Some known results and open questions about eventually dendric shift spaces

France Gheeraert

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Left, right and bi-extensions

$$E_X^L(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}(X)\}, \quad E_X^R(w) = \{b \in \mathcal{A} \mid wb \in \mathcal{L}(X)\},$$

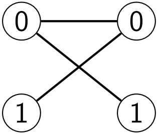
$$E_X(w) = \{(a,b) \in E_X^L(w) \times E_X^R(w) \mid awb \in \mathcal{L}(X)\}$$

Definition

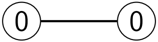
The *extension graph* of $w \in \mathcal{L}(X)$ is the bipartite graph $\mathcal{E}_X(w)$ with vertices $E_X^L(w) \sqcup E_X^R(w)$ and edges $E_X(w)$.

If X is the Fibonacci shift space,

$\mathcal{E}_X(\varepsilon)$



$\mathcal{E}_X(1)$

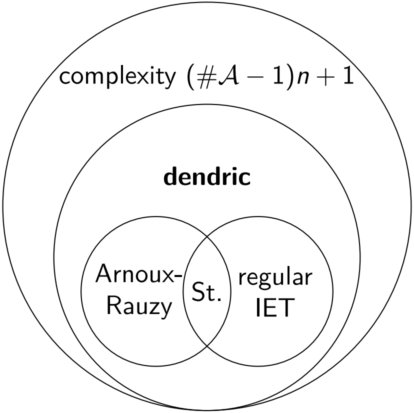


Dendricity

Definition (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone)

A word $u \in \mathcal{L}(X)$ is *dendric* (in X) if $\mathcal{E}_X(u)$ is a tree.

A shift space X is *dendric* if all the elements of $\mathcal{L}(X)$ are dendric.



Derived shift space

Definition

A *return word* for $w \neq \varepsilon$ in X is a word u such that

$$uw \in \mathcal{L}(X), \quad |uw|_w = 2, \quad uw \in w\mathcal{A}^*.$$

The set of return words for w is denoted $R_X(w)$.

Definition

If X is minimal, let $\mathcal{B} = \{0, \dots, \#R_X(w) - 1\}$ and let σ be a morphism such that $\sigma(\mathcal{B}) = R_X(w)$. The *derived shift (with respect to w)* is the shift space

$$\{x \in \mathcal{B}^{\mathbb{Z}} : \sigma(x) \in X\}.$$

Stability under derivation and Return Theorem

Theorem (Berthé *et al.*)

The family of minimal dendric shift spaces is stable under derivation.

Theorem (Berthé *et al.*)

Let X be a minimal dendric shift space. For all $w \in \mathcal{L}(X) \setminus \{\varepsilon\}$, the set $R_X(w)$ is a basis of the free group over $\mathcal{A}$.

What about conjugacy?

Two shift spaces X and Y are conjugate if there exists a bijective factor map $\varphi\colon X \to Y$.

Theorem (Curtis, Hedlund, Lyndon)

A surjective map $\varphi\colon X \to Y$ is a factor map if and only if there exists $N \geq 1$, $k < N$ and a letter-to-letter morphism f such that

$$\varphi(x) = S^{-k} f(x^{(N)})$$

for all $x \in X$, where $x^{(N)} = (x_{[n,n+N)})_{n\in\mathbb{Z}}$.

The *N-th higher block shift* is $X^{(N)} = \{x^{(N)} : x \in X\}$.

The "simplest" conjugacies are between X and its higher block shifts.

Dendric higher block shifts?

Proposition

Let $u \in \mathcal{L}(X^{(N)})$.

- $\mathcal{E}_{X^{(N)}}(u) \cong \mathcal{E}_X(v)$ *for some well-chosen* $v \in \mathcal{L}_{N+|u|-1}(X)$ *if* $u \neq \varepsilon$;
- *if* $u = \varepsilon$, *then* $\mathcal{E}_{X(N)}(u) \cong \sqcup_{v \in \mathcal{L}_{N-1}(X)} \mathcal{E}_X(v)$.

If $N \geq 2$, then ε is not dendric in $\mathcal{L}(X^{(N)})$.

Corollary

The family of dendric shift spaces is not stable under topological conjugacy.

The many definitions of eventually dendric

Definition

A shift space X is *eventually dendric with threshold N* if all the elements of $\mathcal{L}_{\geq N}(X)$ are dendric and N is minimal.

Proposition

The following are equivalent:

1. *X is eventually right-ordinary with threshold K;*
   right-ordinary: $\exists a$ st. $E_X^L(wa) = E_X^L(w)$ and $\forall b \neq a, \# E_X^L(wb) \leq 1$
2. *X is eventually dendric with threshold L;*
3. *X is eventually neutral with threshold M;*
   neutral: $\# E_X(w) = \# E_X^L(w) + \# E_X^R(w) - 1$
4. *X is eventually acyclic with threshold M';*
   acyclic: $\mathcal{E}_X(w)$ is acyclic
5. *X is eventually weak or neutral with threshold N.*
   weak or neutral: $\# E_X(w) \leq \# E_X^L(w) + \# E_X^R(w) - 1$

Moreover, $K \geq L \geq M, M' \geq N$.

The many definitions of eventually dendric (2)

Proposition

The following are equivalent:

1. *X is eventually dendric with threshold L;*
2. *X is eventually connected with threshold M and $p_X(n) \in O(n)$;*
   connected: $\mathcal{E}_X(w)$ is connected
3. *X is eventually strong or neutral with threshold N and $p_X(n) \in O(n)$.*
   strong or neutral: $\#E_X(w) \geq \#E_X^L(w) + \#E_X^R(w) - 1$

Moreover, $L \geq M \geq N$.

Stability under derivation

Theorem (G., Leroy)

The family of minimal eventually dendric shift spaces is stable under derivation.

More precisely, if X is a minimal eventually dendric shift space of threshold N, then for all $w \neq \varepsilon$, the derived shift space is eventually dendric of threshold at most $\max\{0, N - |w|\}$.

Corollary

Let X be a minimal eventually dendric shift space of threshold N. For all w such that $|w| \geq N$, the derived shift space is dendric.

Return words

We consider the shift space generated by

$$00110022001100200110022001100011\cdots$$

It is eventually dendric of threshold 2 and we have

$$R(02) = \{0200110, 02200110, 022001100110, 0220011000110\}.$$

The elements of $R(02)$ generate $\rangle 0, 11, 2 \langle$ so

- $R(02)$ is not free,
- $R(02)$ does not generate the free group.

Back to conjugacy

Proposition

Let X be an eventually dendric shift space. For all $n \geq 1$, the n-th higher block shift is eventually dendric.

Proposition

Let X be an eventually dendric shift space and α a letter-to-letter morphism. If α is injective over X, then $\alpha(X)$ is eventually dendric.

Theorem (Dolce-Perrin)

The family of eventually dendric shift spaces is stable under topological conjugacy.

Recognizable morphisms

Proposition (G., Leroy, Stas)

Let X be an eventually dendric shift space. If σ is non-erasing and recognizable on X, then the image Z of X under σ is eventually dendric.

Idea of proof:

- apply the morphism $a \mapsto a_1 a_2 \cdots a_{|\sigma(a)|}$, the obtained shift space Y is eventually dendric (B. Espinoza)
- $Z = f(Y)$ where $f \colon a_i \mapsto \sigma(a)_i$ so Z is a factor of Y, and as σ is recognizable, it is even a conjugate

What about factorization?

It "suffices" to ...

- consider images under arbitrary letter-to-letter morphisms,
- show that the image is eventually connected.

Proposition (G., Leroy, Stas)

Let X be an eventually dendric shift space and let α be a letter-to-letter morphism. There exists N such that, for all $v \in \mathcal{L}_{\geq N}(\alpha(X))$, if $\mathcal{E}_{\alpha(X)}(v)$ is not connected, then $\alpha^{-1}(v)$ contains no left-special and no right-special word.

Final questions

- Is the family of eventually dendric shift spaces stable under topological factorization? Or equivalently, is it stable under application of any non-erasing morphism?
- What can we still say about the sets of return words?
- Does every eventually dendric shift space have a dendric conjugate? More generally, what can we say about the conjugacy classes?

Thank you for your attention!