Robustness under missing data: a comparison with special attention to inference

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Problem statement

Goal: Evaluate different robust regression methods in inference when missing data are present

Procedure: Simulations

Framework: Let $X \in \mathbb{R}^{n \times p}$ be the design matrix containing n observations and p predictor variables. Let $\beta \in \mathbb{R}^p$ be the vector of regression coefficients.
The dependent variable $y \in \mathbb{R}^n$ is defined according to the linear model as

$$y = X\beta + \varepsilon$$

where ε is a vector from $\mathbb{R}^n$ with entries independent and identically distributed with $\mathbb{E}[\varepsilon] = 0$ and $Var[\varepsilon] = \sigma^2$.

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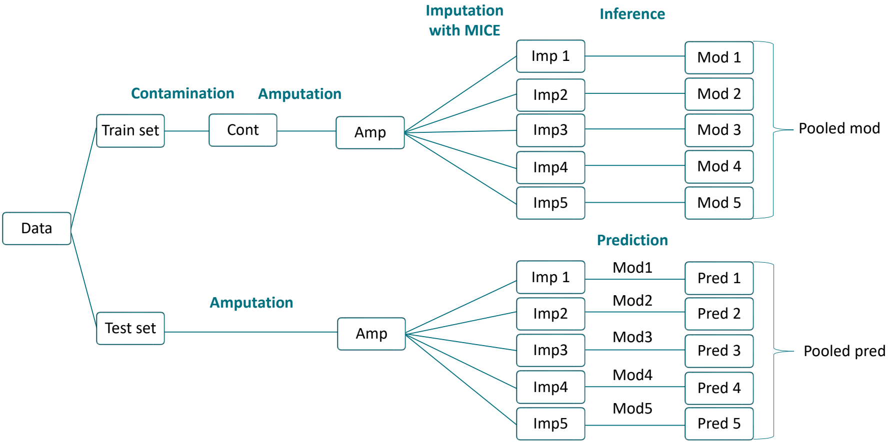
Comparison criteria:

- Bias of the regression coefficients
- Standard error of the regression coefficients
- Mean Squared Prediction Error

Simulation procedure

1. *Generate* the starting train and test sets
2. *Contaminate* the train set
3. *Ampute* the train set and the test set by deleting a selection of values
4. *Impute* the *train set* using MICE to obtain 5 complete imputed train sets
5. *Impute* the *test set* using the same models as created in the previous step to obtain 5 complete imputed test sets
6. *Fit* a *linear model* on the 5 train sets to get 5 models and pool the results $\rightarrow$ inference performance
7. *Predict* the *response* variable for the 5 test sets with the 5 fitted models from the previous step and pool the results $\rightarrow$ prediction performance

Overview of the simulation procedure



Generation of the train and test sets

The simulation setup follows the one proposed by Öllerer, Alfons and Croux [10]:

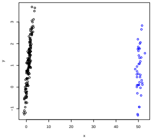
- $p = 15$ variables; $n = 300$ observations for the train set and $n = 100$ for the test set
- Regression coefficients: $\beta_j = j/p$ for $j = 1, \ldots, p$
- Correlation:
  - Independent case: $\Sigma = I_p$ the covariance matrix, $\sigma^2 = 0.5^2$ the error variance
  - Dependent case: $\Sigma_{ij} = 0.5^{|i-j|}$, $\sigma^2 = 0.81^2$
- Generate X according to $\mathcal{N}_p(0, \Sigma)$
- Generate ε according to $\mathcal{N}(0, \sigma^2)$
- Define $y_i = x_i'\beta + \varepsilon_i$ for $i = 1, \ldots, n$

Contamination of the train set

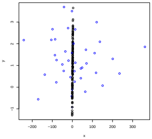
- **Cellwise contamination**
  - Dense cluster: $x_{ij}^{cont} \sim \mathcal{N}(50, 1)$
  - Dispersed outliers: $x_{ij}^{cont} \sim \mathcal{N}(0, 100^2)$
  - Wide cluster: $x_{ij}^{cont} \sim \mathcal{N}(50, 10^2)$
- **Rowwise contamination**
  - Dense cluster: $x_{i}^{cont} \sim \mathcal{N}_p(50, \Sigma)$
  - Dispersed outliers: $x_{i}^{cont} \sim \mathcal{N}_p(0, 100^2\Sigma)$
  - Wide cluster: $x_{i}^{cont} \sim \mathcal{N}_p(50, 10^2\Sigma)$
- **Response contamination**
  - Vertical outliers: $\varepsilon_{i}^{cont} \sim \mathcal{N}(50, \sigma^2)$

$\rightarrow$ Percentage of contamination : 0%, 5% or 10%.

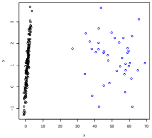
Dense cluster



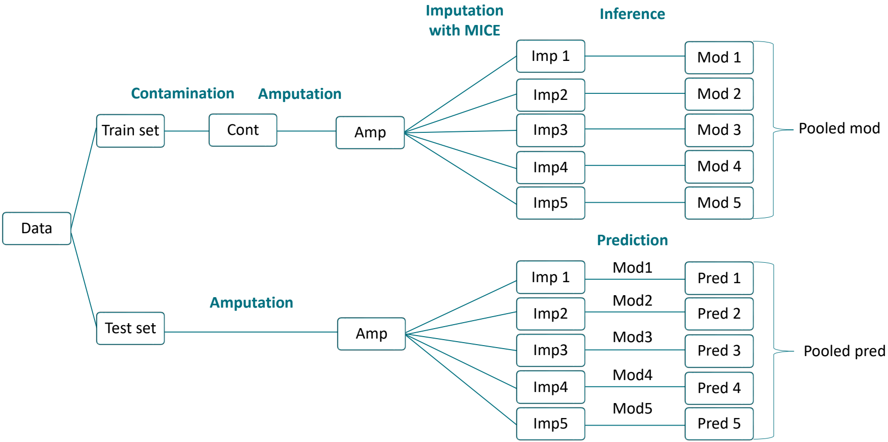
Dispersed outliers



Wide cluster



Overview of the simulation procedure



- **S-estimators** [6]

$$\hat{\beta}_S = \arg \min_{\beta \in \mathbb{R}^p} s(r_1(\beta), \ldots, r_n(\beta))$$

where $s(r_1, \ldots, r_n)$ is a solution of $\frac{1}{n} \sum_{i=1}^n \rho(r_i/s) = \delta$, $\delta = \mathbb{E}_\phi[\rho]$ and $(r_1, \ldots, r_n)$ are the residuals.

- **MM-estimators** [9]

$\hat{\beta}_{MM}$ is any solution of

$$\sum_{i=1}^n \rho' \left(\frac{r_i(\beta)}{s(r_1(\beta_S), \ldots, r_n(\beta_S))} \right) x_i = 0$$

which verifies $S(\beta_{MM}) \leq S(\beta_S)$ where $S(\beta) = \sum_{i=1}^n \rho \left(\frac{r_i(\beta)}{s(r_1(\beta_S), \ldots, r_n(\beta_S))} \right)$

- **Least trimmed squares** [4, 5]

For a fixed h with $[\frac{n+p+2}{2}] \leq h \leq n$,

$$\hat{\beta}_{LTS} = \underset{\hat{\beta}}{\text{argmin}} \sum_{i=1}^h (r^2)_{i:n}$$

where $(r^2)_{1:n} \leq \ldots \leq (r^2)_{n:n}$ are the ordered squared residuals

- **Shooting-S** [10]
  Combine the coordinate descent algorithm with regression S-estimation to obtain the following objective function, $\forall j \in \{1, \ldots, p\}$:

$$\hat{\beta}_j = \underset{\beta \in \mathbb{R}}{\arg\min}\ \hat{\sigma}_j(\beta)$$

$$\text{with } \frac{1}{n}\sum_{i=1}^{n} \rho\left(\frac{\tilde{y}_i^{(j)} - x_{ij}\beta}{\hat{\sigma}_j(\beta)}\right) = \delta$$

$$\text{and } w_{ik} = w\left(\frac{|\tilde{y}_i^{(j)} - x_{ik}\hat{\beta}_k|}{\hat{\sigma}_k(\hat{\beta}_k)}\right)$$

where

$$\tilde{y}_i^{(j)} = y_i - \sum_{k \neq j} \left(w_{ik}x_{ik} + (1 - w_{ik})\hat{x}_{ik}\right)\hat{\beta}_k$$

$$\text{and } \hat{x}_{ik} = \frac{\tilde{y}_i^{(k)}}{\hat{\beta}_k}$$

- **Cellwise robust M regression (CRM)** [2]
  1. Center and scale the data
  2. Use a starting robust regression estimator (MM) to flag observations as casewise outliers if their absolute standard residuals exceed $z_{0.95}$. Apply SPADIMO to separate cellwise and casewise outliers. Cellwise outliers are deleted and then imputed
  3. Apply an IRLS procedure to improve efficicency of the estimates

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Inference methods

- S-estimators
- MM-estimators
- Least trimmed squares
- Shooting-S estimator
- Cellwise robust M regression
- Linear regression (classical)

But How to compute the standard deviation of the regression parameters ?

$\rightarrow$ For LTS and the classical regression : given in the output

$\rightarrow$ Other methods: use bootstrap samples $\Rightarrow$ Fast and robust bootstrap [7]

- Already computed and implented in R for S and MM-estimators
- Need to be adapted for CRM and Shooting-S

Fast and robust bootstrap (FRB) [7]

Let $\hat{\theta}_n \in \mathbb{R}^p$ be the robust parameter estimates of interest and Z_n be a sample. FRB can be used if $\hat{\theta}_n$ can be represented as a solution of fixed-point equations :

$$\hat{\theta}_n = g_n(\hat{\theta}_n)$$

where g_n generally depends on the sample Z_n.
Given a boostrap sample Z_n^*, the recalculated estimates $\hat{\theta}_n^*$ then solves

$$\hat{\theta}_n^* = g_n^*(\hat{\theta}_n^*).$$

Instead of computing $\hat{\theta}_n^*$, we can compute the approximation

$$\hat{\theta}_n^{1*} := g_n^*(\hat{\theta}_n).$$

But $\hat{\theta}_n^{1*}$ underestimate the variability of $\hat{\theta}_n$. To remedy this, they define a linearly corrected version of $\hat{\theta}_n^{1*}$:

$$\hat{\theta}_n^{R*} := \hat{\theta}_n + [I + \nabla g_n(\hat{\theta}_n)]^{-1}(\hat{\theta}_n^{1*} - \hat{\theta}_n)$$

FRB applied to shooting-S

Note $\tilde{y}_i^{(j)} = y_i - \sum_{k \neq j} \left(w_{ik} x_{ik} + (1 - w_{ik}) \hat{x}_{ik} \right) \hat{\beta}_k$, the objective function of shooting-S, $\forall j \in \{1, \ldots, p\}$, can be written as

$$\frac{1}{n} \sum_{i=1}^{n} \rho \left(\frac{\tilde{y}_i^{(j)} - x_{ij} \hat{\beta}_j}{\hat{\sigma}_j(\hat{\beta}_j)} \right) = \delta \tag{1}$$

$$\frac{1}{n} \sum_{i=1}^{n} \rho' \left(\frac{\tilde{y}_i^{(j)} - x_{ij} \hat{\beta}_j}{\hat{\sigma}_j(\hat{\beta}_j)} \right) x_{ij} = 0 \tag{2}$$

Equations (1) and (2) can be rewritten as

$$\hat{\beta}_j = \left(\sum_{i=1}^{n} z_{ij} x_{ij}^2 \right)^{-1} \sum_{i=1}^{n} z_{ij} x_{ij} \tilde{y}_i^{(j)} \tag{3}$$

$$\hat{\sigma}_j = \sum_{i=1}^{n} v_{ij} (\tilde{y}_i^{(j)} - x_{ij} \hat{\beta}_j) \tag{4}$$

where $z_{ik} = \dfrac{\rho'(r_{ik}/\hat{\sigma}_k)}{r_{ik}}$, $v_{ik} = \dfrac{\hat{\sigma}_k}{n\delta} \dfrac{\rho(r_{ik}/\hat{\sigma}_k)}{r_{ik}}$ and $r_{ik} = \tilde{y}_i^{(j)} - x_{ij} \hat{\beta}_j$.

FRB applied to cellwise robust M regression

Cellwise robust M regression:

1. Center and scale the data
2. Use a starting robust regression estimator (MM) to flag observations as casewise outliers if their absolute standard residuals exceed $z_{0.95}$.
   Apply SPADIMO to separate cellwise and casewise outliers. Cellwise outliers are deleted and then imputed
3. Apply an IRLS procedure to improve efficicency of the estimates

Output of the method:

- $\hat{\beta}$: The regression coefficients
- $\tilde{X}$: The imputed design matrix
- Ω : A diagonal matrix containing the weights on the diagonal

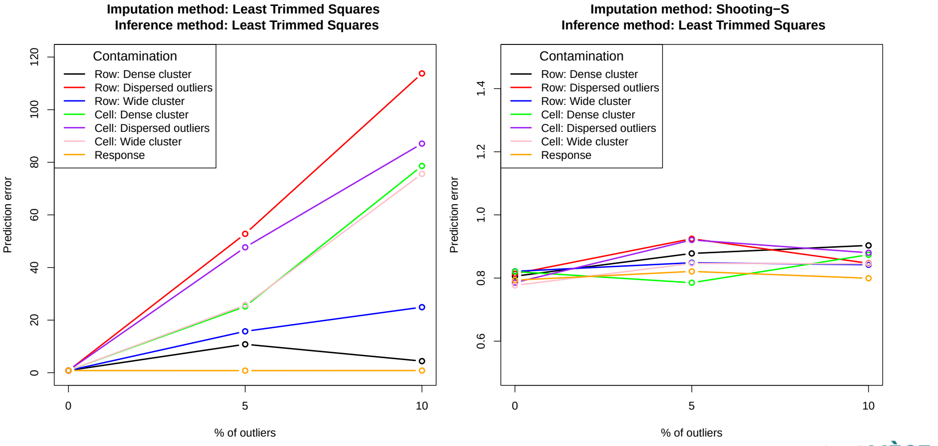
Bootstrap procedure:

1. Compute the clean and weighted sample : $y_\omega = \Omega y$ and $X_\omega = \Omega \tilde{X}$
2. Draw bootstrap samples (X_ω^*, y_ω^*) from (X_ω, y_ω)
3. Bootstrap regression estimates are obtained by fitting a linear regression on that clean bootstrap sample:

$$y_\omega^* = X_\omega^* \hat{\beta}^*$$

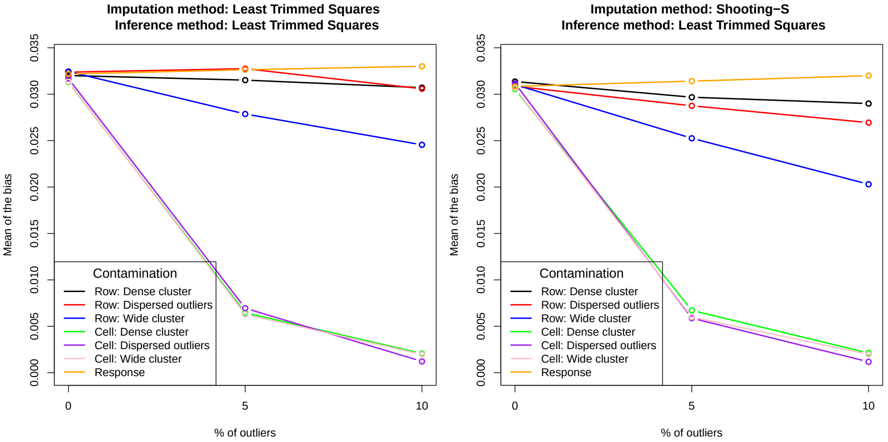
Preliminary results

Prediction error



Preliminary results

Bias



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