Sturmian and dendric words

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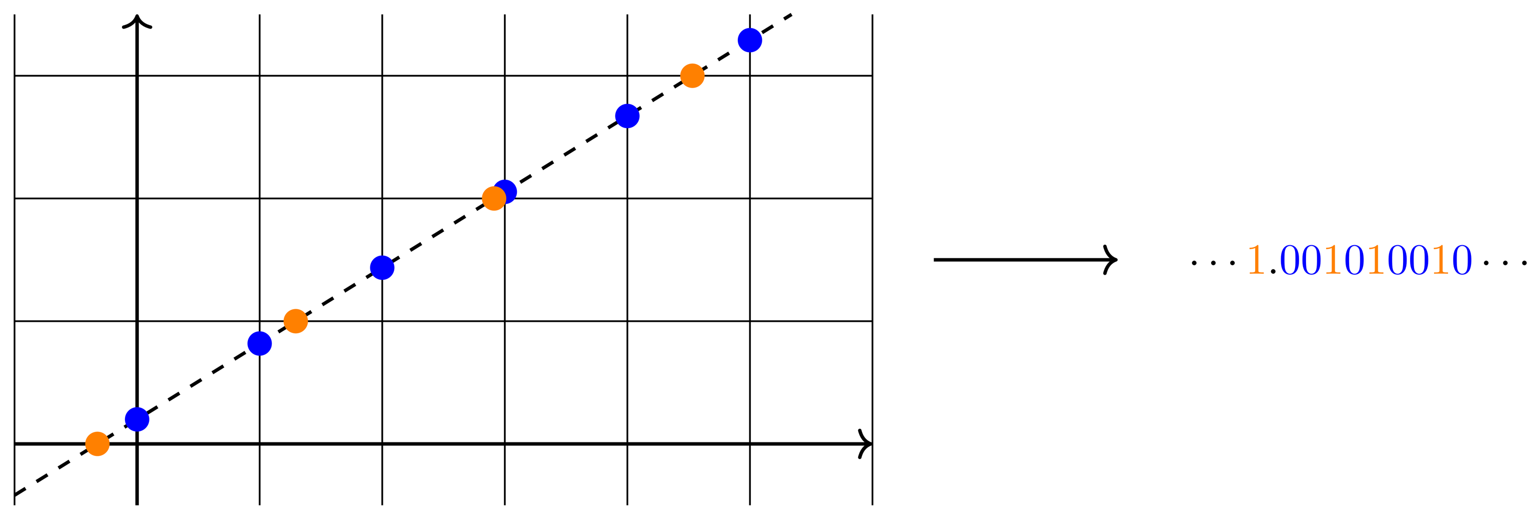
Sturmian words

Trajectory coding

For an irrational slope α and an intercept c, we can code the intersections of the line $y = \alpha x + c$ with the $\mathbb{Z}^2$ grid by a bi-infinite sequence.



These sequences are called (bi-infinite) **Sturmian words**.

Language

Sturmian words are **binary**, **aperiodic**, and **uniformly recurrent**.

Given a word x and a factor (i.e. consecutive subsequence) u of x, the **left-extensions** of u are the letters a such that au appears in x. Similarly, the **right-extensions** of u are the letters a such that ua appears in x.

Example: For the Sturmian word x drawn above, we have

$$\mathrm{LE}_x(\varepsilon) = \{0,1\} \quad \mathrm{LE}_x(10) = \{0\} \quad \mathrm{RE}_x(0) = \{0,1\}.$$

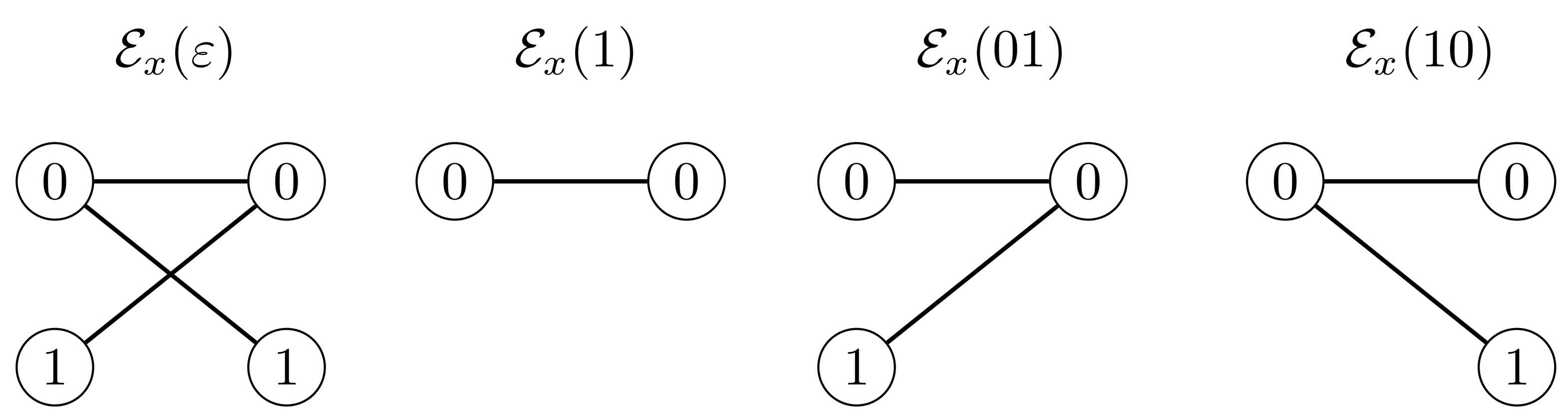
Proposition. *If x is a Sturmian word, then for each length, there is a unique factor u of that length with both 0 and 1 as left-extensions (resp., right-extensions).*

Dendric words

Extension graph

Instead of looking at the left- and right-extensions of u independently, we can look at the pairs of letters (a, b) such that aub appears in x. This defines a relation between the left- and the right-extensions of u that we can represent in a bipartite graph called the **extension graph** of u.

Example: For the Sturmian word x drawn above, we have



Dendricity

A factor u of x is **dendric** in x if its extension graph is a tree.

An infinite word x is **dendric** if all of its factors are dendric.

Factor complexity

The **factor complexity** of a word x is the function $p_x : \mathbb{N} \to \mathbb{N}$ such that $p_x(n)$ is the number of different factors of length n in x.

For Sturmian words

The Sturmian words are exactly the words with $p_x(n) = n+1$ for all $n \geq 0$.

For dendric words

If x is dendric, then $p_x(n) = (\#A-1)n+1$ where A is the minimal alphabet (i.e. the set of letters occuring in x).

In particular, the dendric words on two letters are exactly the Sturmian words.

Preserving morphisms

For Sturmian words

We define the morphisms

$$L : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \end{cases} \qquad R : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 10 \end{cases} \qquad \pi : \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}.$$

Let σ be a morphism. The following are equivalent:

1. σ is a composition of L, R and π;
2. for every Sturmian word x, $\sigma(x)$ is Sturmian;
3. there exists a Sturmian word x such that $\sigma(x)$ is Sturmian.

For dendric words

Let $a \in A$. We define the morphisms

$$L_a : \begin{cases} a \mapsto a \\ b \mapsto ab & \text{if } b \neq a \end{cases} \qquad R_a : \begin{cases} a \mapsto a \\ b \mapsto ba & \text{if } b \neq a \end{cases}.$$

Let σ be a morphism (defined on A^*). The following are equivalent:

1. σ is a composition of L_a, R_a and of permutations of A;
2. for every dendric word x over A, $\sigma(x)$ is dendric.

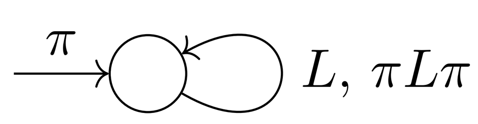
S-adic representations

An S-adic representation of a (uniformly recurrent) bi-inifinite word x is a sequence of morphisms $(\sigma_n : A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ such that the factors of x are exactly the elements of

$$\bigcup_{n\in\mathbb{N}} \mathrm{Fact}(\sigma_0 \dots \sigma_n(A_{n+1})).$$

For Sturmian words

Theorem. *An infinite word x is sturmian if and only if it has a (non-eventually constant) S-adic representation labeling an infinite path in the following graph.*



For dendric words

Theorem. *Let A be an alphabet. There exists a graph G_A such that an infinite word x is uniformly recurrent dendric over A if and only if it has a (primitive) S-adic representation labeling an infinite path in G_A.*

Open questions

1. Given a dendric word x, can we characterize the morphisms σ such that $\sigma(x)$ is dendric?
2. If we consider eventually dendric words (only long enough factors have to be dendric) instead, are all morphisms preserving?

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