a mass near 1.7 GeV corresponds only to a string excitation of $n \approx 3$. While the $n \to \infty$ asymptotic function quoted in Eq. (4) is remarkably accurate even for relatively small values of n, it differs from the true string degeneracies g_n for $n \leq 3$ by as much as 95% for the scalar string. We thus require from string theory a more precise functional form, one which includes a sufficient number of subleading terms so that the true values of g_n for the known strings are accurately reproduced. It is a straightforward matter [12] to determine the first set of these subleading terms, however, and together these yield the following improved "asymptotic" form for the state degeneracies g_M and corresponding density $\rho(M)$:

$$\begin{aligned} g_M \quad &\sim \quad \sqrt{2\pi}\, A\, \xi^\nu\, I_{|\nu|}(\xi) \qquad \text{where } \xi \equiv M/T_H \\ &\equiv \quad (2\alpha' M)^{-1}\, \rho_{\rm string}(M)~. \end{aligned} \tag{7}$$

Here $I_{|\nu|}$ is the modified Bessel function of order $|\nu|$, with $\nu \equiv 1/2 - 2B$; note that this result reproduces Eq. (4) in the limit $M \to \infty$ [13]. By substituting the proper values of A, B, and T_H for known strings, we have verified that Eq. (7) is indeed accurate over the required range to within 2%. This is fortunate, since the forms of any additional subleading terms are dependent on model-specific parameters other than B and T_H.

We then performed a fit comparing the experimental and theoretical values of $\int_{m_i}^{m} \rho(M)dM$ as a function of m, with $m_i = 0.3$ GeV. Our results are as follows. Taking $B = 5/4$ (*i.e.*, $D_\perp = 2$) resulted in a best fit with

$$T_H \;=\; 300 \text{ MeV} \quad\Longrightarrow\quad c \;=\; 1.97 \;, \tag{8}$$

where we have taken $\alpha' = 0.85\ (\text{GeV})^{-2}$. This is remarkably close to the central charge $c = 2$ of the $D_\perp = 2$ scalar string, and demonstrates that a string picture is indeed consistent with the data obtained from counting the numbers of hadronic states. Note, in this regard, that the original 1967 fits by Hagedorn [3] yielded $T_H \approx 160$ MeV, or $c \approx 7$, in clear contradiction with the results from the static inter-quark potential. However, the fundamental difference is the functional form to which the fits are made, for Hagedorn's bootstrap-motivated functional form [3] $\rho(M) = a(M^2 + \mu^2)^{-5/4} e^{M/T_H}$ with $\mu \approx 500$ MeV has no connection to the string-theory result in Eq. (7) and implicitly assumes $B = 7/4$ in the $M \to \infty$ limit. Indeed, it is only upon taking the string-motivated functional form in Eq. (7) that agreement is obtained.

In light of interesting recent proposals [14] for *non*-scalar QCD strings with values $B \neq 5/4$, it is nevertheless important to place general constraints in (B, T_H) parameter space. In Fig. 2, the singly- and doubly-shaded regions indicate the (B, T_H) values for which fits to the data with a χ^2/d.o.f. ≤ 1 can be obtained: the dashed line indicates the central value of the fits (*i.e.*, the "best fits"), and the heavy solid lines indicate the border of the region allowed by string theory (corresponding to the constraints $B \geq 3/4$, $c \geq D_\perp$ [15]). We see that there indeed exists a region of overlap between the data-allowed and string-allowed regions; furthermore, the "best fit" line passes directly through this overlap region in the range $3/4 \leq B \leq 5/4$. In the case of scalar strings for general $D_\perp$ (which lie along the upper curved boundary of this string region), we see that the "best fit" line intersects this scalar string line almost exactly at the expected value $B = 5/4$, or $D_\perp = 2$.

The overlap region can be further narrowed if we constrain the normalization constant A in Eq. (7). Within the allowed region in Fig. 2, we find that A_{fit} ranges from $\mathcal{O}(1)$ to $\geq \mathcal{O}(10^3)$. String theory, however, places the precise limit $A_{\rm string} \leq \sqrt{2\pi}\,(4\pi\alpha' {T_H}^2)^\nu$, with equality occurring for the scalar string. Thus, if we assume that there are 36 independent strings which contribute to the (u,d,s)-quark meson spectrum (corresponding to 36 quark degrees of freedom: 9 possible quark/anti-quark flavor combinations, 4 spin states, and one color singlet state), we can require $r \equiv A_{\rm fit}/A^{\rm max}_{\rm string} \leq 36$. This then restricts us to the lower (doubly-shaded) portion of the data-allowed region in Fig. 2. In Fig. 3 we have plotted versus $(c, D_\perp)$ that region of allowed parameter space which satisfies all three of these string constraints. The best fit line is superimposed.

We have already seen in Eq. (8) that the $c = D_\perp = 2$ scalar string lies directly on the best fit line; we now see from Fig. 2 that this point is also exactly on the $r = 36$ border. This implies that the scalar string should accurately model the *absolute* numbers of states in the meson spectrum, and not merely their rate of growth. In Fig. 1, for example, we have superimposed on the meson data the actual numbers of states predicted by 36 copies of the scalar string (solid line). The agreement is excellent.

In conclusion, we have seen that the density of meson states is consistent with string-theoretic predictions; moreover, estimations of the central charge of the QCD string obtained via measurements of the appropriately-defined Hagedorn temperature are now consistent with those independently obtained via static quark potential methods. This latter agreement is especially significant, for these two methods depend separately on quantities which are *a priori* independent: the rate of growth of the numbers of mesons, and the energy of the ground state of the corresponding flux tube. Indeed, as originally noticed in Ref. [16], only an underlying effective two-dimensional conformal invariance — in particular, modular invariance and the associated symmetry under $x \to -1/x$ in Eq. (3) — serve to relate them. Our results thus constitute strong additional evidence that the confinement phase of QCD is consistent with an effective string theory in which conformal symmetry and modular invariance play a significant role. This certainly warrants further study.

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[13] Since the states in a non-interacting string theory are infinitely narrow and populate discrete energy levels M_n, their density is $\rho(M) = g_M \sum_n \delta(M - M_n) = g_M(2\alpha' M) \sum_n \delta(\alpha' M^2 + \tilde{c}/24 - n)$. We replace the latter sum of δ-functions with unity in order to provide a more realistic estimate of the generic broadening effects that interactions would have on the string spectrum. Furthermore, in the asymptotic limit $n \to \infty$, this yields $\rho(M) = g_M/\Delta M$ where ΔM is the mass difference between adjacent string levels.

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Fig. 1: Number of meson states with masses $\leq M$ as function of M, compared with scalar-string result.

Fig. 2: Data- and string-allowed values of (B, T_H), as discussed in text. Best data fits (dashed line) and $D_\perp = 2$ scalar string point (dot) also shown.

Fig. 3: Values of $(D_\perp, c)$ satisfying both data and string constraints, with best fit line superimposed and scalar string shown (dot).