String attractors of fixed points of k-bonacci-like morphisms

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String attractors

Definition

A *string attractor* for a (finite) word w is a set of positions $\Gamma \subseteq \{1, \ldots, |w|\}$ such that each factor has an occurrence in w crossing one of the positions in Γ.

Outline

We want to find a string attractor for each **prefix** of an infinite word.

- Start from the Fibonacci word...
- extract the ideas...
- and apply them on other words.

Prefixes of the Fibonacci word

Construction of the string attractors:

$$\mathbf{f} = 0100101001001\cdots$$

Prefixes of the Fibonacci word

Construction of the string attractors:

$$\mathbf{f} = \underline{0}100101001001\cdots$$

Prefixes of the Fibonacci word

Construction of the string attractors:

$$\mathbf{f} = \underline{01}00101001001\cdots$$

Prefixes of the Fibonacci word

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Prefixes of the Fibonacci word

Construction of the string attractors:

$$\mathbf{f} = \underline{01}\textcolor{blue}{00}\textcolor{gray}{101001001}\cdots$$

Prefixes of the Fibonacci word

Construction of the string attractors:

$$\mathbf{f} = \underline{010}0101001001\cdots$$

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Prefixes of the Fibonacci word

Construction of the string attractors:

$$\mathbf{f} = 0\underline{1}\underline{0}0\underline{1}01001001\cdots$$

Prefixes of the Fibonacci word

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Three main steps:

- increasing the length of the prefix (no new "small" factor)
- adding a "well-chosen" position
- removing the first position when it becomes redundant

Increasing the length of the prefix (no new “small” factor)

Lemma

Let $\mathbf{x}$ *be an infinite word. If*

- Γ *is a string attractor of* $\mathbf{x}[1, n]$,
- $\mathbf{x}[1, n+1]$ *is periodic of period* p,
- $p \in \Gamma$,

then Γ *is a string attractor of* $\mathbf{x}[1, n+1]$.

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<u>Proof</u>:

$$x_1 \cdots \underline{x_p} x_{p+1} \cdots x_n x_{n+1}$$

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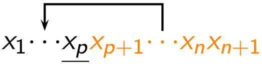

Adding a “well-chosen” position

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence ($F_0 = 1$, $F_1 = 2$).

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Lemma

For all $n \geq 0$, $\mathbf{f}[1, F_{n+1} - 1]$ is periodic of period F_n.

Corollary

Every prefix of $\mathbf{f}$ has a string attractor made of elements of $(F_n)_{n\geq 0}$.

Proof:

It is true for the prefix of length $0 = F_0 - 1$.

Induction:

$$\begin{aligned} & \Gamma \text{ s.a. of } \mathbf{f}[1, F_m - 1] \\ \Rightarrow\ & \Gamma \cup \{F_m\} \text{ s.a. of } \mathbf{f}[1, F_m] \\ \Rightarrow\ & \Gamma \cup \{F_m\} \text{ s.a. of } \mathbf{f}[1, n],\ \forall\, n \in [F_m, F_{m+1} - 1] \end{aligned}$$

Adding a “well-chosen” position: general case

Proposition

Let $\mathbf{x}$ *be an infinite word and let* $(U_n)_{n\geq 0}$ *be an increasing sequence such that* $U_0 = 1$*. If, for all* $n \geq 0$*,*

(Per) the word $\mathbf{x}[1, U_{n+1} - 1]$ *is periodic of period* U_n*,*

then every prefix of $\mathbf{x}$ *has a string attractor composed of elements of* $(U_n)_{n\geq 0}$*.*

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But the size of the string attractors might not be bounded...

Removing the first position when it becomes redundant

$$\mathbf{f} = 01\underline{0}0\underline{1}01\underline{0}01001\cdots$$

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For all $n \geq 0$,

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(Len) $F_{n+3} - F_{n+2} \geq F_{n+1} - F_n$;

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(Per2) $\mathbf{f}[1, F_{n+2} + F_{n+1} - F_n - 2]$ *is periodic of period* F_{n+1}*.*

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(Per2) $\mathbf{f}[1, F_{n+2} + F_{n+1} - F_n - 2]$ *is periodic of period* F_{n+1}.

Proposition

Every prefix of $\mathbf{f}$ *has a string attractor of size (at most) 2 made of consecutive elements of* $(F_n)_{n\geq 0}$.

Removing the first position: general case

Proposition

Let $\mathbf{x}$ *be an infinite word, let* $(U_n)_{n\geq 0}$ *be an increasing sequence such that* $U_0 = 1$*. If there exists* $k \geq 2$ *such that, for all* $n \geq 0$*,*

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then every prefix of $\mathbf{x}$ *has a string attractor of size at most* $k+1$ *made of consecutive elements of* $(U_n)_{n\geq 0}$*.*

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Let $\mathbf{x}$ *be an infinite word, let* $(U_n)_{n\geq 0}$ *be an increasing sequence such that* $U_0 = 1$. *If there exists* $k \geq 2$ *such that, for all* $n \geq 0$,

- *(Per)* $\mathbf{x}[1, U_{n+1} - 1]$ *is periodic of period* U_n,
- *(Suff)* $\mathbf{x}[1, U_n]$ *is a suffix of* $\mathbf{x}[1, U_{n+k}]$,
- *(Len)* $U_{n+k+1} - U_{n+k} \geq U_{n+1} - U_n$,

then every prefix of $\mathbf{x}$ *has a string attractor of size at most* $k+1$ *made of consecutive elements of* $(U_n)_{n\geq 0}$.

Moreover, if for all $n \geq k$,

- *(Per2)* $\mathbf{x}[1, U_{n+1} + U_{n-k+1} - U_{n-k} - 2]$ *is periodic of period* U_n,

then that string attractor can be chosen of size at most k.

k-bonacci-like morphisms

For $k \geq 2$ and for $c_1, \ldots, c_k \geq 0$ with $c_1, c_k \geq 1$, we define

$$\mu_c : \begin{cases} 0 \mapsto 0^{c_1}1 \\ 1 \mapsto 0^{c_2}2 \\ \cdots \\ k-2 \mapsto 0^{c_{k-1}}(k-1) \\ k-1 \mapsto 0^{c_k} \end{cases} \quad \text{and} \quad \mathbf{x}_c = \mu_c^{\omega}(0)$$

If $c_1 = c_2 = \cdots = c_k = 1$, we obtain the k-bonacci word.

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If $c_1 = c_2 = \cdots = c_k = 1$, we obtain the k-bonacci word.

Question

Can we use our techniques to find a string attractor for every prefix of $\mathbf{x}_c$?

Associated sequence U_n

Proposition

For all $n \geq 0$, we have

$$\mu^n(0) = \begin{cases} \mu^{n-1}(0)^{c_1} \cdot \mu^{n-2}(0)^{c_2} \cdots \mu^0(0)^{c_n} \cdot n & \text{if } n \leq k-1; \\ \mu^{n-1}(0)^{c_1} \cdot \mu^{n-2}(0)^{c_2} \cdots \mu^{n-k}(0)^{c_k} & \text{if } n \geq k. \end{cases}$$

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In particular, if $U_n = |\mu^n(0)|$,

$$U_n = \begin{cases} c_1 U_{n-1} + c_2 U_{n-2} + \cdots + c_n U_0 + 1 & \text{if } n \leq k-1; \\ c_1 U_{n-1} + c_2 U_{n-2} + \cdots + c_k U_{n-k} & \text{if } n \geq k. \end{cases}$$

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Remark

For all $n \geq 0$, we have

(Suff) $\mathbf{x}_c[1, U_n]$ is a suffix of $\mathbf{x}_c[1, U_{n+k}]$,

(Len) $U_{n+k+1} - U_{n+k} \geq U_{n+1} - U_n$.

Periodic prefixes

Proposition

For all n, the longest prefix of $\mathbf{x}_c$ which is periodic of period U_n is given by

$$q_n = \mu^n(0)^{a_1} \cdot \mu^{n-1}(0)^{a_2} \cdots \mu^1(0)^{a_n} \cdot \mu^0(0)^{a_{n+1}}$$

where $a_1 a_2 \cdots$ is the infinite concatenation of the longest prefix of $c_1 \cdots c_{k-1}$ maximal among its conjugates.

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n	$\mu^n(0)$	q_n
0	0	
1	01	
2	012	
3	01203	
4	012030100	

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Periodic prefixes

Proposition

For all n, the longest prefix of $\mathbf{x}_c$ which is periodic of period U_n is given by

$$q_n = \mu^n(0)^{a_1} \cdot \mu^{n-1}(0)^{a_2} \cdots \mu^1(0)^{a_n} \cdot \mu^0(0)^{a_{n+1}}$$

where $a_1 a_2 \cdots$ is the infinite concatenation of the longest prefix of $c_1 \cdots c_{k-1}$ maximal among its conjugates.

For $c = 1012$, we have $c_1 \cdots c_{k-1} = 101$ so $a_1 a_2 \cdots = (10)^\omega$.

n	$\mu^n(0)$	q_n
0	0	0
1	01	01
2	012	$012 \cdot 0$
3	01203	01203010
4	012030100	

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4	012030100	$012030100 \cdot 012 \cdot 0$

Some equivalences

Proposition

The following are equivalent:

(Per) *for all* $n \geq 0$, $\mathbf{x}_c[1, U_{n+1} - 1]$ *is periodic of period* U_n;

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- $c_1 \cdots c_{k-1}(c_k - 1)$ *is maximal among its conjugates;*
- *for all* $m \geq 1$, *if* $\mathrm{rep}_{U_n}(m) = b_\ell \cdots b_1 b_0$, *then* $\mathbf{x}_c[1, m] = \mu^{\ell}(0)^{b_\ell} \cdots \mu^{1}(0)^{b_1} \mu^{0}(0)^{b_0}$.

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Theorem

If $c_1 \cdots c_{k-1}(c_k - 1)$ *is maximal among its conjugates, then every prefix of* $\mathbf{x}_c$ *has a string attractor of size* $\leq k+1$ *made of elements of* $(|\mu^n(0)|)_{n \geq 0}$.

Otherwise, there is a prefix which has no string attractor made of elements of $(|\mu^n(0)|)_{n \geq 0}$.

Stronger condition

Proposition

If $c_1 \cdots c_{k-1}$ *is maximal among its conjugates, and* $c_k = 1$*, then every prefix of* $\mathbf{x}_c$ *has a string attractor of size* $\leq k$ *made of consecutive elements of* $(|\mu^n(0)|)_{n \geq 0}$*.*

Proposition (Bernat, Masáková, Pelantová)

The word $\mathbf{x}_c$ *has complexity* $p_{\mathbf{x}_c}(n) = (k-1)n + 1$ *if and only if* $c_1 \cdots c_{k-1}$ *is maximal among its conjugates and* $c_k = 1$*.*

Open questions

- Can we apply this for other words?
- Given a sequence $(U_n)_{n\geq 0}$, can we find an infinite word $\mathbf{x}$ whose prefixes have string attractors (of bounded size) made of U_n's?
- Given a infinite word $\mathbf{x}$, can we find a numeration system $\mathcal{S}$ such that $\mathbf{x}$ is $\mathcal{S}$-automatic and string attractors of $\mathbf{x}$'s prefixes can be easily described using $\mathcal{S}$?

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Thank you for your attention!