Summing the sum of digits

Jean-Paul Allouche[1 [0000−0002−9060−0784]] and Manon Stipulanti[2 [0000−0002−2805−2465]]

[1] CNRS, IMJ-PRG, Sorbonne, Paris, France
jean-paul.allouche@imj-prg.fr
[2] Department of Mathematics, ULiège, Liège, Belgium
m.stipulanti@uliege.be

We dedicate this work to Christiane Frougny on the occasion of her 75th birthday.

Abstract. We revisit and generalize inequalities for the summatory function of the sum of digits in a given integer base. We prove that several known results can be deduced from a theorem in a 2023 paper by Mohanty, Greenbury, Sarkany, Narayanan, Dingle, Ahnert, and Louis, whose primary scope is the maximum mutational robustness in genotype-phenotype maps.

Keywords: Combinatorics on words · Sum of digits · Blancmange curve
2020 Mathematics Subject Classification: 68R15, 11A63, 11B99

1 Introduction

Looking at the sum of digits of integers in a given base has been the subject of numerous papers. In particular, the summatory function of the sum of digits, i.e., the sum of all digits of all integers up to some integer, has received much attention. Such "sums of sums" can be viewed through several prisms, two of them being to obtain (optimal) inequalities on the one hand and asymptotic formulas on the other. For the latter approach we only cite the study *par excellence*, namely the 1975 paper of Delange [7]. The results about these sums sometimes occur in unexpected domains. The most prominent of them is the link with fractal functions, in particular with the Takagi function (a continuous function that is nowhere differentiable [19]) and the blancmange curve—see the nice surveys of Lagarias [12] and Allaart and Kawamura [3].

The present paper will concentrate on inequalities satisfied by these sums of sums. Knowing the inspiring paper of Graham (see [8,10]; also see [14,4]), we first found the 2011 paper of Allaart [1]. Then, we came across the paper [15] about *maximum mutational robustness in genotype–phenotype maps*: that paper drew our attention because it contains the expressions "blancmange-like curve", "Takagi function", and "sums of digits". In particular, the authors of [15] prove the following theorem (they indicate that this generalizes the case $b = 2$ addressed in Graham's paper [8]).

Theorem 1 (Theorem 5.1, pp. 12–13 in [15]). *Let b be an integer ≥ 2. Let $s_b(n)$ denote the sum of digits in the base-b expansion of the integer n. Define $S_b(n) := \sum_{1 \leq j \leq n-1} s_b(j)$. Let $n_1, n_2, \ldots, n_b$ be integers such that $0 \leq n_1 \leq n_2 \leq \cdots \leq n_b$. Then the following equality holds:*

$$\sum_{1 \leq i \leq b} S_b(n_i) + \sum_{1 \leq i \leq b-1} (b-i) n_i \leq S_b \left(\sum_{1 \leq i \leq b} n_i \right). \tag{1}$$

The 2011 paper by Allaart [1] goes in a similar direction somehow. Namely, Allaart proves the following (see [1, Ineq. 4]).

Theorem 2 ([1]). *Let p be a real number. For each integer n, let its binary expansion be $n = \sum_{i=0}^{+\infty} d_i 2^i$ where $d_i \in \{0,1\}$ for all $i \geq 0$ and $d_i = 0$ for all large enough i. Define $\mathfrak{w}_p(n) := \sum_{i=0}^{+\infty} 2^{pi} d_i$ and its summatory function by $\mathfrak{W}_p(n) := \sum_{m=0}^{n-1} \mathfrak{w}_p(m)$. Then, for all $p \in [0,1]$ and all integers $\ell \in [0,m]$,*

$$\mathfrak{W}_p(m+\ell) + \mathfrak{W}_p(m-\ell) - 2\mathfrak{W}_p(m) \leq \ell^{p+1}. \tag{2}$$

When reading the literature on the subject we have noted that the most recent papers do not always cite more ancient ones, confirming a remark of Stolarsky [18, p. 719]: *Whatever its mathematical virtues, the literature on sums of digital sums reflects a lack of communication between researchers.* As one might add, reasons for this could be that there are a very large number of papers dealing with sums of digits, and many of them are not directly interested in these sums *per se*, but because they occur in seemingly unrelated questions.

In this paper, we will first answer a question of Allaart about the case $p = 0$ in [1]; see Theorem 2 above. Then, in Section 4 we will give a corollary and two generalizations of the result in [15] (Theorem 1 above): we will prove that several results that we found in the literature can be actually deduced from this corollary and these two generalizations. Finally, we will ask a few questions about possible sequels to this work.

2 A quick lemma that will be used several times

In this short section we give an easy useful lemma (the first equality can be found, e.g., as Lemma 7 on page 683 in [2], or as Exercise 3.11.5 on page 112 in [5]; the other equalities are immediate consequences of the first one).

Lemma 1. *(i) For all integers $b \geq 2$ and $n \geq 1$, we have*

$$S_b(bn) = bS_b(n) + \frac{b(b-1)}{2} n.$$

(ii) For all integers $b \geq 2$, $n \geq 1$, and $x \geq 0$, we have

$$S_b(b^x n) = b^x \, S_b(n) + \frac{b-1}{2} \, x \, b^x \, n \quad \text{and} \quad S_b(b^x) = \frac{b-1}{2} \, x \, b^x.$$

3 Graham's result implies the case $p = 0$ of Allaart's result

As mentioned above, the result in the paper of Graham [8] (also see [10] and [14], and a less elegant but possibly more natural proof in [4]) corresponds to the case $b = 2$ of Theorem 1, namely (with the usual convention on empty sums):

Theorem 3 ([8]). *For all n_1, n_2 with $0 \leq n_1 \leq n_2$, we have*

$$S_2(n_1) + S_2(n_2) + n_1 \leq S_2(n_1 + n_2). \tag{3}$$

The author of [1] writes (middle of page 690) about Ineq. (2): *"It seems that for $0 < p < 1$ the inequality may be new. In fact, even for the case $p = 0$ the author has not been able to find a reference"*. In this section we indicate that the case $p = 0$ appears, in a slightly disguised form, in the 1970 paper of Graham.

Proof. First, we note that, taking $b = 2$ in Lemma 1 (i) above, we obtain,

$$S_2(2t) = 2S_2(t) + t \tag{4}$$

for all positive integers t. Now let m, ℓ be two integers with $0 \leq \ell \leq m$. Define $n_1 := m - \ell$ and $n_2 = m + \ell$. Then $0 \leq n_1 \leq n_2$. Graham's theorem and Eq. (4) yield

$$S_2(m - \ell) + S_2(m + \ell) + m - \ell \leq S_2(2m) = 2S_2(m) + m,$$

and hence

$$S_2(m - \ell) + S_2(m + \ell) - 2S_2(m) \leq \ell. \quad \square$$

Remark 1. Having showed that Graham's inequality implies the case $p = 0$ in Allaart's inequality, one can ask whether Allaart's inequality for $p = 0$ gives Graham's. For two integers n_1, n_2 with $0 \leq n_1 \leq n_2$, we want to prove that Ineq. (3) holds. If n_1 and n_2 have the same parity, then we can define m and ℓ by

$$m := \frac{n_1 + n_2}{2} \quad \ell := \frac{n_2 - n_1}{2}.$$

Then Allaart's inequality applied to m, ℓ gives

$$S_2(n_2) + S_2(n_1) - 2S_2\left(\frac{n_1 + n_2}{2}\right) \leq \frac{n_2 - n_1}{2}.$$

Now, this inequality after applying Eq. (4) with $t = \frac{n_1+n_2}{2}$, can be written

$$S(n_2) + S_2(n_1) - S_2(n_1 + n_2) + \frac{n_1 + n_2}{2} \leq \frac{n_2 - n_1}{2},$$

and hence

$$S_2(n_2) + S_2(n_1) + n_1 \leq S_2(n_1 + n_2).$$

It seems that Allaart's inequality for $p = 0$, in which both $m + \ell$ and $m - \ell$ (necessarily of the same parity) occur, does not immediately imply Graham's, where n_1 and n_2 may have opposite parities. This suggests the possible existence of a "Graham-Allaart" inequality.

4 A variation on Theorem 1 and two generalizations

In this section we state and prove three results inspired by Theorem 1, from which, together with Theorem 1 itself, most of the results that we found in the literature can be deduced (except Allart's in [1], i.e., Theorem 2, for $p \neq 0$, and a sharp inequality due to Allaart [2], see Remark 3).

4.1 A variation on Theorem 1

We begin with a variation of Theorem 5.1, pp. 12–13 in [15] (stated as Theorem 1 above).

Theorem 4. *Let $b \geq 2$. Let $k_1, k_2, \ldots, k_b$ be nonnegative integers such that $k_j \leq k_b$ for all $j \in [1, b-1]$. Then*

$$S_b(k_1 + k_2 + \cdots + k_b) + \sum_{1 \leq j \leq b-1} S_b(k_b - k_j) - b\, S_b(k_b) \leq \sum_{1 \leq i \leq b-1} i\, k_i.$$

Proof. Define the integers n_i by $n_i := k_b - k_{b-i}$, for all $i \in [1, b-1]$, and $n_b := \sum_{1 \leq i \leq b} k_i$. Since $0 \leq n_1 \leq n_2 \cdots \leq n_{b-1} \leq n_b$ we can apply Ineq. (1). Thus we obtain

$$S_b(k_1 + k_2 + \cdots + k_b) + \sum_{1 \leq j \leq b-1} S_b(k_b - k_{b-j}) + \sum_{1 \leq i \leq b-1} (b-i)(k_b - k_{b-i}) \leq S_b(bk_b).$$

But $S_b(bk_b) = b\, S_b(k_b) + \frac{b(b-1)}{2}\, k_b$ (see [2] or see Lemma 1 above). Hence

$$S_b(k_1 + k_2 + \cdots + k_b) + \sum_{1 \leq j \leq b-1} S_b(k_b - k_{b-j}) - b\, S_b(k_b) \leq \sum_{1 \leq i \leq b-1} (b-i)\, k_{b-i}$$

i.e.,

$$S_b(k_1 + k_2 + \cdots + k_b) + \sum_{1 \leq j \leq b-1} S_b(k_b - k_j) - b\, S_b(k_b) \leq \sum_{1 \leq i \leq b-1} i\, k_i. \quad \square$$

4.2 Generalizations of Theorems 1 and 4

In [15, Theorem 5.1] (see Theorem 1 above) we can drop the hypothesis that the number of n_i is equal to the base b and replace it with the assumption that the number of n_i is at most equal to the base b.

Theorem 5. *Let b be an integer ≥ 2. Let $s_b(n)$ denote the sum of digits in the base-b expansion of the integer n. Define $S_b(n) := \sum_{1 \leq j \leq n-1} s_b(j)$. Let r be an integer in $[1, b]$. Let $n_1, n_2, \ldots, n_r$ be integers such that $0 \leq n_1 \leq n_2 \cdots \leq n_r$. Then the following equality holds:*

$$\sum_{1 \leq i \leq r} S_b(n_i) + \sum_{1 \leq i \leq r-1} (r-i) n_i \leq S_b\left(\sum_{1 \leq i \leq r} n_i\right). \qquad (5)$$

Proof. In Ineq. (1) of Theorem 1, take $n_1 = n_2 = \cdots = n_{b-r} = 0$. This gives

$$\sum_{b-r+1\le i\le b} S_b(n_i) + \sum_{b-r+1\le i\le b-1} (b-i)n_i \le S_b\left(\sum_{b-r+1\le i\le b} n_i\right),$$

and hence, with the change of indices $b - r + j = i$,

$$\sum_{1\le j\le r} S_b(n_{b-r+j}) + \sum_{1\le j\le r-1} (r-j)n_{b-r+j} \le S_b\left(\sum_{1\le j\le r} n_{b-r+j}\right).$$

Now, define $m_j := n_{b-r+j}$ for all $j \in [1, r]$. Then $0 \le m_1 \le m_2 \le \cdots \le m_r$ and

$$\sum_{1\le j\le r} S_b(m_j) + \sum_{1\le j\le r-1} (r-j)m_j \le S_b\left(\sum_{1\le i\le r} m_j\right). \quad \square$$

In the same spirit one can extend Theorem 4.

Theorem 6. *Let $b \ge 2$. Let $r \in [1, b]$, and let $m_1, \ldots, m_r$ be non-negative integers satisfying $m_j \le m_r$ for all $j \in [1, r-1]$. Then*

$$S_b(m_1+m_2+\cdots+m_r) + \sum_{1\le j\le r-1} S_b(m_r - m_j) - rS_b(m_r) \le \sum_{1\le j\le r-1} (b-r+j)m_j. \tag{6}$$

Proof. Let $k_1, k_2, \ldots, k_b$ be integers such that $k_1 = k_2 = \cdots = k_{b-r} := 0$ and $0 \le k_{b-r+1} \le k_{b-r+2} \le \cdots \le k_b$. Applying Theorem 4 yields

$$S_b(k_{b-r+1} + \cdots + k_b) + (b-r)S_k(k_b) + \sum_{b-r+1\le j\le b-1} S_b(k_b - k_j) - b\,S_b(k_b) \le \sum_{b-r+1\le j\le b-1} j\,k_j.$$

Changing the indices in the last two sums gives

$$S_b(k_{b-r+1} + \cdots + k_b) + (b-r)S_k(k_b) + \sum_{1\le j\le r-1} S_b(k_b - k_{b-r+j}) - b\,S_b(k_b) \le \sum_{1\le j\le r-1} (b-r+j)k_{b-r+j},$$

and hence, by grouping the terms in $S_b(b_k)$ and letting $m_j := k_{b-r+j}$,

$$S_b(m_1 + \cdots + m_r) - r\,S_b(m_r) + \sum_{1\le j\le r-1} S_b(m_r - m_j) \le \sum_{1\le j\le r-1} (b-r+j)m_j.$$

as desired. $\square$

4.3 (Non-)Optimality in the theorems of this section

One can ask, e.g., whether Theorem 5 can be further generalized by taking $r > b$. The answer is no: one can show that Theorem 5 is optimal in the sense that for any integer $r > b$ there do not exist constants $\alpha_j \geq 1$ such that for all $n_1, n_2, \ldots, n_r$ one has the inequality

$$\sum_{1 \leq i \leq r} S_b(n_i) + \sum_{1 \leq i \leq r-1} \alpha_i n_i \leq S_b\left(\sum_{1 \leq i \leq r} n_i\right).$$

Namely, we prove the following theorem.

Theorem 7. *Let b be an integer ≥ 2 and let $r \geq b+1$. Then there exist integers $n_1, n_2, \ldots, n_r$ with $0 \leq n_1 \leq \cdots \leq n_r$ such that*

$$\sum_{1 \leq i \leq r} S_b(n_i) + \sum_{1 \leq i \leq r} n_i > S_b\left(\sum_{1 \leq i \leq r} n_i\right).$$

Proof. Take $n_1 = n_2 = \cdots = n_{r-b-1} := 0$, $n_{r-b} = n_{r-b+1} = \cdots = n_{r-1} := 1$, and $n_r = b^x$ where x is an integer ≥ 2. Then, on the one hand,

$$\begin{aligned}\sum_{1 \leq i \leq r} S_b(n_i) + \sum_{1 \leq i \leq r} n_i &= \sum_{r-b+1 \leq i \leq r} S_b(n_i) + b + b^x \\ &= bS_b(1) + S_b(b^x) + b + b^x = S_b(b^x) + b + b^x\end{aligned}$$

and on the other,

$$\begin{aligned}S_b\left(\sum_{1 \leq i \leq r} n_i\right) &= S_b\left(\sum_{r-b+1 \leq i \leq r} n_i\right) = S_b(b + b^x) \\ &= S_b(b^x) + s_b(b^x) + s_b(b^x + 1) + \cdots + s_b(b^x + b - 1) \\ &= S_b(b^x) + 1 + 2 + \cdots + b \\ &= S_b(b^x) + \frac{b(b+1)}{2} < S_b(b^x) + b + b^x,\end{aligned}$$

where we use the fact that $x \geq 2$. □

Remark 2. The right-hand term of Ineq. (6) in Theorem 6 is not optimal: e.g., take $b \geq 4$, $r = 2$, and see Remark 3 below.

5 Graham's inequality and its first generalizations by Allaart and Cooper are consequences of Theorem 1

Graham's theorem was given above as Theorem 3. The following generalization for any base $b \geq 2$ and two integers n_1, n_2, was proved by Allaart in [2] and again quite recently by Cooper [6].

Theorem 8 ([2]). *Let $b \geq 2$ be an integer. For all n_1, n_2 with $0 < n_1 \leq n_2$, we have*

$$S_b(n_1) + S_b(n_2) + n_1 \leq S_b(n_1 + n_2). \tag{7}$$

It is immediate that this statement is implied by Theorem 5 by taking $r = 2$. Hence so is Graham's result by taking $b = r = 2$.

Another result is proved in [2], namely:

Theorem 9 ([2]). *For any integers k, ℓ and m with $0 \leq \ell \leq k \leq m$, we have*

$$S_3(m + k + \ell) + S_3(m - k) + S_3(m - \ell) - 3S_3(m) \leq 2k + \ell. \tag{8}$$

This theorem is an easy consequence of our Theorem 4 (and hence of Theorem 5.1 in [15], see Theorem 1 above): indeed, take $b = 3$.

Remark 3. On page 680 of [2], Allaart notes that, by taking $\ell = 0$, Ineq. (8) gives: for all k, m with $0 \leq k \leq m$, one obtains

$$S_3(m + k) + S_3(m - k) - 2S_3(m) \leq 2k. \tag{9}$$

Then, Allaart proves the following (sharp) inequality in [2, th. 3, p. 681] :

Theorem 10 ([2]). *If $0 \leq k \leq m$, then*

$$S_b(m + k) + S_b(m - k) - 2S_b(m) \leq \left\lfloor \frac{b+1}{2} \right\rfloor k. \tag{10}$$

For $b \geq 4$ this inequality is stronger than Ineq. (6) for $r = 2$, which only gives

$$S_b(m + k) + S_b(m - k) - 2S_b(m) \leq (b - 1)k.$$

We did not succeed in deducing Ineq. (10) from the result of [15] or variations thereof.

6 A binomial digression

An easy inequality mentioned on page 682 of [2] reads: for any nonnegative integers n, k we have: $s_b(n + b^k) \leq s_b(n) + 1$. A more general, probably well-known, inequality, is that for any nonnegative integers n, m, we have $s_b(n + m) \leq s_b(n) + s_b(m)$ (see, e.g., [9, Prop. 2.1]). A way of proving this inequality *when b is prime*, is to use a result of Legendre [13, p. 10–12]: $\nu_b(n!) = \frac{n - s_b(n)}{b-1}$, where $\nu_b(k)$ is the b-adic valuation of the positive integer k. This implies easily $s_b(m) + s_b(n) - s_b(n+m) = \nu_b(\binom{n+m}{n})$. Since $\nu_b(\binom{n+m}{n}) \geq 0$ we are done. This inequality rises the question of whether something similar (at least when b is prime) could be done for, say, Theorem 1 and/or Theorem 2 above. For the second one, we note that it might be necessary to introduce a kind of generalized binomial coefficient.

7 How to generalize Allaart's Theorem 2?

It is tempting to try to generalize Theorem 2. A reasonable idea seems to try to replace the sequence $(2^{pi})_{i\geq 0}$ with a sequence $(\lambda_i)_{i\geq 0}$, where the latter sequence is well chosen. This leads to the following definition.

Definition 1. *Let $(\lambda_i)_{i\geq 0}$ be a sequence of positive real numbers. If we let $(d_i)_{i\geq 0}$ be the binary digits of n, we define $\omega_{(\lambda)}(n) = \sum_{i=0}^{+\infty} \lambda_i d_i$ and its summatory function $\mathcal{W}_{(\lambda)}(n) = \sum_{m=0}^{n-1} \omega_{(\lambda)}(m)$.*

Example 1. If $\lambda_i := 2^{pi}$, with $p \in [0,1]$, then $\omega_{(\lambda)}$ and $\mathcal{W}_{(\lambda)}$ are exactly the quantities ω_p and $\mathcal{W}_p$ in Theorem 2 above.

Of course, not every sequence of λ_i's would work. It seems that some extra conditions should be imposed on the sequence $(\lambda_i)_i$; possibly that this sequence is non-decreasing, but not too much, e.g., one could need to impose that $\lambda_i \leq \lambda_{i+1} \leq C\lambda_i$ for some constant $C \geq 2$ and for all i. However, even this is not enough. Namely, studying, e.g., the quantity $\sum_{i=0}^{+\infty} 3^{pi} d_i$ for $p \in [0,1]$ is equivalent to studying the quantity $\sum_{i=0}^{+\infty} 2^{p'i} d_i$ for some p', which can be greater than 1, since we have

$$\sum_{i=0}^{+\infty} 3^{pi} d_i = \sum_{i=0}^{+\infty} 2^{\log_2(3)pi} d_i.$$

But the remark below shows that the condition $p \in [0,1]$ in Allaart's inequality is crucial.

Remark 4. Let us take $p > 1$ and $\ell = 1$. If Allaart's inequality were true, we would have

$$\mathcal{W}_p(m+1) + \mathcal{W}_p(m-1) - 2\mathcal{W}_p(m) \leq 1,$$

which is equivalent to saying that

$$\omega_p(m) - \omega_p(m-1) \leq 1.$$

Now, let m be a power of 2, say 2^k with k large. The binary expansion of m is 10^k and that of $(m-1)$ is 1^k (where, for $a \in \{0,1\}$, a^k means that the digit a is repeated k times). Therefore

$$\omega_p(m) - \omega_p(m-1) = 2^{pk} \cdot 1 - \sum_{0\leq i\leq k-1} 2^{pi} \cdot 1 = \frac{2^{pk}(2^p - 1) - 2^{pk} + 1}{2^p - 1},$$

which behaves like $\frac{2^{pk}(2^p-1)}{2^p-1} = 2^{pk}$ when k goes to infinity (recall that $p > 1$, and hence $2^p - 1 > 1$). This contradicts the inequality $\omega_p(m) - \omega_p(m-1) \leq 1$.

8 Questions and expectations

We propose the following questions or/and expectations.

* Generalize Theorem 2: is there a generalized Ineq. (2) and/or a generalized Ineq. (1)? In doing so, recall Section 7 above.
* Give a proof of Ineq. (1) or even of Ineq. (5) using the method of [4].
* Is there a "Graham-Allaart inequality" (see end of Section 3)?
* To what extent is it possible to address inequalities mentioned in this paper, through the use of (generalized) binomial coefficients (see end of Section 6)?
* Are there similar inequalities if the sum of digits is replaced with another "block counting-function" (e.g., the number of 11 in the binary expansion of the integer n)? It is possible that the papers [17,11,16] yield some hints in this direction.

Acknowledgements

Jean-Paul Allouche wants to thank Alain Denise for discussions about the biology part of [15]. Both authors thank Jeff Shallit for his comments, and Jia-Yan Yao and Pieter Allaart for providing them with Reference [2]. Manon Stipulanti is an FNRS Research Associate supported by the Research grant 1.C.104.24F.

References

1. Allaart, P.C.: An inequality for sums of binary digits, with application to Takagi functions. J. Math. Anal. Appl. **381**(2), 689–694 (2011), https://doi.org/10.1016/j.jmaa.2011.03.039
2. Allaart, P.C.: Digital sum inequalities and approximate convexity of Takagi-type functions. Math. Inequal. Appl. **17**(2), 679–691 (2014), https://doi.org/10.7153/mia-17-50
3. Allaart, P.C., Kawamura, K.: The Takagi function: a survey. Real Anal. Exchange **37**(1), 1–54 (2011/12), http://projecteuclid.org/euclid.rae/1335806762
4. Allouche, J.-P.: On an inequality in a 1970 paper of R. L. Graham. Integers **21A**, Paper No. A2, 6 (2021)
5. Allouche, J.-P., Shallit, J.: Automatic sequences. Theory, applications, generalizations. Cambridge University Press, Cambridge (2003), https://doi.org/10.1017/CBO9780511546563
6. Cooper, C.: A generalization of an inequality by Graham. Integers **22**, Paper No. A50, 9 (2022)
7. Delange, H.: Sur la fonction sommatoire de la fonction"somme des chiffres". Enseign. Math. (2) **21**(1), 31–47 (1975)
8. Graham, R.L.: On primitive graphs and optimal vertex assignments. Ann. New York Acad. Sci. **175**, 170–186 (1970)
9. Hare, K.G., Laishram, S., Stoll, T.: Stolarsky's conjecture and the sum of digits of polynomial values. Proc. Amer. Math. Soc. **139**(1), 39–49 (2011), https://doi.org/10.1090/S0002-9939-2010-10591-9

10. Jones, J.C., Torrence, B.F.: The case of the missing case: the completion of a proof by R. L. Graham. Pi Mu Epsilon J. **10**, 772–778 (1999), https://www.jstor.org/stable/24340445
11. Kirschenhofer, P.: Subblock occurrences in the q-ary representation of n. SIAM J. Algebraic Discrete Methods **4**(2), 231–236 (1983), https://doi.org/10.1137/0604025
12. Lagarias, J.C.: The Takagi function and its properties. In: Functions in number theory and their probabilistic aspects, pp. 153–189. RIMS Kôkyûroku Bessatsu, B34, Res. Inst. Math. Sci. (RIMS), Kyoto (2012)
13. Legendre, A.M.: Théorie des nombres. Paris: Firmin Didot Frères, Libraires, 3rd edition of the book published previously under the title Essai sur la théorie des nombres edn. (1830), https://gallica.bnf.fr/ark:/12148/bpt6k426107?rk=42918;4
14. McIlroy, M.D.: The number of 1's in binary integers: Bounds and extremal properties. SIAM Journal on Computing **3**(4), 255–261 (1974), https://doi.org/10.1137/0203020
15. Mohanty, V., Greenbury, S., Sarkany, T., Narayanan, S., Dingle, K., Ahnert, S., Louis, A.: Maximum mutational robustness in genotype–phenotype maps follows a self-similar blancmange-like curve. J. R. Soc. Interface **20**, 20230169 (2023), https://doi.org/10.1098/rsif.2023.0169
16. Muramoto, K., Okada, T., Sekiguchi, T., Shiota, Y.: An explicit formula of subblock occurrences for the p-adic expansion. Interdiscip. Inform. Sci. **8**(1), 115–121 (2002), https://doi.org/10.4036/iis.2002.115
17. Prodinger, H.: Generalizing the sum of digits function. SIAM J. Algebraic Discrete Methods **3**(1), 35–42 (1982), https://doi.org/10.1137/0603004
18. Stolarsky, K.B.: Power and exponential sums of digital sums related to binomial coefficient parity. SIAM J. Appl. Math. **32**(4), 717–730 (1977), https://doi.org/10.1137/0132060
19. Takagi, T.: A simple example of the continuous function without derivative. Tokio Math. Ges. (Proc. Phys.-Math. Soc. Japan) **1**, 176–177 (1903)