Interpolation with a function parameter from the category point of view

T. Lamby*and Samuel Nicolay†

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Abstract

We generalize here the notion of interpolation space of given exponent by replacing this exponent with Boyd functions. In particular, this approach leads to the usual interpolation method with a function parameter. We present some results in this general setting. Some are well-known, others not so well.

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1 Introduction

The origin of the theory of interpolation can be traced back to Marcinkiewicz [26] and the Riesz-Thorin theorem [33, 35], which states that if a linear function is continuous on L^p and L^q, then it is also continuous on L^r for r between p and q. Later, as it was shown that Sobolev spaces were constituted of functions that have a non-integer order of differentiability [21, 1, 34], various techniques were conceived to generate similar spaces. Among them were the interpolation methods. Let us briefly recall the basic definitions (see Section 3 for more details). Let A, A_0, A_1, B, B_0 and B_1 be Banach spaces. The pair (A, B) is called an interpolation pair if we have

$$A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1 \quad \text{and} \quad B_0 \cap B_1 \hookrightarrow B \hookrightarrow B_0 + B_1,$$

where $\hookrightarrow$ is the symbol for the continuous embedding and if any linear operator $T : A_0 + A_1 \to B_0 + B_1$ which maps continuously A_0 to B_0 and A_1 to B_1 also maps A to B continuously. Moreover, (A, B) is said to be of exponent $\theta \in [0, 1]$ if there exists a constant $C > 0$ such that

$$\|T\|_{A,B} \leq C\|T\|^{1-\theta}_{A_0,B_0}\|T\|^{\theta}_{A_1,B_1}, \tag{1}$$

for any operator T as above, where $\|T\|_{X,Y}$ is the norm of $T : X \to Y$.

The real interpolation methods [21, 4, 36, 17, 1, 34] have been generalized using a function parameter (see [31, 14, 7, 12, 16, 30, 27, 25, 32] and references

*Université de Liège, Département de mathématique – zone Polytech 1, 12 allée de la Découverte, Bât. B37, B-4000 Liège. thomas.lamby@uliege.be
†Université de Liège, Département de mathématique – zone Polytech 1, 12 allée de la Découverte, Bât. B37, B-4000 Liège. Orcid ID: 0000-0003-0549-0566. S.Nicolay@uliege.be

therein). Most of the times, these authors start from the K-method. Let A_0 and A_1 be two Banach spaces continuously embedded into a Hausdorff topological vector space so that $A_0 \cap A_1$ and $A_0 + A_1$ are well defined Banach spaces. One defines the K-functional by

$$K(t,a) := \inf_{a=a_0+a_1} \{\|a_1\|_{A_0} + t\|a_1\|_{A_1}\},$$

for $t > 0$ and $a \in A_0 + A_1$. Given $0 < \theta < 1$ and $q \in [1,\infty]$, a belongs to the interpolation space $(A_0, A_1)_{\theta,q}$ if $a \in A_0 + A_1$ and

$$(2^{-j\theta} K(2^j, a))_{j\in\mathbb{Z}} \in \ell^q. \tag{2}$$

The generalized version is obtained by replacing the sequence $(2^{-j\theta})_{j\in\mathbb{Z}}$ appearing in (2) with a Boyd function (see Section 2). Similar definitions have been proposed in [19, 23] and the relations between these techniques have been studied in [20]. The J-method is defined in a similar way: one sets

$$J(t,a) := \max\{\|a\|_{A_0}, t\|a\|_{A_1}\},$$

for $t > 0$ and $a \in A_0 \cap A_1$. This time, one considers

$$(2^{-j\theta} J(2^j, b_j))_{j\in\mathbb{Z}} \in \ell^q,$$

with $a = \sum_{j\in\mathbb{Z}} b_j$ and $b_j \in A_0 \cap A_1$ (for all j), the convergence being in $A_0 + A_1$. This approach can be generalized in the same way and one can show that both methods give rise to the same spaces, in the sense of Theorem 6.2 (with equivalence of norms). The Boyd functions form a natural apparatus for studying function spaces [2, 13, 28, 15, 29, 19, 23, 24] and interpolation methods with a function parameter provide an interesting tool in this context [14, 27, 32, 10]. For example, they lead to a definition of the Besov spaces of generalized smoothness based on the usual Sobolev spaces [23]. Other examples are given in [27].

In this work, we show that this generalized approach still allows a functorial interpretation. We introduce the Boyd functions earlier in the process, that is in the notion of interpolation of real exponent. We are thus naturally led to consider results such as the Aronszajn-Gagliardo theorem. We also consider real interpolation methods, but without confining ourselves to the K- and J-methods.

We begin by introducing the usual notions related to the Boyd functions in order to replace $t \mapsto t^\theta$ in (1) with a Boyd function ϕ, which leads to (3), as a starting point. Next, we explore the basic properties of this theory. We show that, under the right hypothesis, the usual results can be formulated in this more generalized setting. For example, the real interpolation methods are still equivalent in this context (this is a generalization of Theorem 2.2 from [14] for example) and a reiteration theorem still holds.

As we often rephrase standard theory exposed in classical textbooks (see [4, 8] for instance) using Boyd functions, proofs are given with a minimum of details and, in some cases, are omitted. Also, we tried to keep the notation as standard as possible. Throughout the paper, we use the letter C for a generic positive constant, while d is the dimension of the space if it makes sense.

2 Boyd functions

We recall here some basic properties of the Boyd functions. The interested reader can consult [27, 20] and the references therein for the details.

Definition 2.1. A function $\phi : (0,\infty) \to (0,\infty)$ is a Boyd function if

- $\phi(1) = 1$,
- ϕ is continuous,
- for any $t > 0$,
$$\bar{\phi}(t) := \sup_{s>0} \frac{\phi(ts)}{\phi(s)} < \infty.$$

The set of Boyd functions will be denoted by $\mathcal{B}$.

The function $\bar{\phi}$ is sub-multiplicative, Lebesgue-measurable and we have both $1/\bar{\phi}(1/t) \leq \phi(t) \leq \bar{\phi}(t)$ and $1/\phi(t) \leq 1/\bar{\phi}(1/t)$.

Definition 2.2. Given $\phi \in \mathcal{B}$, the lower and upper Boyd indices [6] are defined by
$$\underline{b}(\phi) := \sup_{t\in(0,1)} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to 0^+} \frac{\log \bar{\phi}(t)}{\log t}$$
and
$$\overline{b}(\phi) := \inf_{t\in(1,\infty)} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to \infty} \frac{\log \bar{\phi}(t)}{\log t},$$
respectively.

Let us give an family of Boyd functions that naturally generalize the function $t \mapsto t^\theta$ appearing in (2).

Example 2.3. Let ψ be a slowly varying function on $(0,\infty)$:
$$\lim_{t\to 0} \frac{\psi(tx)}{\psi(t)} = 1,$$
for any $x > 0$. For $\theta \in \mathbb{R}$, the function $\psi : t \mapsto t^\theta \psi(t)$ is a Boyd function such that $\underline{b}(\phi) = \overline{b}(\phi) = \theta$ [18]. Such functions are known as Karamata regularly varying functions [5]. A standard choice for the slowly varying function is $\psi = |\ln|^\gamma$, for $\gamma > 0$.

Such functions naturally appear when dealing with the law of the iterated logarithm [18], but logarithmic corrections are also commonly needed in interpolation theory [11].

If $\phi \in \mathcal{B}$, for $\varepsilon > 0$ and $R > 0$, there exists a constant $C > 0$ such that
$$C^{-1} r^{\overline{b}(\phi)+\varepsilon} \leq \phi(r) \leq C r^{\underline{b}(\phi)-\varepsilon},$$
for any $r \in (0,R)$ and a constant $C > 0$ such that
$$C^{-1} r^{\underline{b}(\phi)-\varepsilon} \leq \phi(r) \leq C r^{\overline{b}(\phi)+\varepsilon},$$
for any $r \geq R$. Moreover, for such a function ϕ, we have the following properties:

- $\underline{b}(\phi) > 0 \Leftrightarrow \bar{\phi} \in L^1_*(0,1) \Leftrightarrow \lim_{t\to 0^+} \bar{\phi}(t) = 0$,
- $\underline{b}(\phi) < 0 \Leftrightarrow \bar{\phi} \in L^1_*(1,\infty) \Leftrightarrow \lim_{t\to \infty} \bar{\phi}(t) = 0$,
- $\bar{b}(\phi) > 0 \Rightarrow \phi \in L^\infty(0,1)$,
- $\bar{b}(\phi) < 0 \Rightarrow \phi \in L^\infty(1,\infty)$,

where $L^q_*(a,b) = L^q(a,b,dt/t)$.

Definition 2.4. A function $\phi : (0,\infty) \to (0,\infty)$ is Boyd-regular if

- $\phi(1) = 1$,
- $\phi \in C^1(0,\infty)$,
- we have
$$0 < \inf_{t>0} t\frac{|\phi'(t)|}{\phi(t)} \leq \sup_{t>0} t\frac{|\phi'(t)|}{\phi(t)} < \infty.$$

The set of Boyd-regular functions will be denoted by $\mathcal{B}^*$.

We have $\mathcal{B}^* \subset \mathcal{B}$. The set of functions $\phi \in \mathcal{B}^*$ that are strictly increasing (resp. strictly decreasing) will be denoted by $\mathcal{B}^*_+$ (resp. $\mathcal{B}^*_-$).

Given two functions f and g defined on $(0,\infty)$, we write $f \sim g$ if there exists a constant $C > 0$ such that $C^{-1}g(t) \leq f(t) \leq Cg(t)$ for any $t > 0$. If $\phi \in \mathcal{B}$ is such that $\underline{b}(\phi) > 0$ (resp. $\bar{b}(\phi) < 0$), then there exists $\xi \in \mathcal{B}^*_+$ (resp. $\xi \in \mathcal{B}^*_-$) such that $\xi^{-1} \in \mathcal{B}^*_+$ (resp. $\xi^{-1} \in \mathcal{B}^*_-$) and $\phi \sim \xi$.

3 Interpolation and Boyd functions

We present here a generalization of the interpolation spaces of real exponent using Boyd functions. We shall reserve $\mathcal{N}$ for the category of all normed vector spaces (the objects of $\mathcal{N}$ are normed vector spaces and the morphisms are the bounded linear operators) and $\mathcal{B}$ for the sub-category of all Banach spaces.

Let us first briefly recall the basic theory of interpolation (see [4, 8] for example). If $(A_0, \|\cdot\|_{A_0})$ and $(A_1, \|\cdot\|_{A_1})$ are two normed topological vector spaces, A_0 and A_1 are compatible if they are both subspaces of a Hausdorff topological vector space. In this case, $A_0 \cap A_1$ is a normed vector space for the norm
$$\|a\|_{A_0\cap A_1} := \max\{\|a\|_{A_0}, \|a\|_{A_1}\}$$
and $A_0 + A_1$ is a normed vector space for the norm
$$\|a\|_{A_0+A_1} := \inf_{a=a_0+a_1}\{\|a_0\|_{A_0}, \|a_1\|_{A_1}\}.$$

Moreover, if A_0 and A_1 are both complete, so are $A_0 \cap A_1$ and $A_0 + A_1$. Let $\mathcal{C}$ be a sub-category of $\mathcal{N}$ and denote by $\mathcal{C}_c$ a category of compatible couples $\bar{A} = (A_0, A_1)$ (such that $A_0 \cap A_1$ and $A_0 + A_1$ are in $\mathcal{C}$). The morphisms $T : (A_0, A_1) \to (B_0, B_1)$ in $\mathcal{C}_c$ are bounded linear mappings from $A_0 + A_1$ to $B_0 + B_1$ such that both $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are morphisms in $\mathcal{C}$. The two basic functors Σ and Δ from $\mathcal{C}_c$ to $\mathcal{C}$ are defined as follows: $\Sigma(T) = \Delta(T) = T$ and

$$\Delta(\bar{A}) = A_0 \cap A_1 \quad \text{and} \quad \Sigma(\bar{A}) = A_0 + A_1.$$

In the sequel, $\mathcal{C}$ will stand for any subcategory of $\mathcal{N}$ such that $\mathcal{C}$ is closed under the operations sum and intersection, while $\mathcal{C}_c$ will stand for the category of all compatible couples $\bar{A}$ of spaces in $\mathcal{C}$. Given a couple $\bar{A} = (A_0, A_1)$ in $\mathcal{C}_c$, a space $A \in \mathcal{C}$ is an intermediate space between A_0 and A_1 (or with respect to $\bar{A}$) if

$$\Delta(\bar{A}) \hookrightarrow A \hookrightarrow \Sigma(\bar{A}).$$

Such a space A is called an interpolation space between A_0 and A_1 (or with respect to $\bar{A}$) if in addition $T : \bar{A} \to \bar{A}$ implies $T : A \to A$. Now, if $\bar{B}$ is another couple in $\mathcal{C}_c$, two spaces A and B in $\mathcal{C}$ are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$ if A and B are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$ respectively and if $T : \bar{A} \to \bar{B}$ implies $T : A \to B$. These interpolation spaces are uniform interpolation spaces if

$$\|T\|_{A,B} \leq C \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}$$

always holds for some constant $C > 0$. If $C = 1$ in the previous inequality, A and B will be called exact interpolation spaces. Of course, in the case $B = A$, we will omit any reference to the second interpolation space B; in particular, we set $\|T\|_X := \|T\|_{X,X}$. An interpolation functor (or interpolation method) on $\mathcal{C}$ is a functor $F : \mathcal{C}_c \to \mathcal{C}$ such that if $\bar{A}$ and $\bar{B}$ are two couples in $\mathcal{C}_c$, then $F(\bar{A})$ and $F(\bar{B})$ are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$, with $F(T) = T$ for all $T : \bar{A} \to \bar{B}$. Naturally, the descriptive terms related to the interpolation spaces can be transposed to the interpolation functors; we shall say that F is a uniform (exact) interpolation functor if $F(\bar{A})$ and $F(\bar{B})$ are uniform (exact) interpolation spaces with respect to $\bar{A}$ and $\bar{B}$.

Given $\phi \in \mathcal{B}$, we will denote by ϕ_* the function explicitly defined by

$$\phi_*(t) := \frac{t}{\phi(t)},$$

for $t > 0$.

Definition 3.1. Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two couples in $\mathcal{C}_c$; two interpolation spaces A and B with respect to $\bar{A}$ and $\bar{B}$ respectively are of exponent $\phi \in \mathcal{B}$ if, for any $T : \bar{A} \to \bar{B}$,

$$\|T\|_{A,B} \leq C \bar{\phi}_*(\|T\|_{A_0,B_0}) \bar{\phi}(\|T\|_{A_1,B_1}) \tag{3}$$

always holds for some constant $C > 0$. If $C = 1$, we say that A and B are exact of exponent ϕ.

Most of the time, we will assume $\underline{b}(\phi) > 0$ and $\bar{b}(\phi) < 1$, which corresponds to the classical assumption $0 < \theta < 1$ in (2) for example. The extreme cases (0 and 1) are not always meaningful, even in the classical setting [9, 4, 11].

Let us remark that A and B are of exponent ϕ if and only if they are of exponent $1/\phi(1/\cdot)$.

Theorem 3.2. *Let $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, A be an interpolation space with respect to $\bar{A}$ such that $A_\phi \hookrightarrow \Sigma(\bar{A})$, where A_ϕ is the set of all $x \in \Sigma(\bar{A})$ admitting a representation of the form*

$$x = \sum_j T_j a_j,$$

with $T_j : \bar{A} \to \bar{A}$ and $a_j \in A$ for all j, the convergence being in $\Sigma(\bar{A})$ and with the norm

$$\|x\|_{A_\phi} := \inf_{x=\sum_j T_j a_j} \sum_j \bar{\phi}_*(\|T_j\|_{A_0})\bar{\phi}(\|T_j\|_{A_1})\|a_j\|_A.$$

Then A_ϕ is a minimal exact interpolation space of exponent ϕ with respect to $\bar{A}$ that contains A.

Proof. We directly get $A \hookrightarrow A_\phi$, which implies that A_ϕ is an interpolation space with respect to $\bar{A}$. From (3) with $C = 1$, it is easy to check that A_ϕ is exact of exponent ϕ. Finally, if B is an exact interpolation space of exponent ϕ with respect to $\bar{A}$ that contains A, the same formula leads to $A_\phi \hookrightarrow B$. □

4 Aronszajn-Gagliardo-type theorems

In 1965, Aronszajn and Gagliardo showed that any interpolation space of a given Banach couple could be realized as the value of a minimal or maximal interpolation functor on the category of all Banach couples [3]. Later, connections between important methods of interpolation and this result were identified [7, 16], highlighting the importance of this theorem.

Theorem 4.1. *Let A be an interpolation space of exponent ϕ with respect to $\bar{A}$, where $\phi \in \mathcal{B}$ is such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$. If $\bar{X}$ is a given couple, the set $F(\bar{X})$ consists of those $x \in \Sigma(\bar{X})$ which admit a representation $x = \sum_j T_j a_j$ (with convergence in $\Sigma(\bar{X})$), with $T_j : \bar{A} \to \bar{X}$ and $a_j \in A$ for all j. Define*

$$N_\phi(\sum_j T_j a_j) := \sum_j \bar{\phi}_*(\|T_j\|_{A_0,X_0})\bar{\phi}(\|T_j\|_{A_1,X_1})\,\|a_j\|_A,$$

so that $F(\bar{X})$ can be equipped with the norm

$$\|x\|_{F(\bar{X})} := \inf_{x=\sum_j T_j a_j} N_\phi(\sum_j T_j a_j).$$

If $F(\bar{X}) \hookrightarrow \Sigma(\bar{X})$ for all couples $\bar{X}$, then F gives a minimal interpolation functor which is exact of exponent ϕ such that $F(\bar{A}) = A$.

Proof. The classical proof can be adapted without any difficulty to this context (see [4] for example). □

Theorem 4.2. *Let A be an interpolation space of exponent ϕ with respect to $\bar{A}$, where $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$. If $\bar{X}$ is a given couple, the set $F(\bar{X})$ consists of those $x \in \Sigma(\bar{X})$ such that $Tx \in A$ for all $T : \bar{X} \to \bar{A}$ with norm*

$$\|x\|_{F(\bar{X})} := \sup\{\|Tx\|_A : \bar{\phi}_*(\|T\|_{X_0,A_0})\bar{\phi}(\|T\|_{X_1,A_1}) \leq 1\}.$$

If $\Delta(\bar{X}) \hookrightarrow F(\bar{X})$ for all couples $\bar{X}$, then F gives a maximal interpolation functor which is exact of exponent ϕ such that $F(\bar{A}) = A$.

Proof. Let $x \in F$ and $f \in \Sigma(\bar{X})'$ such that $\|f\| = 1$ and $f(x) = \|x\|_{\Sigma(\bar{X})}$. If C_A is the constant corresponding to the continuous inclusion $\Delta(\bar{A}) \hookrightarrow A$, assuming that $\Delta(\bar{A}) \neq \{0\}$, for a given $\varepsilon > 0$, there exists an element $a \in \Delta(\bar{A})$ with $\|a\|_{\Delta(\bar{A})} = 1$ such that $\|a\|_A > C_A - \varepsilon$.
Consider the operator $P \in L(\bar{X}, \bar{A})$ defined by $P(y) = f(y)a$. One has

$$\bar{\phi}_*(\|P\|_{X_0,A_0})\bar{\phi}(\|P\|_{X_1,A_1}) \leq \bar{\phi}_*(\|a_0\|_{A_0})\bar{\phi}(\|a_1\|_{A_1}) \leq 1.$$

Therefore,

$$C_A\|x\|_{\Sigma(\bar{X})} \leq \|x\|_{F(\bar{X})}.$$

As a consequence, $F(\bar{X})$ is an interpolation space with respect to $\bar{X}$.

It is easy to check that F is an exact interpolation functor of exponent ϕ such that $F(\bar{A}) = A$ that is maximal. □

5 The K-method

Mimicking the usual K-method [31], one can construct here a family of interpolation functors on $\mathcal{N}$; we obtain the real interpolation spaces with a function parameter (see [27, 32] and references therein).

Let us recall that given a couple $\bar{A}$ and $t > 0$,

$$K(t, a) = K(t, a; \bar{A}) := \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}),$$

for $a \in \Sigma(\bar{A})$. The function $t \mapsto K(t, a)$ is positive, increasing and concave. We also have $K(t, a) \leq \max\{1, t/s\}K(s, a)$.

For $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, let $K_{\phi,q}(\bar{A})$ be the space of all $a \in \Sigma(\bar{A})$ such that

$$\|a\|_{K_{\phi,q}} = \|a\|_{K_{\phi,q}(\bar{A})} := (\int_0^\infty (\frac{1}{\phi(t)} K(t, a))^q \frac{dt}{t})^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

Theorem 5.1. *For $\phi \in \mathcal{B}$, $K_{\phi,q}$ is an exact interpolation functor of exponent ϕ on $\mathcal{N}$. Moreover, we have*

$$K(t, a) \leq C\phi(t)\|a\|_{K_{\phi,q}}.$$

Proof. The fact that $K_{\phi,q}$ is an exact interpolation functor is well known [27, 32]. The classical proof (see [4], Section 3.1) can be easily adapted to show that this functor is of exponent ϕ. □

Obviously, $K_{1/\phi(1/\cdot),q}$ is also exact of exponent ϕ on $\mathcal{N}$.

Remark 5.2. From the fact that, for $a \in \Sigma(\bar{A})$, $t \mapsto K(t, a)$ is non-decreasing and $t \mapsto K(t, a)/t$ is non-increasing, since ϕ is a Boyd function, a belongs to $K_{\phi,q}(\bar{A})$ if and only if $\big(K(2^j, a)/\phi(2^j)\big)_{j\in\mathbb{Z}} \in \ell^q$.

The usual results can be revised to get the following propositions. The proofs are omitted, since they can be easily obtained from the original ones (see [4], Section 3.4).

Proposition 5.3. *Given $\phi \in \mathcal{B}$, for $1 \leq p \leq q \leq \infty$, we have the continuous inclusion $K_{\phi,p}(\bar{A}) \hookrightarrow K_{\phi,q}(\bar{A})$.*

Proposition 5.4. *For $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, we have*

$$K_{\phi,q}(A_0, A_1) = K_{\phi_*,q}(A_1, A_0).$$

Proposition 5.5. *Let $\phi, \phi_0, \phi_1 \in \mathcal{B}$ and $q, q_0, q_1 \in [1, \infty]$; if $\overline{b}(\phi_0) < \underline{b}(\phi)$ and $\overline{b}(\phi) < \underline{b}(\phi_1)$, then*

$$K_{\phi_0,q_0}(\bar{A}) \cap K_{\phi_1,q_1}(\bar{A}) \hookrightarrow K_{\phi,q}(\bar{A}).$$

Proposition 5.6. *Given $\phi_0, \phi_1 \in \mathcal{B}$ such that $\overline{b}(\phi_0) < \underline{b}(\phi_1)$ and $q \in [1, \infty]$, if $A_1 \hookrightarrow A_0$ then $K_{\phi_1,q}(A_0, A1) \hookrightarrow K_{\phi_0,q}(A_0, A_1)$.*

Proposition 5.7. *Let $\phi \in \mathcal{B}$ and $q \in [1, \infty]$; if A_0 and A_1 are complete, so is $K_{\phi,q}(A_0, A_1)$.*

Let us also recall the duality theorem and the power theorem, which are also well-known in this context (and can be easily obtained from the proofs in [4], Sections 3.7 and 3.11).

Theorem 5.8. *Let $\bar{A} = (A_0, A_1)$ be a couple of Banach spaces such that $\Delta(\bar{A})$ is dense in both A_0 and A_1; for $1 \leq q < \infty$, $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, we have*

$$K_{\phi,q}(\bar{A})' = K_{\phi^-,q'}(\overline{A'}),$$

where q' is the exponent conjugate to q and $\phi^-(t) = 1/\phi(1/t)$.

Remark 5.9. For the case $q = \infty$, one has

$$\bar{K}_{\phi,\infty}(\bar{A})' = K_{\phi^-,1}(\overline{A'}),$$

where $\bar{K}_{\phi,\infty}(\bar{A})$ denotes the closure of $K_{\phi,\infty}(\bar{A})$ in $\Delta(\bar{A})$.

Theorem 5.10. *For $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, we have*

$$K_{\phi,q}(A_0^p, A_1^p)^{1/p} = K_{\phi_p,pq}(\bar{A}).$$

6 The J-method

We can also consider the J-method to get a second family of explicit interpolation functors.

Let us recall that given a couple $\bar{A}$ and $t > 0$,

$$J(t, a) = J(t, a; \bar{A}) := \max\{(\|a\|_{A_0}, t\|a\|_{A_1}),$$

for $a \in \Delta(\bar{A})$.

For $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, let $J_{\phi,q}(\bar{A})$ be the space of all $a \in \Sigma(\bar{A})$ which can be represented by $a = \int_0^\infty b(t)\, dt/t$, with convergence in $\Sigma(\bar{A})$, where b is measurable, takes its values in $\Delta(\bar{A})$ for $t > 0$ and

$$t \mapsto \frac{J(t, b(t))}{\phi(t)} \in L_*^q. \tag{4}$$

This space is equipped with the norm

$$\|a\|_{J_{\phi,q}} = \|a\|_{J_{\phi,q}(\bar{A})} := \inf_{b\in J_{\phi,q}(\bar{A})} \|\frac{J(t,b(t))}{\phi(t)}\|_{L^q_*}.$$

It is well known that the equivalence theorem still holds [14]; however as we use here slightly different arguments, we sketch a proof.

Lemma 6.1. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$ and $q \in [1,\infty]$, we have $K_{\phi,q}(\bar{A}) \hookrightarrow J_{\phi,q}(\bar{A})$.*

Proof. Let a be an element of $J_{\phi,q}(\bar{A})$, so that $a = \int_0^\infty b(s)ds/s$ with condition (4) satisfied. From the trivial decomposition $b = b + 0 = 0 + b$, we get $b \in \Sigma(\bar{A})$ and we have

$$K(t,a) \leq \int_0^\infty \min\{\|b\|_{A_0}, t\|b\|_{A_1}\} \frac{ds}{s}.$$

We get

$$\begin{aligned}\frac{K(t,a)}{\phi(t)} &\leq \int_0^\infty \min\{(\frac{\phi(t)}{\phi(s)})^{-1}, \frac{t}{s}(\frac{\phi(t)}{\phi(s)})^{-1}\} \frac{J(s,b(s))}{\phi(s)} \frac{ds}{s} \\ &\leq \int_0^\infty \min\{\bar{\phi}(\frac{s}{t}), \frac{t}{s}\bar{\phi}(\frac{s}{t})\} \frac{J(s,b(s))}{\phi(s)} \frac{ds}{s}.\end{aligned}$$

The last expression is a convolution product (for the multiplicative group $\mathbb{R}_+$ and the Haar measure ds/s) of the function $s \mapsto J(s,b(s))/\phi(s)$ from L^q_* and $s \mapsto \min\{\bar{\phi}(1/s), \bar{\phi}(1/s)s\}$. This last function belongs to L^1_* if $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$. By Young's inequality, we get

$$\|\frac{K(t,a)}{\phi(t)}\|_{L^q_*} \leq C\|\frac{J(t,b(t))}{\phi(t)}\|_{L^q_*},$$

which is sufficient to conclude. □

Theorem 6.2. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$ and $q \in [1,\infty]$, we have*

$$J_{\phi,q}(\bar{A}) = K_{\phi,q}(\bar{A}).$$

Proof. Let a be an element of $K_{\phi,q}(\bar{A})$; one has $K(t,a) \leq C\phi(t)$ for any $t > 0$. For $j \in \mathbb{Z}$, let $a_0^{(j)} \in A_0$ and $a_1^{(j)} \in A_1$ be such that $a = a_0^{(j)} + a_1^{(j)}$ and

$$\|a_0^{(j)}\|_{A_0} + \|a_1^{(j)}\|_{A_1} \leq 2K(e^j, a),$$

where e is Euler's number.
Since ϕ is a Boyd function, we have

$$0 \leq \|a_0^{(j)}\|_{A_0} \leq C\phi(e^j) \leq C(e^j)^{\underline{b}(\phi)-\underline{b}(\phi)/2},$$

where the right-hand side tends to 0 as j tends to $-\infty$. On the other hand,

$$0 \leq \|a_1^{(j)}\|_{A_1} \leq C(e^j)^{\overline{b}(\phi)+\varepsilon}e^{-j}$$

tends to 0 as j tends to ∞.

For $j \in \mathbb{Z}$, let us set

$$b_j := a_0^{(j+1)} - a_0^{(j)} = a_1^{(j)} - a_1^{(j+1)} \in \Delta(\bar{A}),$$

so that $\sum_{j\in\mathbb{Z}} b_j = a$ with convergence in $\Sigma(\bar{A})$. For $t \in (e^j, e^{j+1})$, we get

$$\|b_j\|_{A_0} \leq 2K(e^{j+1}, a) + 2K(e^j, a)$$

and so $t\|b_j\|_{A_1} \leq CK(t,a)$. Finally, by setting $b(t) = b_j$ for $t \in (e^j, e^{j+1})$, we get $a = \int_0^\infty b(t)dt/t$ and thus $J(t,a) \leq CK(t,a)$. □

Remark 6.3. One can check that a belongs to $J_{\phi,q}(\bar{A})$ if and only if $a = \sum_{j\in\mathbb{Z}} b_j$ in $\Sigma(\bar{A})$ with $b_j \in \Delta(\bar{A})$ for all j and $\big(J(2^j, b_j)/\phi(2^j)\big)_{j\in\mathbb{Z}}$ belongs to ℓ^q.

Considering the classical results, we get the following properties. Once again, the proofs are left to the reader (see [4], Section 3.4).

Proposition 6.4. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$ and $q \in [1,\infty)$, $\Delta(\bar{A})$ is dense in $K_{\phi,q}(\bar{A})$.*

Proposition 6.5. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, the closure of $\Delta(\bar{A})$ in $K_{\phi,\infty}(\bar{A})$ is the space of the elements a such that $K(t,a)/\phi(t)$ tends to 0 as t tends to 0 or ∞.*

The interpolation functors $J_{\psi,1}$ and $K_{\psi,\infty}$ are extremal (in the sense of Theorem 6.6), using the appropriate function ψ. For $\phi \in \mathcal{B}$, let us define

$$\underline{\phi}(t) := \sup_{s>0} \frac{\phi(s)}{\phi(ts)}.$$

Theorem 6.6. *If F is an interpolation functor of exponent $\phi \in \mathcal{B}$ with $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, then, for any compatible Banach couple $\bar{A} = (A_0, A_1)$, one has*

$$J_{1/\underline{\phi},1}(\bar{A}) \hookrightarrow F(\bar{A}).$$

Moreover, if $\Delta(\bar{A})$ is dense in A_0 and A_1, then

$$F(\bar{A}) \hookrightarrow K_{\bar{\phi},\infty}(\bar{A}).$$

Proof. The proof in [4], Section 3.9, can be easily modified to get the desired result. □

7 Other real interpolation methods

As expected, the "espaces de moyennes" [21] and the trace spaces ("espaces de trace") [22] can be generalized in the context of the Boyd functions. The induced methods are equivalent to the K-method.

Given a compatible Banach couple $\bar{A} = (A_0, A_1)$ and $p \in [1,\infty]$, let $X_{\phi,p}(\bar{A})$ be the subspace of $\Sigma(\bar{A})$ defined by the norm

$$\|a\|_{X_{\phi,p}} := \inf_{a=a_0(t)+a_1(t)} (\|\frac{a_0(t)}{\phi(t)}\|^p_{L^p_*(A_0)} + \|\phi_*(t)a_1(t)\|^p_{L^p_*(A_1)})^{1/p}.$$

Theorem 7.1. *For $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, we have*

$$X_{\phi,p}(\bar{A}) = K_{\phi,p}(\bar{A}).$$

Proof. One has

$$\begin{aligned}\|a\|^p_{X_{\phi,p}} &\sim \int_0^\infty \inf_{a=a_0(t)+a_1(t)} \Big(\frac{1}{\phi(t)^p}\|a_0(t)\|^p_{A_0} + \phi_*(t)^p\|a_1(t)\|^p_{A_1}\Big)\frac{dt}{t} \\ &\sim \int_0^\infty \inf_{a=\tilde{a}_0(\tau)+\tilde{a}_1(\tau)} \frac{1}{\phi(\tau^{1/p})^p}(\|\tilde{a}_0(\tau)\|^p_{A_0} + \tau\|\tilde{a}_1(\tau)\|^p_{A_1}))\frac{d\tau}{\tau}.\end{aligned}$$

Using the power theorem, we get

$$X_{\phi,p}(\bar{A})^p = K_{\phi_{1/p},1}(A_0^p, A_1^p) = K_{\phi,p}(\bar{A})^p,$$

as desired. □

If f is an A-valued function on $(0,\infty)$, $f^{(m)}$ will denote the derivative of order m of f in the sense of the distribution theory. The space $X^m_{\phi,p}(\bar{A})$ is the space of $\Sigma(\bar{A})$-valued functions f on $(0,\infty)$ that are locally A_0-integrable and such that $f^{(m)}$ is locally A_1-integrable with

$$\|f\|_{X^m_{\phi,p}} := \max(\|\phi f\|_{L^p_*(A_0)}, \|\frac{1}{\phi_*(t)} f^{(m)}(t)\|_{L^p_*(A_1)}) < \infty.$$

We shall say that f has a trace in $\Sigma(\bar{A})$ if $f(t)$ converges in $\Sigma(\bar{A})$ as $t \to 0^+$; in this case we set

$$\mathrm{trace}(f) := \lim_{t\to 0^+} f(t).$$

The trace space of functions in $X^m_{\phi,p}(\bar{A})$ will be denoted $TX^m_{\phi,p}(\bar{A})$; it is the space of all $a \in \Sigma(\bar{A})$ such that there exists $f \in X^m_{\phi,p}(\bar{A})$ with $\mathrm{trace}(f) = a$. This space is a Banach space for the norm

$$\|a\|_{TX^m_{\phi,p}} := \inf_{\mathrm{trace}(f)=a} \|f\|_{X^m_{\phi,p}}.$$

Theorem 7.2. *For $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, we have*

$$TX^m_{\phi,p}(\bar{A}) = K_{\phi,p}(\bar{A}).$$

Proof. This result can be shown using the same proof as for the corresponding theorem of [4], Section 3.12. □

8 A reiteration theorem

We give here a stability result for the repeated use of the real interpolation method.

Let us recall (see [27]) that given $\psi := \phi_1/\phi_0$ with $\phi_0, \phi_1 \in \mathcal{B}$, $q_0, q_1 \in [1,\infty]$, $E_0 := K_{\phi_0,q_0}(\bar{A})$ and $E_1 := K_{\phi_1,q_1}(\bar{A})$, if $\underline{b}(\psi) > 0$, there exists a bijection $\xi \in \mathcal{B}^*_+$ such that $\psi \sim \xi$ and

$$K(t,a;\bar{E}) \sim \|\frac{K(\cdot,a)}{\phi_0}\|_{L^{q_0}_*(0,\xi^{-1}(t))} + t\|\frac{K(\cdot,a)}{\phi_1}\|_{L^{q_1}_*(\xi^{-1}(t),\infty)},$$

with $\bar{E} = (E_0, E_1)$ as soon as both the following conditions are satisfied:

- $\underline{b}(\phi_0) > 0$ if $q_0 < \infty$ or $\sup_{t \leq 1} \bar{\phi}_0(t) < \infty$ if $q_0 = \infty$,
- $\bar{b}(\phi_1) < 1$ if $q_1 < \infty$ or $\sup_{t \geq 1} \bar{\phi}_1(t)/t < \infty$ if $q_1 = \infty$.

We first need some further results concerning the Boyd functions.

Lemma 8.1. *Let u, v, ϕ be functions from $(0, \infty)$ to $(0, \infty)$; if $u \sim v$ and $\phi \in \mathcal{B}$ is such that $\underline{b}(\phi) > 0$ or $\bar{b}(\phi) < 0$, then $\phi \circ u \sim \phi \circ v$.*

Proof. Let us suppose that $\underline{b}(\phi) > 0$ and let $\xi \in \mathcal{B}_+^*$ be such that $C_1 \phi \leq \xi \leq C_2 \phi$ for two constants $C_1, C_2 > 0$. Now, let $C_1', C_2' > 0$ be two constants such that $C_1' v \leq u \leq C_2' v$. It is easy to check that we have

$$\frac{C_1}{C_2 \bar{\xi}(C_2')} \phi \circ u \leq \phi \circ v \leq \frac{C_2 \bar{\xi}(1/C_1')}{C_1} \phi \circ u.$$

The case $\bar{b}(\phi) < 0$ can be treated in the same way. □

Lemma 8.2. *Let $\phi_1, \phi_2 \in \mathcal{B}$ be such that $\underline{b}(\phi_2) > 0$. If $\underline{b}(\phi_1) > 0$, then $\phi_1 \circ \phi_2$ belongs to $\mathcal{B}$ and $\underline{b}(\phi_1 \circ \phi_2) > 0$. If $\underline{b}(\phi_1) < 0$, then $\phi_1 \circ \phi_2 \in \mathcal{B}$ and $\underline{b}(\phi_1 \circ \phi_2) < 0$.*

Proof. Let us suppose that $\underline{b}(\phi_1) > 0$ and let $\xi, \eta \in \mathcal{B}_+^*$ be such that $\xi \sim \phi_1$ and $\eta \sim \phi_2$. For $t > 0$, one has

$$\overline{\phi_1 \circ \phi_2}(t) \leq C \bar{\xi}(\bar{\eta}(t)),$$

so that $\overline{\phi_1 \circ \phi_2}(t)$ tends to 0 as t tends to 0^+. As a consequence, we have $\bar{b}(\phi_1 \circ \phi_2) > 0$. The case $\bar{b}(\phi_1 \circ \phi_2) < 0$ can be treated in the same way. □

Let us recall the following notions. Let $\bar{A}$ be a couple of normed vector spaces and $\phi \in \mathcal{B}$; if X is an intermediate spaces with respect to $\bar{A}$, X is of class $\mathcal{C}_K(\phi; \bar{A})$ if

$$K(t, a) \leq C \phi(t) \|a\|_X,$$

for all $a \in X$. In the same way, X is of class $\mathcal{C}_J(\phi; \bar{A})$ if

$$\phi(t) \|a\|_X \leq C J(t, a),$$

for all $a \in \Delta(\bar{A})$. Finally, X is of class $\mathcal{C}(\phi; \bar{A})$ if it is both of class $\mathcal{C}_K(\phi; \bar{A})$ and $\mathcal{C}_J(\phi; \bar{A})$.

Proposition 8.3. *Let $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\bar{b}(\phi) < 1$; X is of class $\mathcal{C}_K(\phi; \bar{A})$ if and only if $\Delta(\bar{A}) \hookrightarrow X \hookrightarrow K_{\phi,\infty}(\bar{A})$.*

Proof. This is clear since we have $X \hookrightarrow K_{\phi,\infty}(\bar{A})$ if and only if

$$\sup_{t > 0} \frac{K(t, a)}{\phi(t)} \leq C \|a\|_X,$$

X being an intermediate space with respect to $\bar{A}$. □

Proposition 8.4. *Let $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\bar{b}(\phi) < 1$; X is of class $\mathcal{C}_J(\phi; \bar{A})$ if and only if $K_{\phi,1}(\bar{A}) \hookrightarrow X \hookrightarrow \Sigma(\bar{A})$.*

Proof. Let us suppose that X is Banach space of class $\mathcal{C}_J(\phi;\bar{A})$; for $a=\sum_{j\in\mathbb{Z}} b_j$ in $\Sigma(\bar{A})$, we have

$$\|a\|_X \le \sum_{j\in\mathbb{Z}} \|b_j\|_X \le C\sum_{j\in\mathbb{Z}} \frac{J(2^j,a)}{\phi(2^j)},$$

so that $K_{\phi,1}(\bar{A})$ is included in X.

On the other hand, if $K_{\phi,1}(\bar{A})$ is included in X, let m be an integer and set $b_m=a$ and $b_j=0$ for $j\neq m$. In this case, we have

$$\|a\|_X \le C\|a\|_{K_{\phi,1}} = C\frac{J(2^m,a)}{\phi(2^m)},$$

so that X is of class $\mathcal{C}_J(\phi;\bar{A})$. □

Let us now give a generalization of the reiteration theorem from [4].

Theorem 8.5. *If for $j\in\{0,1\}$, X_j is of class $\mathcal{C}_J(\phi_j;\bar{A})$ with $\underline{b}(\phi_j)\ge 0$ and $\overline{b}(\phi_j)\le 1$, let $\phi\in\mathcal{B}$ be such that $\underline{b}(\phi)>0$ and $\overline{b}(\phi)<1$ and set $\theta=\phi_1/\phi_0$, $\psi=(\phi\circ\theta)\phi_0$; if $\underline{b}(\theta)>0$ or $\overline{b}(\theta)<0$ then $K_{\phi,q}(\bar{X})=K_{\psi,q}(\bar{A})$.*

In particular, if for $\underline{b}(\phi_j)>0$ and $\overline{b}(\phi_j)<1$, the spaces $K_{\phi_j,q_j}(\bar{A})$ are complete ($j\in\{0,1\}$), then

$$K_{\phi,q}(K_{\phi_0,q_0}(\bar{A}),K_{\phi_1,q_1}(\bar{A}))=K_{\psi,q}(\bar{A}).$$

Proof. For $a=a_0+a_1\in K_{\phi,q}(\bar{X})$, we have

$$K(t,a;\bar{A})\le C\phi_0(t)K(\theta(t),a;\bar{X}).$$

Therefore,

$$\|a\|_{K_{\psi,q}(\bar{A})}\le C\Big(\int_0^\infty\Big(\frac{K(\phi_1(t)/\phi_0(t),a;\bar{X})}{\psi(t)/\phi_0(t)}\Big)^q\frac{dt}{t}\Big)^{1/q},$$

so that for $s=\theta(t)$, we get

$$\|a\|_{K_{\psi,q}(\bar{A})}\le C\Big(\int_0^\infty\frac{K(s,a;\bar{X})}{\phi(s)}\Big)^q\frac{ds}{s}\Big)^{1/q}$$

and thus $K_{\phi,q}(\bar{X})\hookrightarrow K_{\psi,q}(\bar{A})$.

Now, for $a=\int_0^\infty b(t)dt/t\in J_{\phi,q}(\bar{X})$, we have

$$\begin{aligned}\phi_0(t)\,K(\theta(t),a;\bar{X}) &\le \int_0^\infty \phi_0(t)\,K(\theta(t),b(s);\bar{X})\,\frac{ds}{s}\\ &\le \int_0^\infty \phi_0(t)\,\min\{1,\frac{\theta(t)}{\theta(s)}\}\,J(\theta(s),b(s);\bar{X})\,\frac{ds}{s}\\ &\le C\int_0^\infty \min\{\bar{\phi}_0(t/s),\bar{\phi}_1(t/s)\}\,J(s,b(s);\bar{A})\,\frac{ds}{s},\end{aligned}$$

so that for $u=\theta(t)$ and $s=\sigma t$, we get

$$\begin{aligned}\|a\|_{K_{\phi,q}(\bar{X})} &\le C\Big(\int_0^\infty\Big(\frac{\phi_0(t)K(\theta(t),a;\bar{X})}{\phi_0(t)\phi(\theta(t))}\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &\le C\Big(\int_0^\infty\Big(\int_0^\infty\frac{1}{\psi(s/\sigma)}\min\{\bar{\phi}_0(1/\sigma),\bar{\phi}_1(1/\sigma)\}\,J(s,b(s);\bar{A})\,\frac{ds}{s}\Big)^q\frac{d\sigma}{\sigma}\Big)^{1/q}\\ &\le C\Big\|\frac{J(s,b(s);\bar{A})}{\psi(s)}\Big\|_{L^q_*}\end{aligned}$$

and thus, using the equivalence theorem, $K_{\psi,q}(\bar{A}) \hookrightarrow K_{\phi,q}(\bar{X})$. $\square$

9 A compactness theorem

Using the previous reiteration theorem, one can show that the classical compactness theorem [4] still holds in the setting of Boyd functions.

Theorem 9.1. *Let $\bar{A}$ be a couple of Banach spaces, B be a Banach space and consider a bounded linear operator T such that $T : A_0 \to B$ is compact and $T : A_1 \to B$ (not necessarily compact); if $E \in \mathcal{C}_K(\phi; \bar{A})$ for some $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, then $T : E \to B$ is also compact.*

Proof. The proof from [4], Section 3.8, can be easily adapted to provide the desired result. $\square$

In the same way, we get the following theorem.

Theorem 9.2. *Let $\bar{A}$ be a couple of Banach spaces, B be a Banach space and consider a bounded linear operator T such that $T : B \to A_0$ is compact and $T : B \to A_1$ (not necessarily compact); if $E \in \mathcal{C}_J(\phi; \bar{A})$ for some $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, then $T : B \to E$ is also compact.*

Corollary 9.3. *Let $\phi_0, \phi_1 \in \mathcal{B}$ be such that*

$$0 < \underline{b}(\phi_0) \leq \overline{b}(\phi_0) < \underline{b}(\phi_1) \leq \overline{b}(\phi_1) < 1;$$

if A_0 and A_1 are two Banach spaces such that $A_1 \hookrightarrow A_0$ compactly, then

$$K_{\phi_1,q_1}(\bar{A}) \hookrightarrow K_{\phi_0,q_0}(\bar{A}),$$

with compact inclusion.

Proof. Since the identity $A_1 \to A_0$ is compact, Theorem 9.1 implies $K_{\phi_1,a_1}(\bar{A}) \hookrightarrow A_0$ with compact inclusion. Now, from Theorem 9.2, we also have

$$K_{\phi_1,q_1}(\bar{A}) \hookrightarrow K_{f,q_0}(A_0, K_{\phi_1,q_1}(\bar{A})),$$

with compact inclusion. Since Theorem 8.5 implies

$$K_{f,q_0}(A_0, K_{\phi_1,q_1}(\bar{A})) = K_{\phi_0,q_0}(\bar{A}),$$

for f such that $\phi_0 = f \circ \phi_1$, we can conclude. $\square$

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