Out-of-time-ordered correlators in Bose-Hubbard systems

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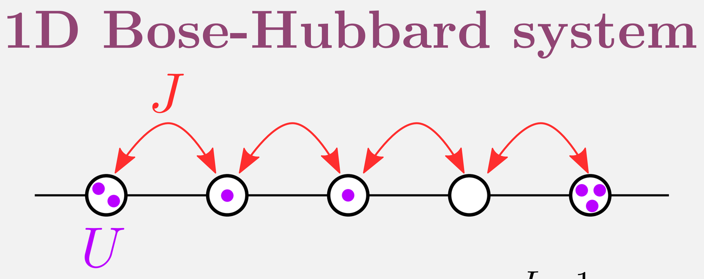
Motivation

Out-of-time-ordered correlators (OTOCs) are objects that measure the loss of quantum information, i.e. scrambling. They can also be used as a probe to **quantum chaos**. More specifically, they can be related to the underlying classical dynamics of the system and give access to the maximum **Lyapunov exponent** and the **Ehrenfest time**, also called the scrambling time.
To shed more light on this, we developped a quasiclassical formalism of OTOCs. We reproduced results established by previous paper [1-4] concerning the short-time behaviour of the OTOC: an exponential growth with the growth rate proportional to the Lyapunov exponent. In addition, our derivations allow us to go beyond the Ehrenfest time a obtain a finite long-time classical limit.
While our formalism is valid for all systems that possess a classical analogue, our numerical simulations will be done in the context of 1D Bose-Hubbard systems.

Ultracold bosonic atoms in optical lattices

1D Bose-Hubbard system



$$\hat{H} = \sum_{l=1}^{L}\left(E_l\hat{n}_l + \frac{U}{2}\,\hat{n}_l\left(\hat{n}_l - 1\right)\right) - J\sum_{l=1}^{L-1}\left(\hat{b}_l^\dagger\hat{b}_{l+1} + \hat{b}_{l+1}^\dagger\hat{b}_l\right)$$

- with:
  - an on-site interaction E_l
  - a hopping parameter J
  - a 2-body interaction U
  - the bosonic creation and annihilation operators $\hat{b}_l^\dagger, \hat{b}_l$
  - the population operator $\hat{n}_l = \hat{b}_l^\dagger\hat{b}_l$

Classical (mean-field) analogue

- From operators to complex numbers:

$$\frac{1}{2}\left(\hat{b}_l^\dagger\hat{b}_l + \hat{b}_l\hat{b}_l^\dagger\right) \to \psi_l^*\psi_l$$

→ discretised Gross-Pitaevskii equation:

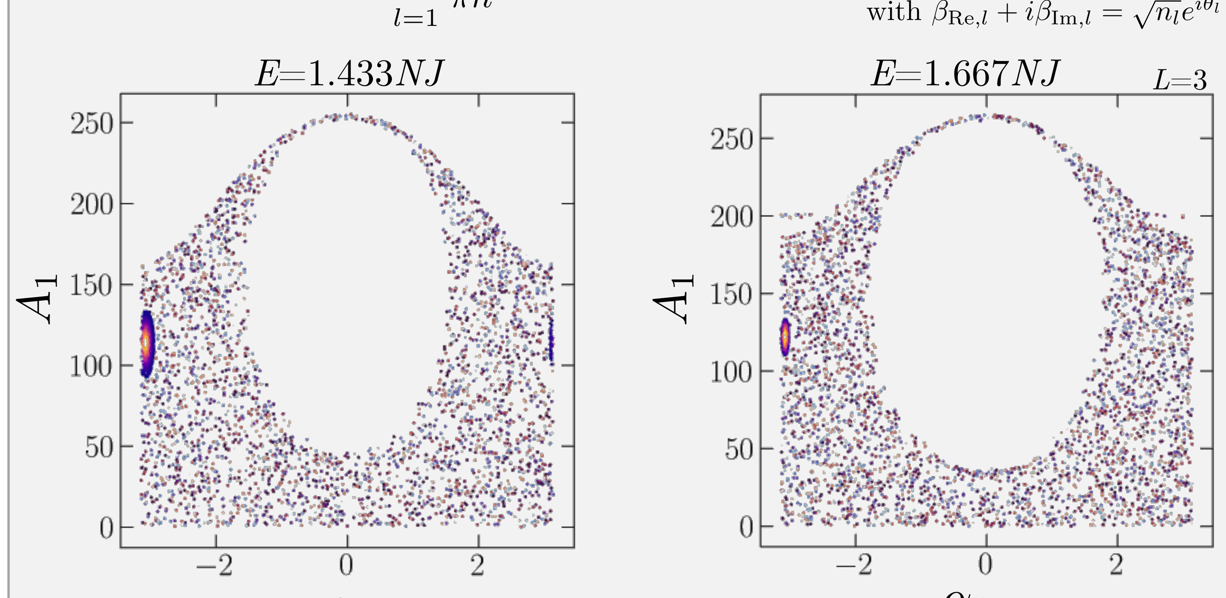
$$i\hbar\frac{d}{dt}\psi_l = E_l\psi_l - J\left(\psi_{l+1} + \psi_{l-1}\right) + U|\psi_l|^2\psi_l$$

Poincaré sections

Coherent state

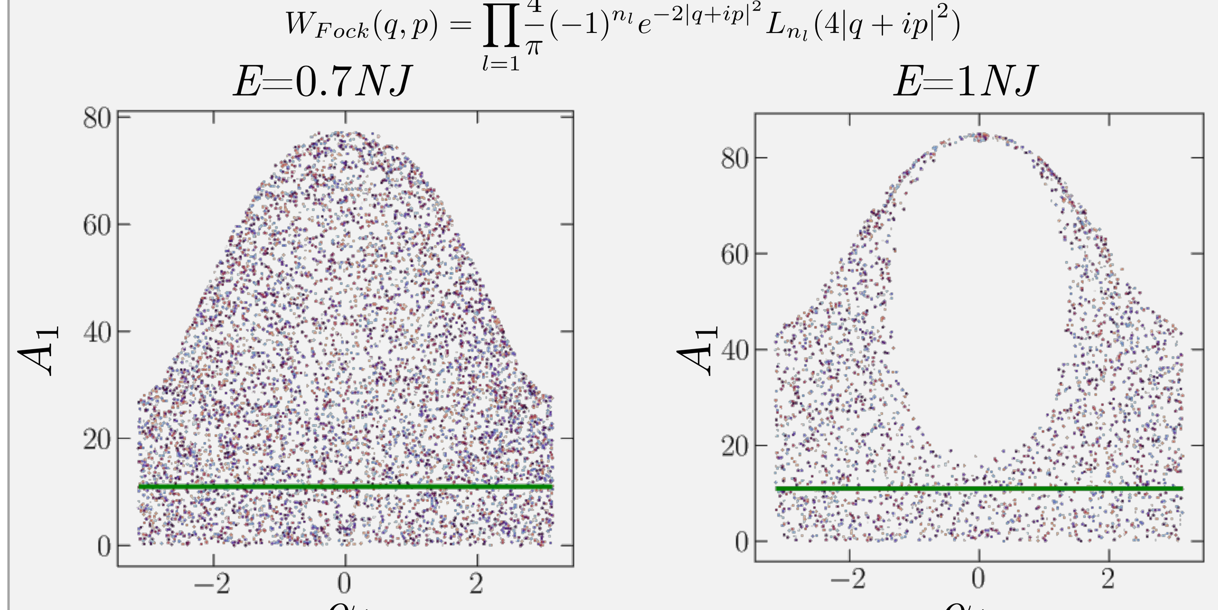
$$W_{CS}(q,p) = \prod_{l=1}^{L}\frac{1}{\pi\hbar}e^{-2(q_l-\varepsilon_q\beta_{\mathrm{Re},l})^2/\varepsilon_q^2}\ e^{-2(p_l-\varepsilon_p\beta_{\mathrm{Im},l})^2/\varepsilon_p^2}$$

with $\beta_{\mathrm{Re},l} + i\beta_{\mathrm{Im},l} = \sqrt{n_l}e^{i\theta_l}$



Fock state

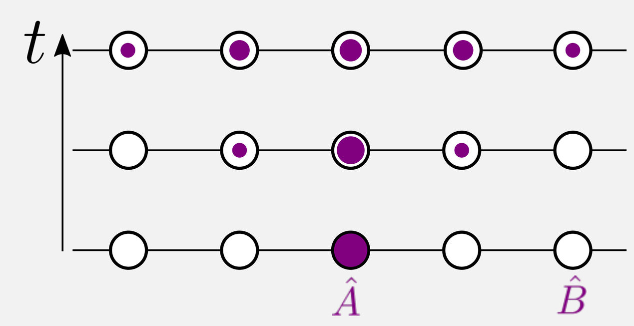
$$W_{Fock}(q,p) = \prod_{l=1}^{L}\frac{4}{\pi}(-1)^{n_l}e^{-2|q+ip|^2}L_{n_l}(4|q+ip|^2)$$



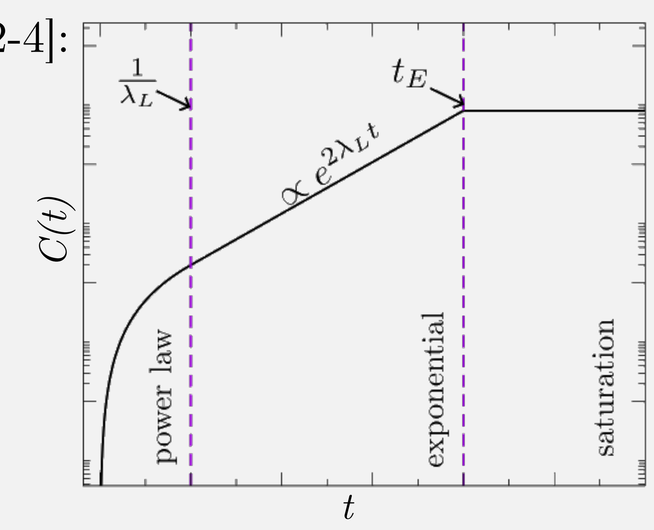
with $\alpha_1 = \theta_1 - \theta_3$, $A_1 = n_1 + \frac{1}{2}$, $\frac{UN}{LJ} \simeq 2$

Out-of-time-ordered correlator

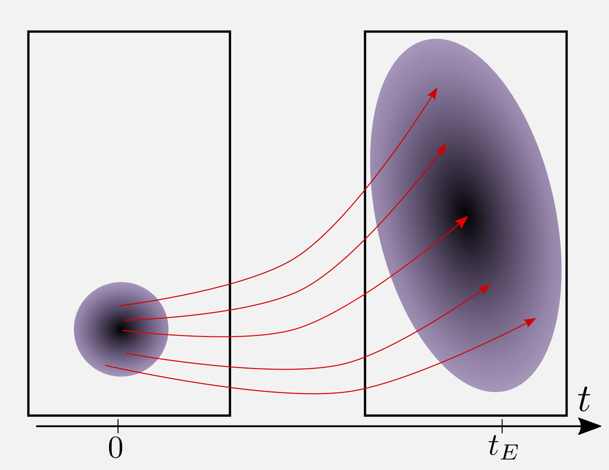
- Defined by: $C(t) = \left\langle\left|\left[\hat{A}(t), \hat{B}\right]\right|^2\right\rangle$
- Characterises the propagation of quantum information in complex quantum system, *i.e.* **scrambling**



- 3 regimes [2-4]:



- At the Ehrenfest time t_E, breakdown of quantum-classical correspondence



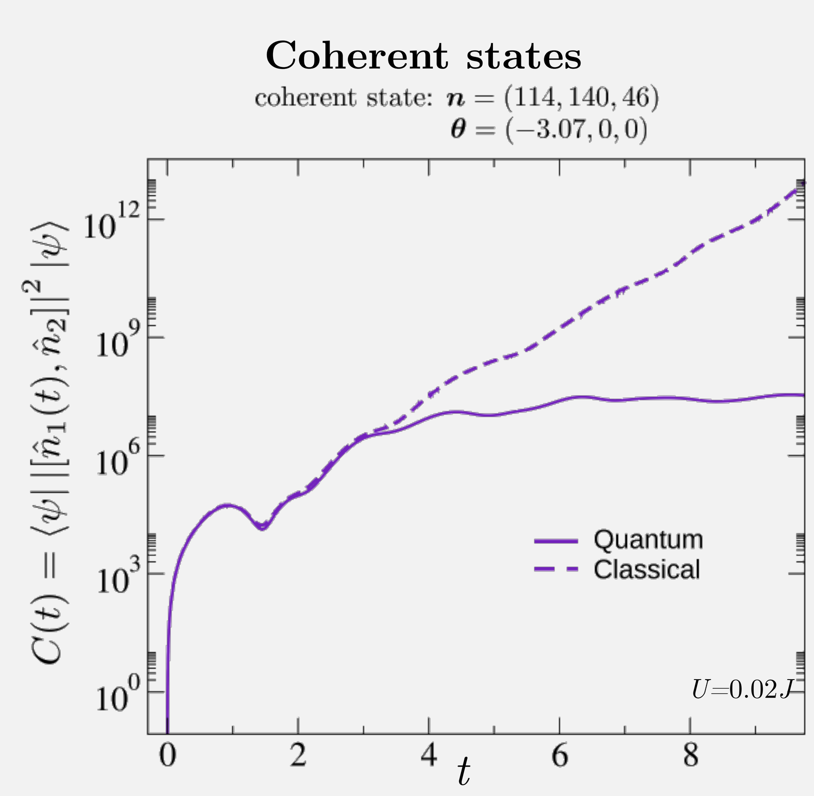
Direct classical limit

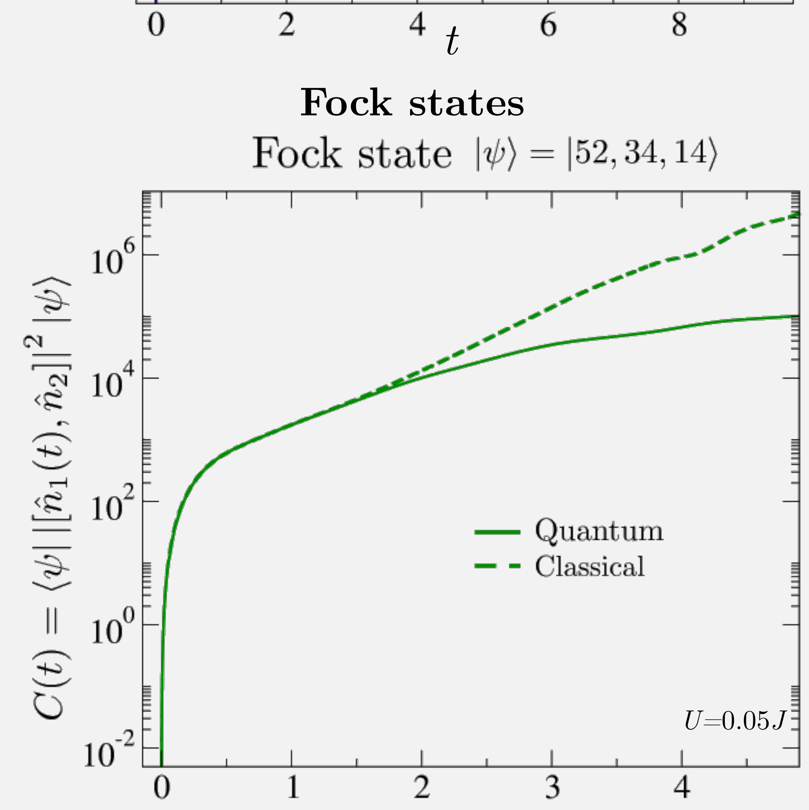
- Direct classical limit: **Wigner-Moyal** formalism
- Consists in taking the strict limit $\hbar \to 0$

$$\langle\psi|\left|\left[\hat{A}(t), \hat{B}\right]\right|^2|\psi\rangle \to \hbar^2\int dqdp\,\left\{\hat{A}_W(t), \hat{B}_W(0)\right\}^2 W_\psi(q,p) + \mathcal{O}(\hbar^3)$$

- Breakdown beyond Ehrenfest time: $t_E \sim \frac{1}{\lambda_L}\log\left(\frac{1}{\hbar}\right) \to \infty$

Quantum vs classical OTOC



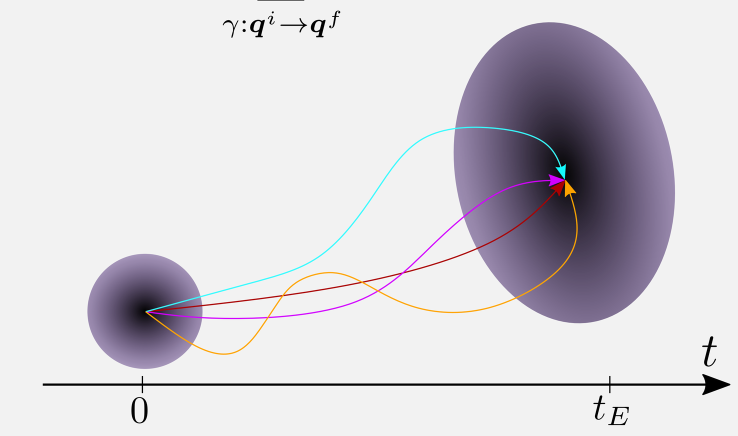


Semiclassical approach

Semiclassical limit

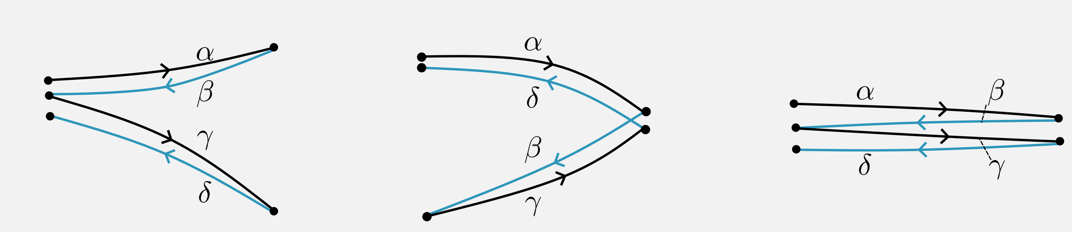
- Semiclassical propagator: **van Vleck-Gutzwiller**

$$\langle\boldsymbol{q}^f|\hat{U}(t)|\boldsymbol{q}^i\rangle \simeq \sum_{\gamma:\boldsymbol{q}^i\to\boldsymbol{q}^f} A_\gamma(\boldsymbol{q}^f, \boldsymbol{q}^i, t)\ e^{\frac{i}{\hbar}R_\gamma(\boldsymbol{q}^f,\boldsymbol{q}^i,t)}$$



Classical limit

- Classical approximation: **Diagonal Approximation**
  Only action-correlated trajectories remain [1]:



- After (very) long calculations...:

$$C(t) = \int \mathrm{d}X_{1,2,3}\mathrm{d}\Delta_{1,2}\ e^{i(\Delta_1\otimes(X_2-X_3)+\Delta_2\otimes(X_1-2X_2+X_3))}A_t\left(X_1 + \tfrac{\hbar\Delta_2}{4}\right)\ A_t\left(X_1 - \tfrac{\hbar\Delta_2}{4}\right)$$
$$\Big[B\left(X_2 + \tfrac{\hbar\Delta_1}{4}\right)\ B\left(X_2 - \tfrac{\hbar\Delta_1}{4}\right) - B\left(2X_1 - 2X_2 + X_3\right)\left(B\left(X_2 + \tfrac{\hbar\Delta_1}{4}\right) + B\left(X_2 - \tfrac{\hbar\Delta_1}{4}\right)\right)$$
$$+ (\hat{B}^2)_W\left(2X_1 - 2X_2 + X_3\right)\Big]\frac{W(X_3)}{(2\pi)^{4L}}$$

- More control towards the classical limit
- No time-limit of validity
- Poisson bracket in leading order in $\hbar$:

$$C(t) = \hbar^2\int dR\ \{A(S(R)), B(R)\}^2\, W(R) + \mathcal{O}(\hbar^3)$$

Classical long-time limit

- Hypotheses of ergodicity and mixing [1]:

$$A_t(q,p) \to \bar{A}_t(H(q,p), N(q,p))$$

- $$C(t\to\infty) = \frac{\hbar^2}{2}\int dX\sum_{i,j=1}^{2}\left(\frac{\partial\bar{A}_t}{\partial c_i}\frac{\partial\bar{A}_t}{\partial c_j} - \bar{A}_t\frac{\partial^2\bar{A}_t}{\partial c_i\partial c_j}\right)\{c_i, B\}\{c_j, B\}\,W + \frac{\hbar^2}{16}\int \mathrm{d}X\,\bar{A}_t^2(X)f(B(X), W(X)) + \mathcal{O}(\hbar^3)$$
- Valid in mixed phase space as well if the wavepacket is completely within the chaotic sea
- Numerical comparison with quantum saturation value: work in progress

Conclusion

Our goal was to find a new and more general classical limit of the OTOC. This was done using the **semiclassical van Vleck-Gutzwiller propagator** and the diagonal approximation. It allows to recover **Wigner-Moyal** in the $\hbar \to 0$ limit.
The result is at the same time valid **beyond the Ehrenfest time** due to the absence of $\hbar$ expansion and can be shown to tend to a finite limit, meaning that we were able to recover a final long-time value of the OTOC in a purely classical scheme. Comparison with quantum OTOC is in progress.

References

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