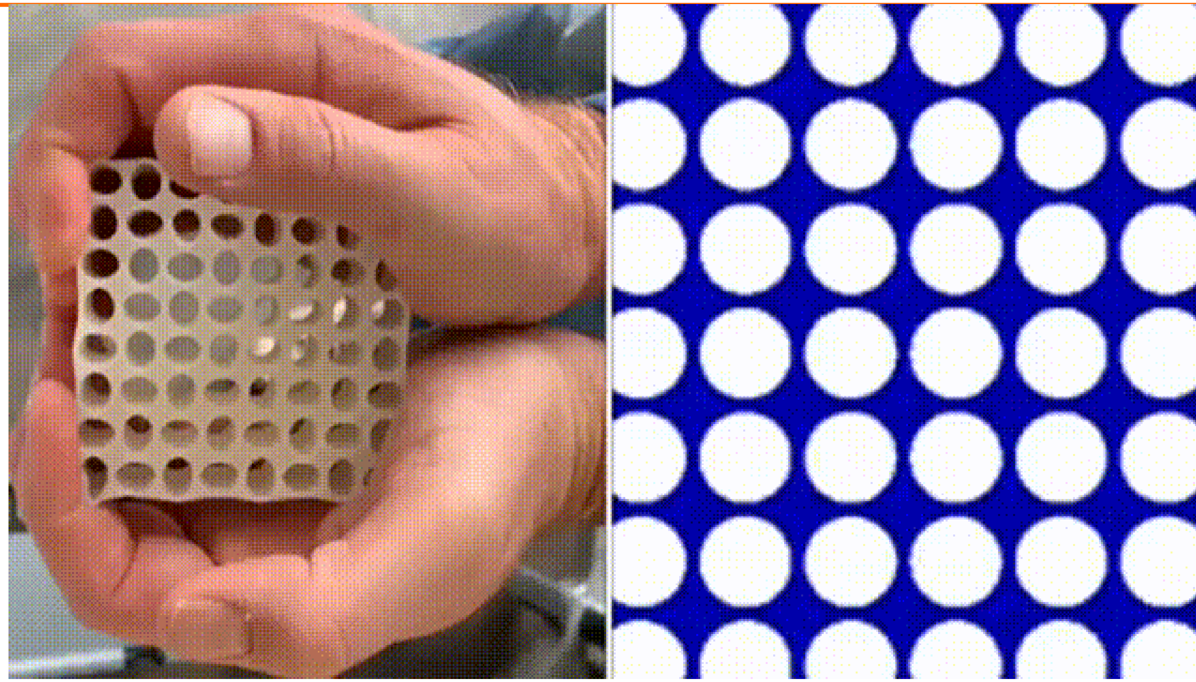
Body-force enhanced second-order computational homogenisation for non-linear cellular materials and metamaterials



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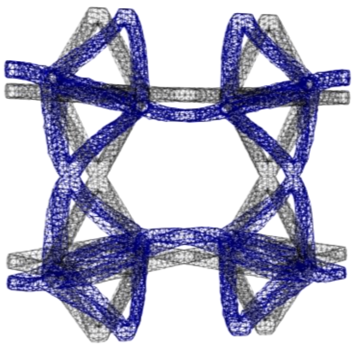
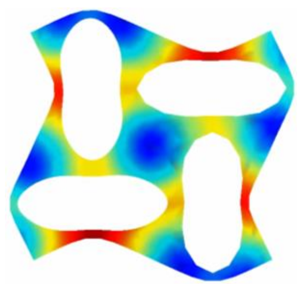
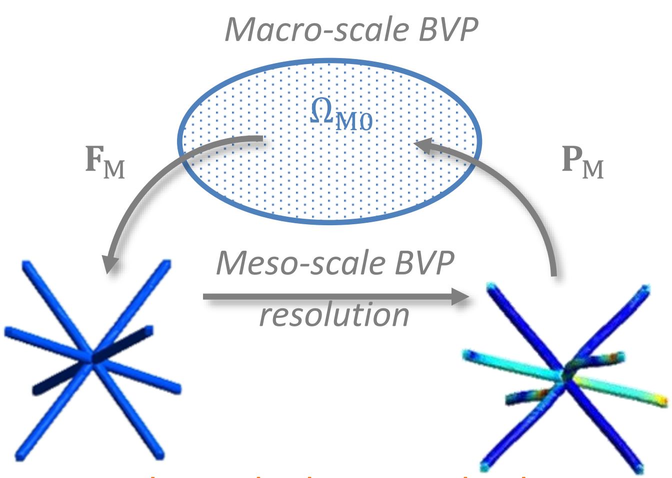
MOAMMM
www.moammm.eu

- First vs. second order homogenisation
  - First order homogenisation
    - Does not prevent spurious localisation
    - No material length-scale
  - Second-order homogenisation
    - High order strain $\mathbf{G}_\mathrm{M}$ and stress $\mathbf{Q}_\mathrm{M}$ at macro-scale
    - Material length scale related to the RVE length

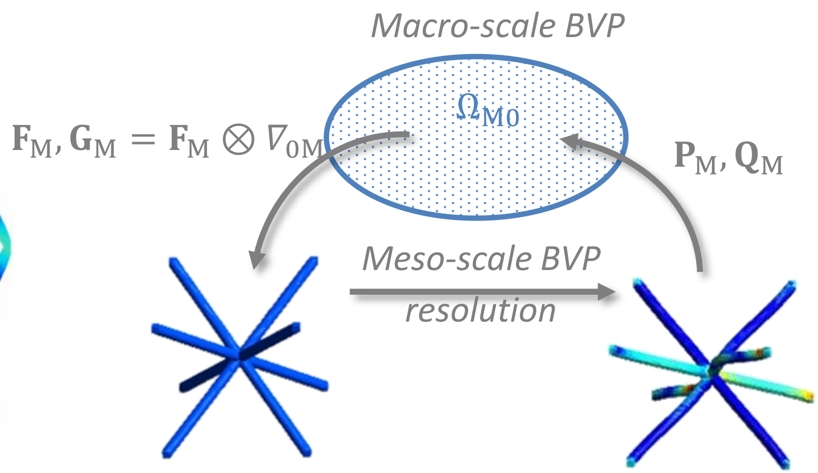




    - Issue for metamaterial: RVE length is larger than unit cell because of patterning change



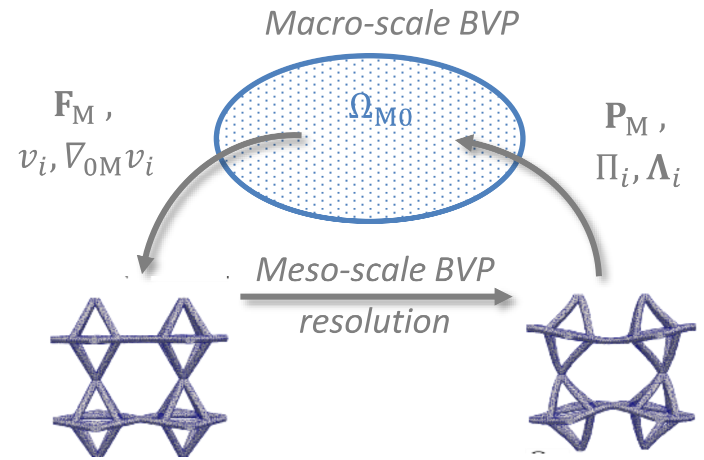
First order homogenisation



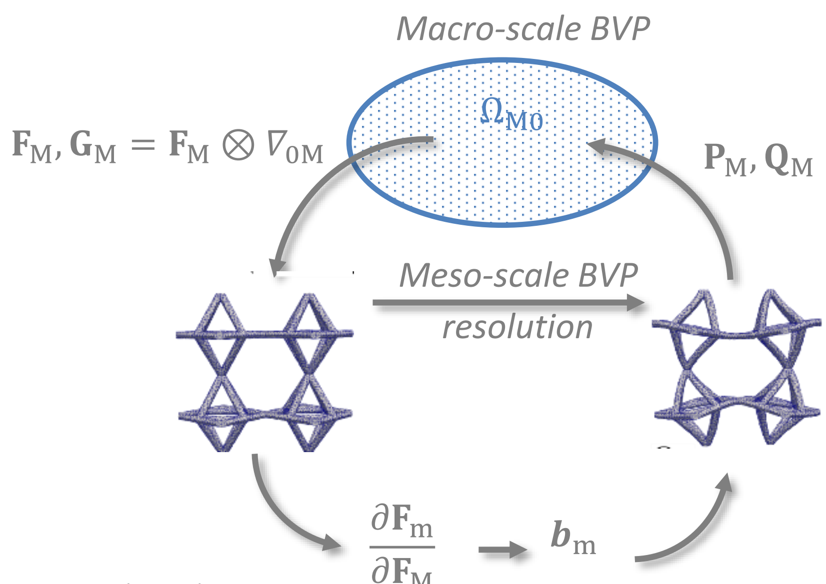
Second order homogenisation

Second order homogenisation for cellular and metamaterials

- Account for patterning change
  - Micromorphic approach*
    - Constrains change of patterning modes
    - Developed in elasticity
  - Enhanced second-order homogenisation
    - Remove cell size dependency using a body-force
    - Arises from asymptotic homogenization in linear elasticity**
    - How to account for finite strain, elasto-plasticity etc…?



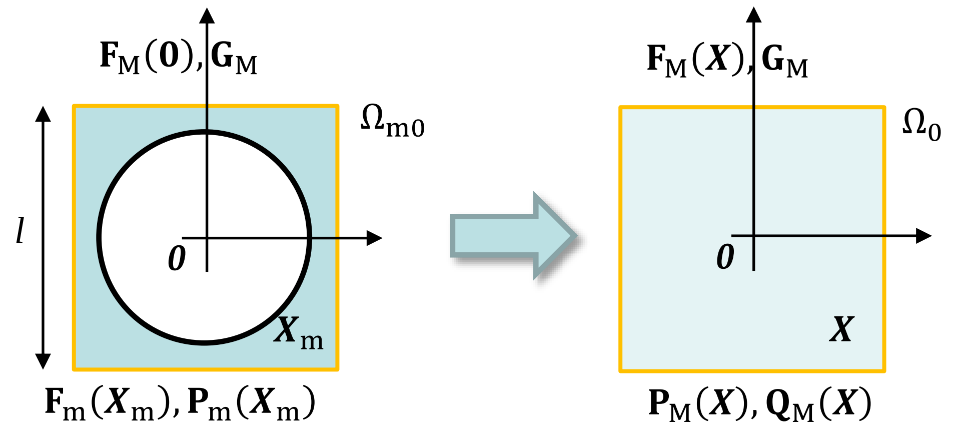
Micromorphic approach



Body-force enhanced second order homogenisation

*O. Rokoš, M. Ameen, R. Peerlings, M. Geers, J. Mech. Phys. Solids 123 (2019)
**V. Monchiet, N. Auffray, J. Yvonnet, Mech. Mater. 143 (2020)
J. Yvonnet, N. Auffray, V. Monchiet, Int. J. Solids Struct. 191–192 (2020)

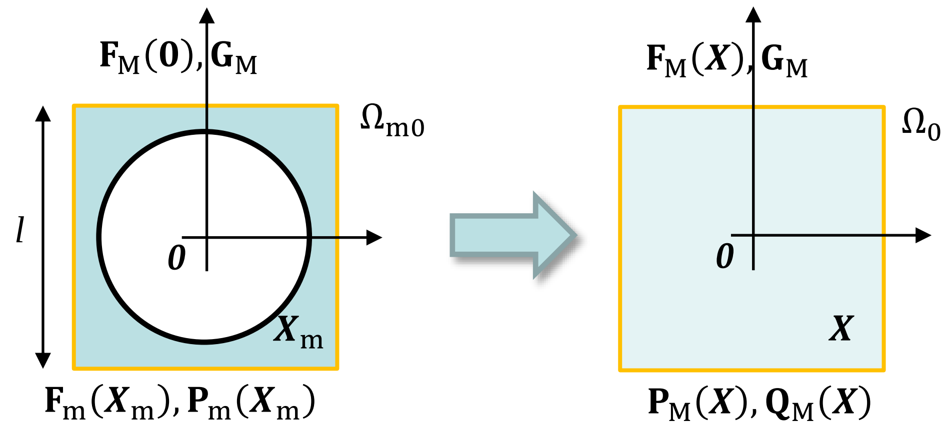
- Second order homogenisation with body force enhancement
  - Consider an equivalent homogeneous volume element
    - Cauchy homogenous / Second order continuum



    - Development of the (no-longer) homogeneous field

$$\begin{cases} \mathbf{F}_\mathrm{M}(\boldsymbol{X}) = \mathbf{F}_\mathrm{M}(0) + \mathbf{G}_\mathrm{M} \cdot \boldsymbol{X} \\ \mathbf{G}_\mathrm{M} = \mathbf{F}_\mathrm{M}(\mathbf{0}) \otimes \nabla_{0\mathrm{M}} \end{cases} \qquad \begin{cases} \mathbf{P}_\mathrm{M}(\boldsymbol{X}) = \mathbf{P}_\mathrm{M}(0) + \left.\dfrac{\partial \mathbf{P}_\mathrm{M}}{\partial \mathbf{F}_\mathrm{M}}\right|_0 : \mathbf{G}_\mathrm{M} \cdot \boldsymbol{X} \\ \mathbf{Q}_\mathrm{M}(\boldsymbol{X}) = \mathbf{Q}_\mathrm{M}(0) + \left.\dfrac{\partial \mathbf{Q}_\mathrm{M}}{\partial \mathbf{F}_\mathrm{M}}\right|_0 : \mathbf{G}_\mathrm{M} \cdot \boldsymbol{X} \end{cases}$$

- Second order homogenisation with body force enhancement
  - Consider an equivalent homogeneous volume element
    - The equivalence of energy (Hill-Mandel condition) with introduction of body forces $\boldsymbol{b}_{\mathrm{m}}(\boldsymbol{X}_{\mathrm{m}})$:



$$\int_{\Omega_{\mathrm{m}0}} [\mathbf{P}_{\mathrm{m}} : \delta\mathbf{F}_{\mathrm{m}} - \mathbf{b}_{\mathrm{m}} \cdot \delta\boldsymbol{u}_{\mathrm{m}}] d\Omega$$

$$\Leftrightarrow$$

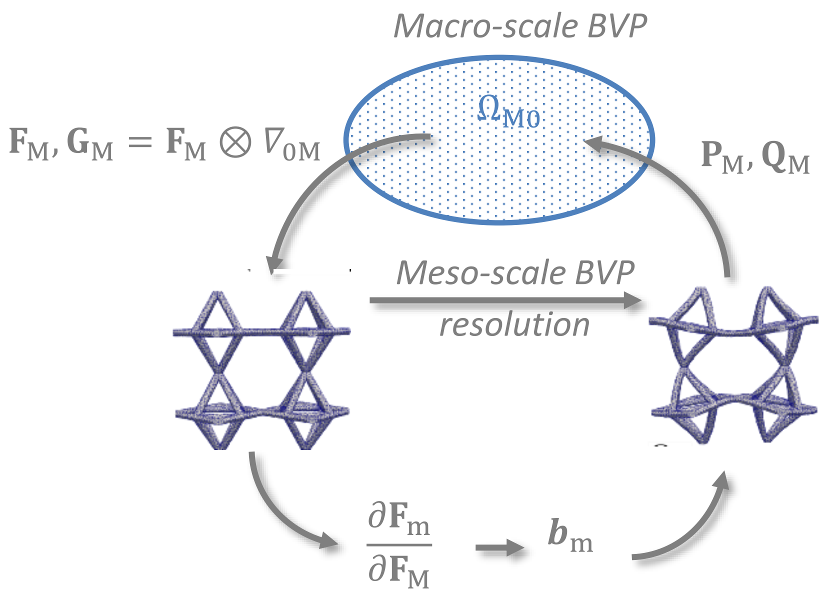
$$\int_{\Omega_0} (\mathbf{P}_{\mathrm{M}} : \delta\mathbf{F}_{\mathrm{M}} + \mathbf{Q}_{\mathrm{M}} \vdots \delta\mathbf{G}_{\mathrm{M}} - \mathbf{B}_{\mathrm{M}} \cdot \delta\boldsymbol{u}_{\mathrm{M}}) d\Omega$$

- Is satisfied by the following introduction of micro-scale body forces and homogenised stresses

$$\begin{cases}
\mathbf{P}_{\mathrm{M}} = \mathbf{P}_{\mathrm{M}}(0) = \dfrac{1}{V_0} \displaystyle\int_{\Omega_{\mathrm{m}0}} (\mathbf{P}_{\mathrm{m}} - \boldsymbol{b}_{\mathrm{m}} \otimes \boldsymbol{X}_{\mathrm{m}})\, d\Omega \\
\mathbf{Q}_{\mathrm{M}} = \mathbf{Q}_{\mathrm{M}}(0) = \frac{1}{2V_0} \int_{\Omega_{\mathrm{m}0}} [\mathbf{P}_{\mathrm{m}} \otimes \boldsymbol{X}_{\mathrm{m}} + (\mathbf{P}_{\mathrm{m}} \otimes \boldsymbol{X}_{\mathrm{m}})^T]\, d\Omega + \frac{1}{2V_0} \int_{\Omega_{\mathrm{m}0}} [\boldsymbol{b}_{\mathrm{m}} \otimes \boldsymbol{X}_{\mathrm{m}} \otimes \boldsymbol{X}_{\mathrm{m}}]\, d\Omega - \\
\qquad \frac{1}{2V_0} \left(\left[\frac{\partial \mathbf{P}_{\mathrm{M}}(0)}{\partial \mathbf{F}_{\mathrm{M}}} : \mathbf{G}_{\mathrm{M}} \cdot \boldsymbol{J}_{\mathrm{M}} + \left(\frac{\partial \mathbf{P}_{\mathrm{M}}(0)}{\partial \mathbf{F}_{\mathrm{M}}} : \mathbf{G}_{\mathrm{M}} \cdot \boldsymbol{J}_{\mathrm{M}} \right)^T \right] - \boldsymbol{B}_{\mathrm{M}} \otimes \boldsymbol{J}_{\mathrm{M}} \right) \\
\displaystyle\int_{\Omega_{\mathrm{m}0}} \mathbf{b}_{\mathrm{m}} d\Omega = \int_{\Omega_0} \mathbf{B}_{\mathrm{M}} d\Omega = -\int_{\Omega_0} \frac{\partial \mathbf{P}_{\mathrm{M}}}{\partial \mathbf{F}_{\mathrm{M}}} : \mathbf{G}_{\mathrm{M}} : \boldsymbol{I} d\Omega = -\int_{\Omega_{m0}} \left(\frac{\partial \mathbf{P}_{\mathrm{m}}}{\partial \mathbf{F}_{\mathrm{m}}} : \frac{\partial \mathbf{F}_{\mathrm{m}}}{\partial \mathbf{F}_{\mathrm{M}}} : \mathbf{G}_{\mathrm{M}} \right) : \boldsymbol{I} d\Omega
\end{cases}$$

- Meso-scale problem
  - Micro-scale weak form

$$\int_{\Omega_{\mathrm{m}0}} \mathbf{P}_{\mathrm{m}} : (\delta \boldsymbol{w} \otimes \nabla_0) - \mathbf{b}_{\mathrm{m}} \cdot \delta \boldsymbol{w} \mathrm{d}\Omega = 0$$



  - Introduction of body forces $\mathbf{b}_{\mathrm{m}}(\mathbf{X}_{\mathrm{m}})$:

$$\mathbf{b}_{\mathrm{m}} = -\left(\frac{\partial \mathbf{P}_{\mathrm{m}}}{\partial \mathbf{F}_{\mathrm{m}}} : \frac{\partial \mathbf{F}_{\mathrm{m}}}{\partial \mathbf{F}_{\mathrm{M}}} : \mathbf{G}_{\mathrm{M}}\right) : \boldsymbol{I}$$

Applied strain

Instantaneous tangent *Strain concentration tensor*

  - Approximation

$$\mathbf{b}_{\mathrm{m}}^{n+1} = -\left(\left(\frac{\partial \mathbf{P}_{\mathrm{m}}}{\partial \mathbf{F}_{\mathrm{m}}}\right)^{n+1} : \left(\frac{\partial \mathbf{F}_{\mathrm{m}}}{\partial \mathbf{F}_{\mathrm{M}}}\right)^{n} : \mathbf{G}_{\mathrm{M}}^{n+1}\right) : \boldsymbol{I}$$

- Meso-scale problem resolution
  - Meso-scale FEM is a constrained problem

$$\begin{cases} \mathbf{f}_\mathrm{m}(\mathbf{u}_\mathrm{m}) - \mathbf{b}_\mathrm{m}(\mathbf{u}_\mathrm{m}) - \boldsymbol{\mathcal{C}}^\mathrm{T}\boldsymbol{\lambda} = -\boldsymbol{r} \to 0 \\ \boldsymbol{\mathcal{C}}\mathbf{u}_\mathrm{m} - \boldsymbol{\mathcal{S}}\begin{bmatrix} \mathbf{F}_\mathrm{M}^\mathbf{T} - \mathbf{I} \\ \mathbf{G}_\mathrm{M}^\mathbf{T} \end{bmatrix} = -\boldsymbol{r}_{\boldsymbol{c}} \to 0 \end{cases}$$

FE discretisation of weak form

$\boldsymbol{\mathcal{C}}$, $\boldsymbol{\mathcal{S}}$ from boundary conditions, e.g. PBC

  - Finite element resolution ($\mathbf{F}_\mathrm{M}, \mathbf{G}_\mathrm{M}$=cst)

$$\delta\mathbf{u}_\mathrm{m} = -\widetilde{\mathbf{K}}^{-1}\left(\mathbf{Q}^\mathrm{T}\mathbf{r} + \left(\boldsymbol{\mathcal{C}}^\mathrm{T} - \mathbf{Q}^\mathrm{T}\left(\frac{\partial\mathbf{f}_\mathrm{m}}{\partial\mathbf{u}_\mathrm{m}} - \frac{\partial\mathbf{b}_\mathrm{m}}{\partial\mathbf{u}_\mathrm{m}}\right)\boldsymbol{\mathcal{C}}^\mathrm{T}\left(\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{C}}^\mathrm{T}\right)^{-\mathbf{1}}\right)\mathbf{r}_{\boldsymbol{c}}\right)$$

with $\widetilde{\mathbf{K}} = \boldsymbol{\mathcal{C}}^\mathrm{T}\boldsymbol{\mathcal{C}} + \mathbf{Q}^\mathrm{T}\left(\frac{\partial\mathbf{f}_\mathrm{m}}{\partial\mathbf{u}_\mathrm{m}} - \frac{\partial\mathbf{b}_\mathrm{m}}{\partial\mathbf{u}_\mathrm{m}}\right)\mathbf{Q}$ & $\mathbf{Q} = \mathbf{I} - \boldsymbol{\mathcal{C}}^\mathrm{T}\left(\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{C}}^\mathrm{T}\right)^{-\mathbf{1}}\boldsymbol{\mathcal{C}}$

  - After convergence ($\boldsymbol{r} = \boldsymbol{r}_{\boldsymbol{c}} = \mathbf{0}$)

$$\frac{\partial\mathbf{u}_\mathrm{m}}{\partial\left[\mathbf{F}_\mathrm{M}^{\boldsymbol{T}} \quad \mathbf{G}_\mathrm{M}^{\boldsymbol{T}}\right]} = \widetilde{\mathbf{K}}^{-1}\left(\boldsymbol{\mathcal{C}}^\mathrm{T}\boldsymbol{\mathcal{S}} - \mathbf{Q}^\mathrm{T}\left(\frac{\partial\mathbf{f}_\mathrm{m}}{\partial\mathbf{u}_\mathrm{m}} - \frac{\partial\mathbf{b}_\mathrm{m}}{\partial\mathbf{u}_\mathrm{m}}\right)\boldsymbol{\mathcal{C}}^\mathrm{T}\left(\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{C}}^\mathrm{T}\right)^{-\mathbf{1}}\boldsymbol{\mathcal{S}} + \mathbf{Q}^\mathrm{T}\frac{\partial\mathbf{b}_\mathrm{m}}{\partial\left[\mathbf{F}_\mathrm{M}^{\boldsymbol{T}} \quad \mathbf{G}_\mathrm{M}^{\boldsymbol{T}}\right]}\right)$$

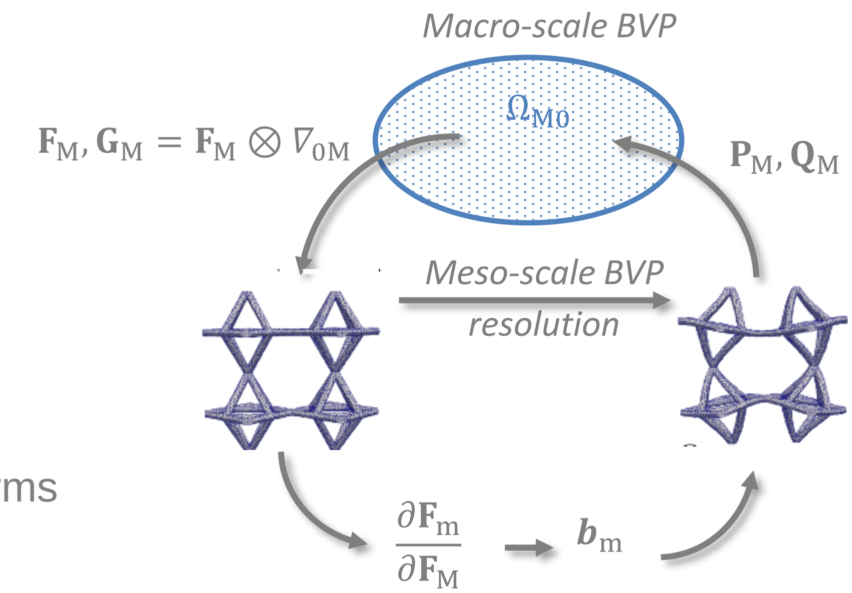
One stiffness matrix $\widetilde{\mathbf{K}}$ to "inverse"

$$\frac{\partial\mathbf{F}_\mathrm{m}}{\partial\mathbf{F}_\mathrm{M}} = \boldsymbol{\nabla}N\frac{\partial\mathbf{u}_\mathrm{m}}{\mathbf{F}_\mathrm{M}} \quad \Rightarrow \quad \mathbf{b}_\mathrm{m} = -\left(\frac{\partial\mathbf{P}_\mathrm{m}}{\partial\mathbf{F}_\mathrm{m}} : \frac{\partial\mathbf{F}_\mathrm{m}}{\partial\mathbf{F}_\mathrm{M}} : \mathbf{G}_\mathrm{M}\right) : \boldsymbol{I}$$

- Macro-scale problem resolution
  - Macro-scale weak form

$$\int_{\Omega_{M0}} \mathbf{P}_M : (\delta \boldsymbol{w} \otimes \nabla_0) + \mathbf{Q}_M : (\delta \boldsymbol{w} \otimes \nabla_0 \otimes \nabla_0) \, d\Omega = 0$$

High-order continuity weakly enforced using DG method



  - Homogenised stress tensors with new terms

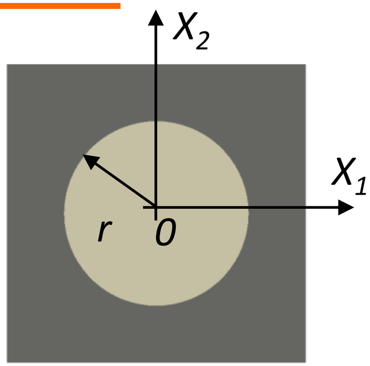
$$\begin{cases} \mathbf{P}_M = \frac{1}{V_0} \int_{\Omega_{m0}} (\mathbf{P}_m - \boldsymbol{b}_m \otimes \boldsymbol{X}_m) \, d\Omega \\ \mathbf{Q}_M = \frac{1}{2V_0} \int_{\Omega_{m0}} [\mathbf{P}_m \otimes \boldsymbol{X}_m + (\mathbf{P}_m \otimes \boldsymbol{X}_m)^T] \, d\Omega + \frac{1}{2V_0} \int_{\Omega_{m0}} [\boldsymbol{b}_m \otimes \boldsymbol{X}_m \otimes \boldsymbol{X}_m] \, d\Omega - \\ \quad \frac{1}{2\mathrm{V}_0} \left(\left[\frac{\partial \mathbf{P}_M}{\partial \mathbf{F}_M} : \mathbf{G}_M \cdot \boldsymbol{J}_M + \left(\frac{\partial \mathbf{P}_M}{\partial \mathbf{F}_M} : \mathbf{G}_M \cdot \boldsymbol{J}_M \right)^T \right] - \boldsymbol{B}_M \otimes \boldsymbol{J}_M \right) \end{cases}$$

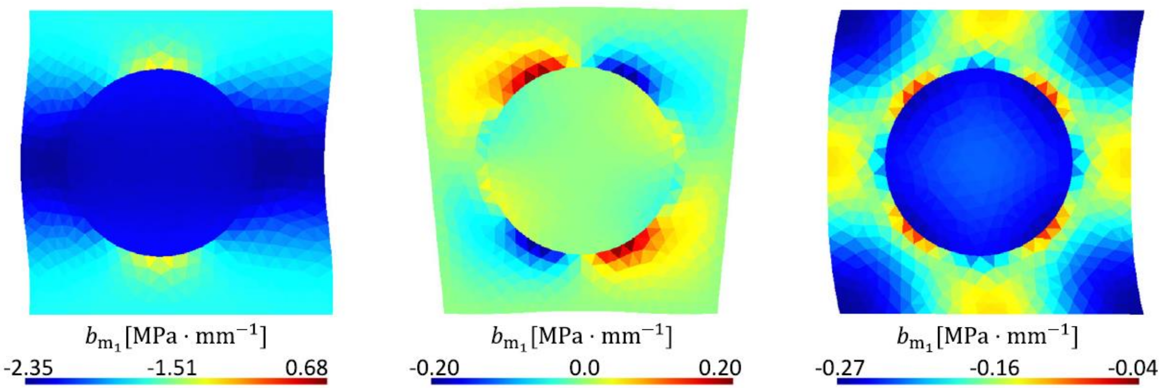
  - Material operator

From RVE resolution

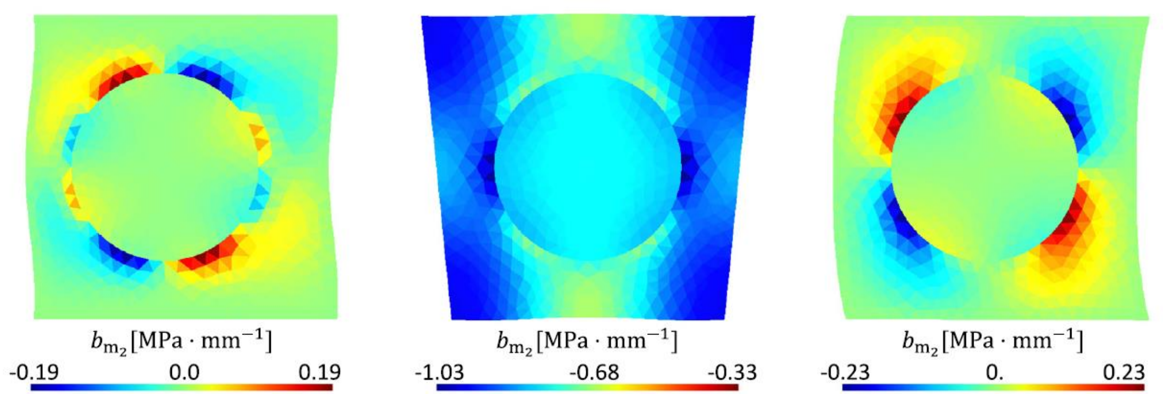
$$\begin{bmatrix} \frac{\partial \mathbf{P}_M}{\partial \mathbf{F}_M} & \frac{\partial \mathbf{P}_M}{\partial \mathbf{G}_M} \\ \frac{\partial \mathbf{Q}_M}{\partial \mathbf{F}_M} & \frac{\partial \mathbf{Q}_M}{\partial \mathbf{G}_M} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \circ \begin{pmatrix} \frac{\partial \mathbf{F}_m}{\partial \mathbf{F}_M} \\ \frac{\partial \mathbf{F}_m}{\partial \mathbf{G}_M} \end{pmatrix} = \left(\bigwedge_e \begin{bmatrix} \mathcal{A}^e \nabla N^e & \mathcal{B}^e \nabla N^e \\ \mathcal{C}^e \nabla N^e & \mathcal{D}^e \nabla N^e \end{bmatrix} \right) \circ \frac{\partial \mathbf{u}_m}{\partial [\mathbf{F}_M \quad \mathbf{G}_M]^{\mathrm{T}}}$$

- Remove boundary effect
  - Linear elasticity
  - With the presented approach, the body forces are not uniform

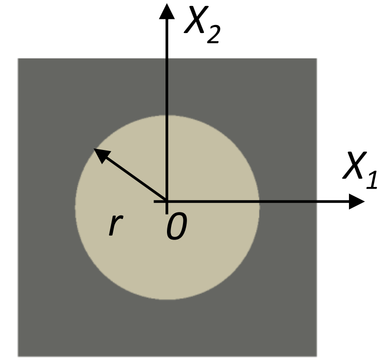


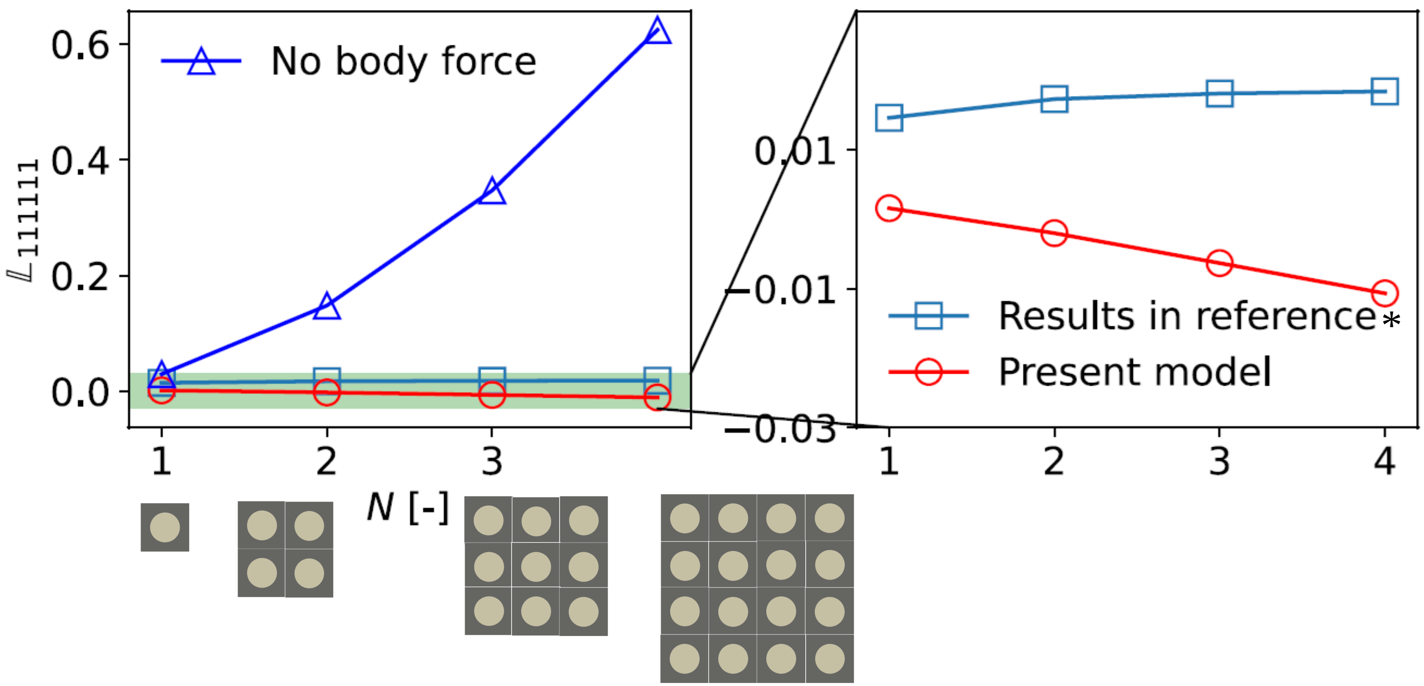


$G_{M_{xxx}} = 0.4\ /\mathrm{mm}$ $\quad G_{M_{xxy}} = 0.2\ /\mathrm{mm}$ $\quad G_{M_{xyy}} = 0.4\ /\mathrm{mm}$



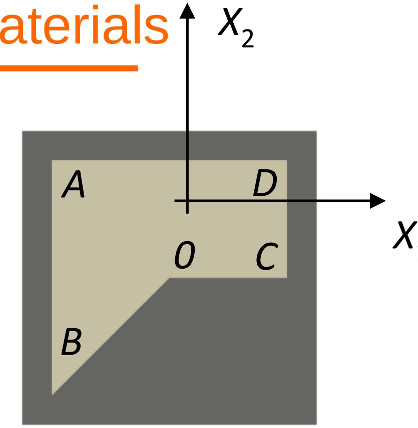
- Remove boundary effect
  - Linear elasticity
  - Comparison with uniform body force*
  - Study of the higher-order operator: $\mathbb{L} = \frac{\partial \mathbf{Q}_\mathrm{M}}{\partial \mathbf{G}_\mathrm{M}}$



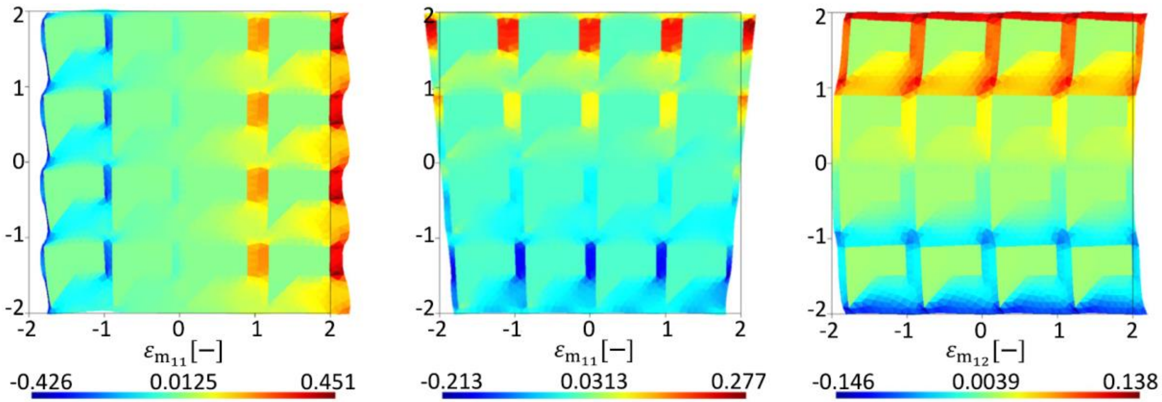


*J. Yvonnet, N. Auffray, V. Monchiet, Int. J. Solids Struct. 191–192 (2020)

Second order homogenisation for cellular and metamaterials

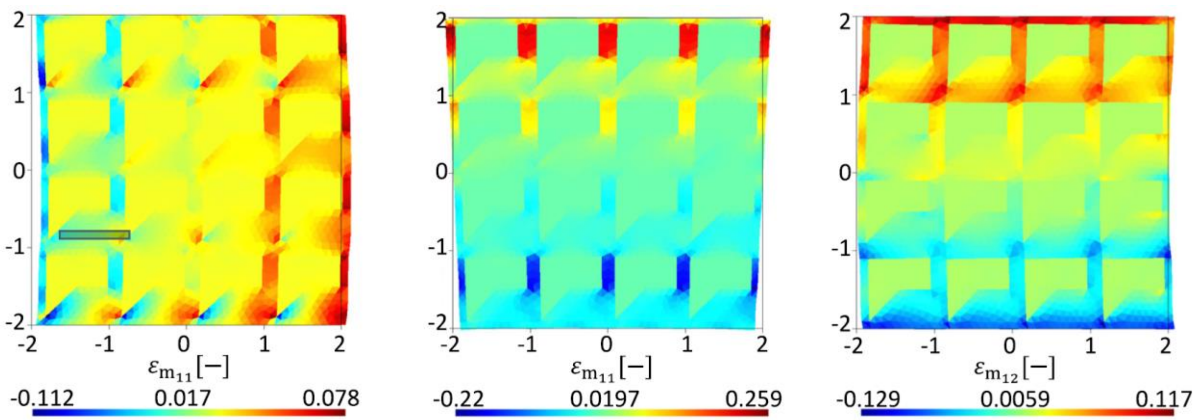


- Remove boundary effect
  - Linear elasticity
  - Comparison with no body force



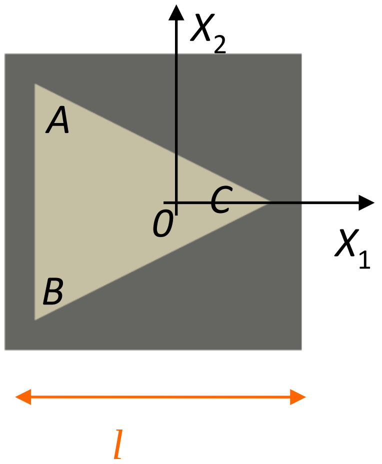
Body-force enhanced second order homogenisation

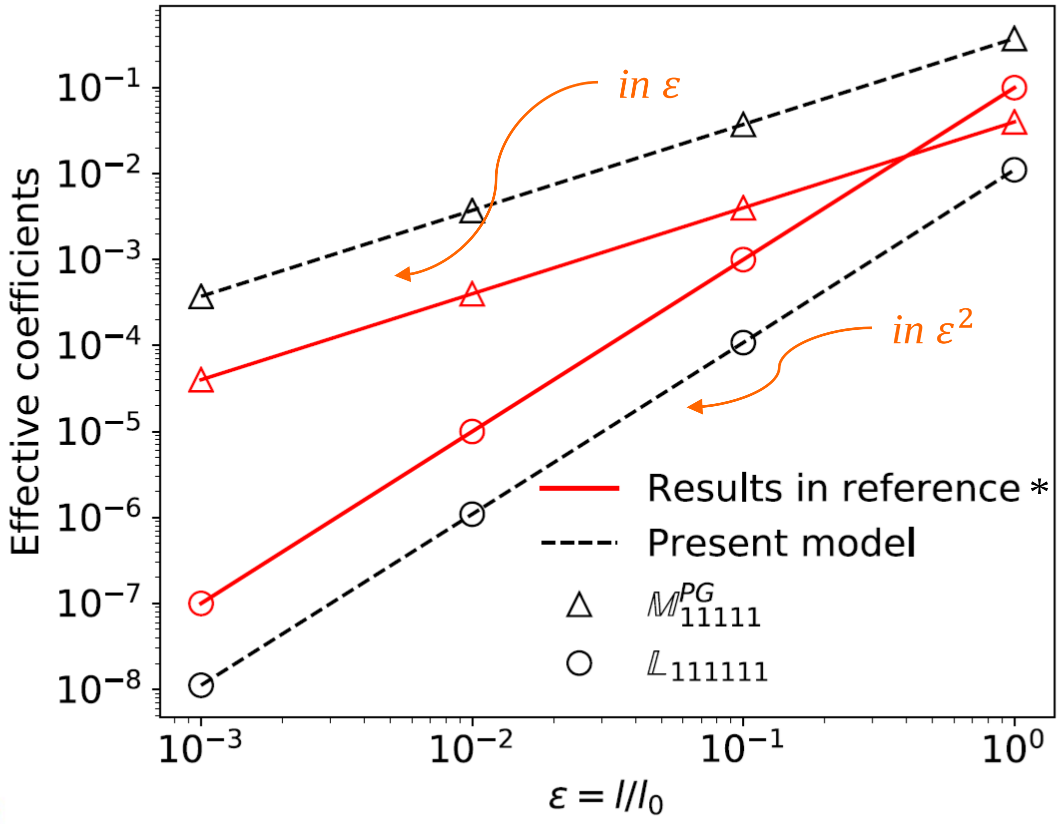
$G_{M_{xxx}} = 0.05\ /\mathrm{mm}$ $G_{M_{xxy}} = 0.025\ /\mathrm{mm}$ $G_{M_{xyy}} = 0.05\ /\mathrm{mm}$



Classical second order homogenisation

- Remove boundary effect
  - Linear elasticity
  - Where does the size effect come from?
  - Study of the higher-order operators: $\mathbb{L} = \frac{\partial \mathbf{Q}_M}{\partial \mathbf{G}_M}$, $\mathbb{M}^{PG} = \frac{\partial \mathbf{P}_M}{\partial \mathbf{G}_M}$





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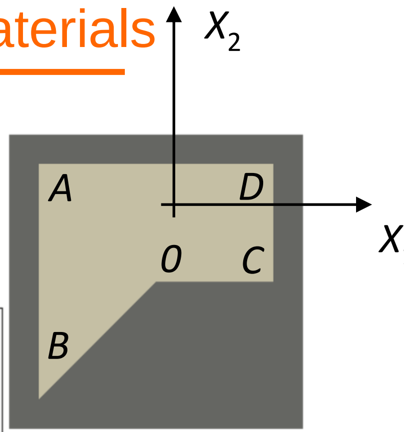
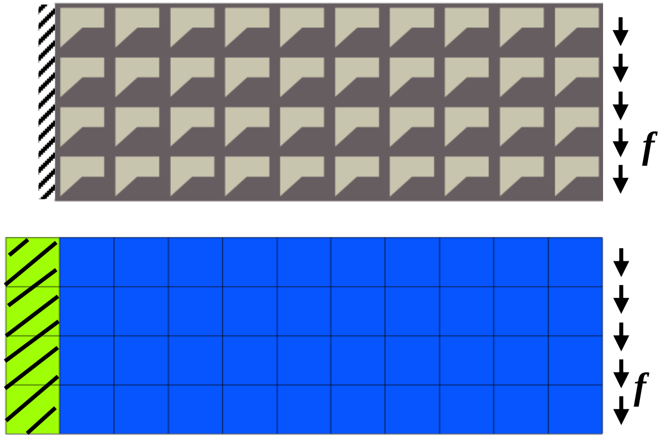
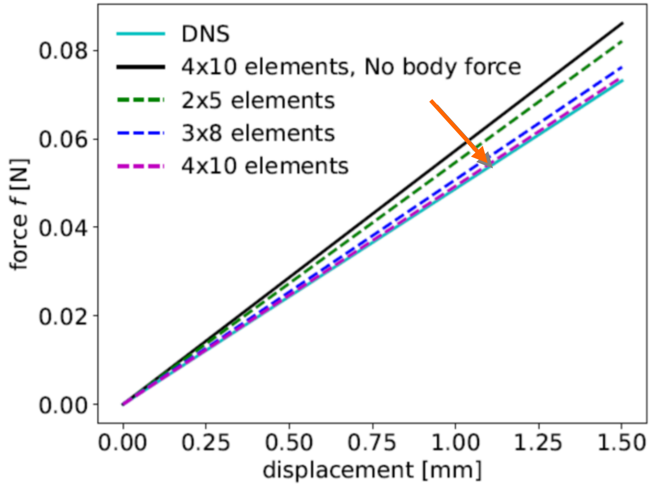
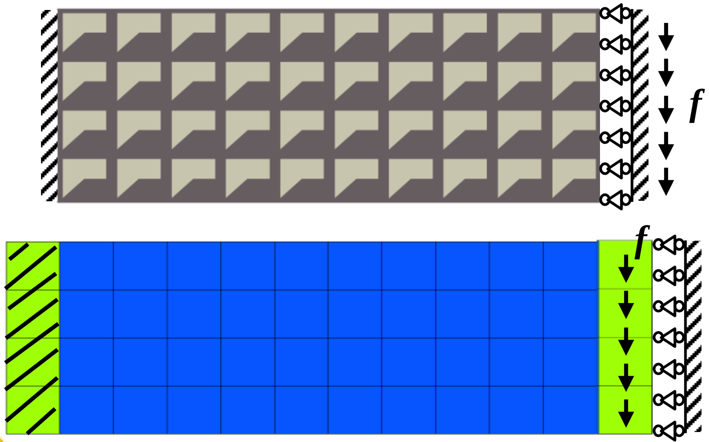


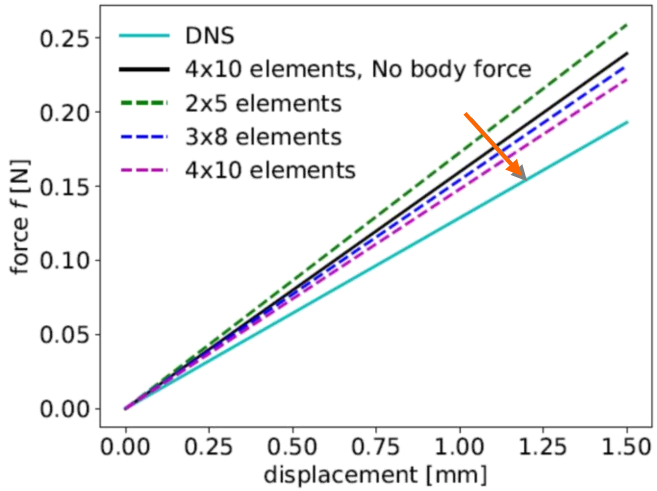
- Convergence toward DNS ?
  - Linear elasticity: Beam bending





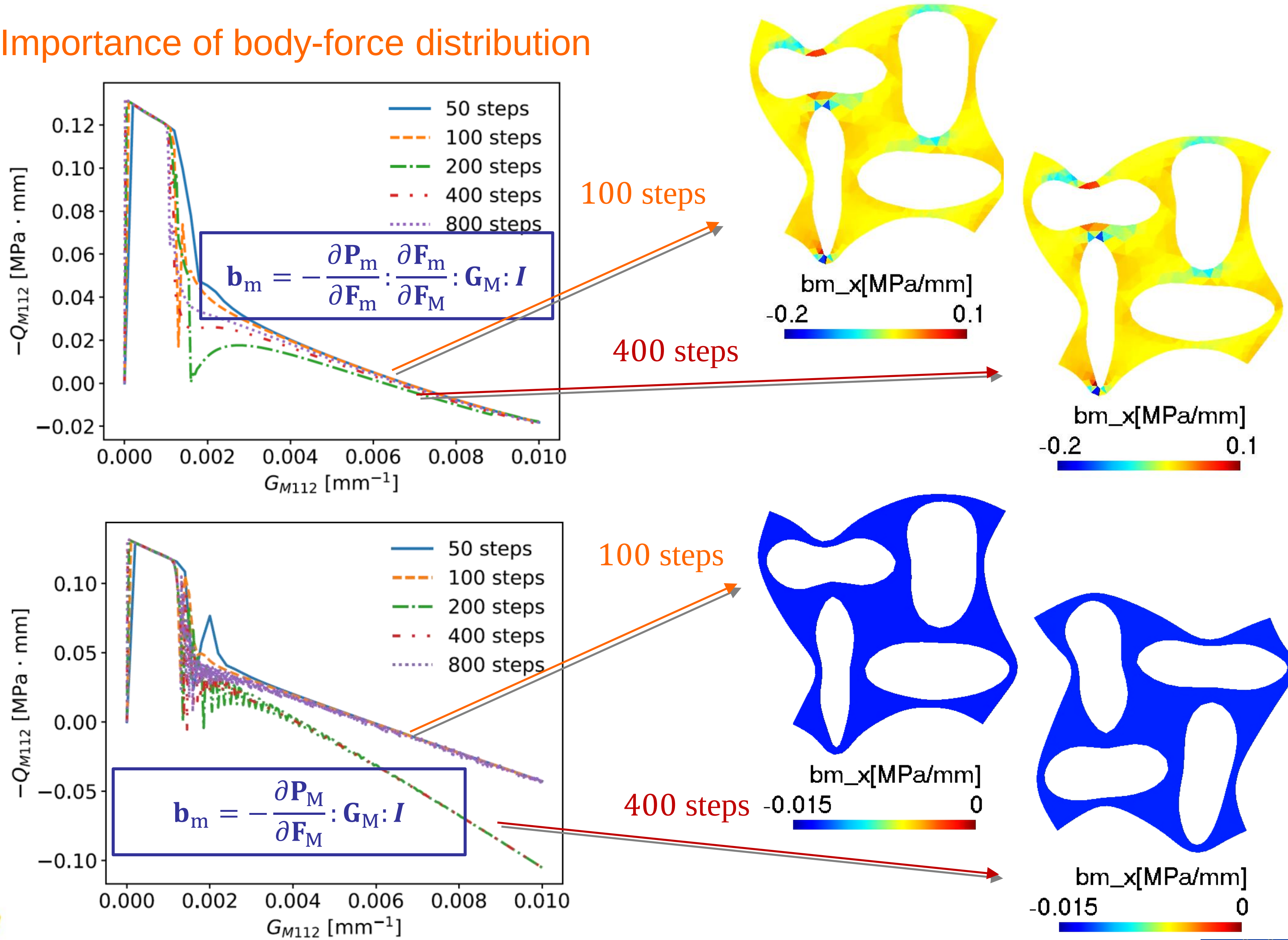
  - Linear elasticity: Beam shearing





- Importance of body-force distribution

$\mathbf{F}_M = 0.9\,\mathbf{I}$, $G_{M_{xxx}} = 0.02$ /mm, $G_{M_{xxy}} = 0.01$ /mm

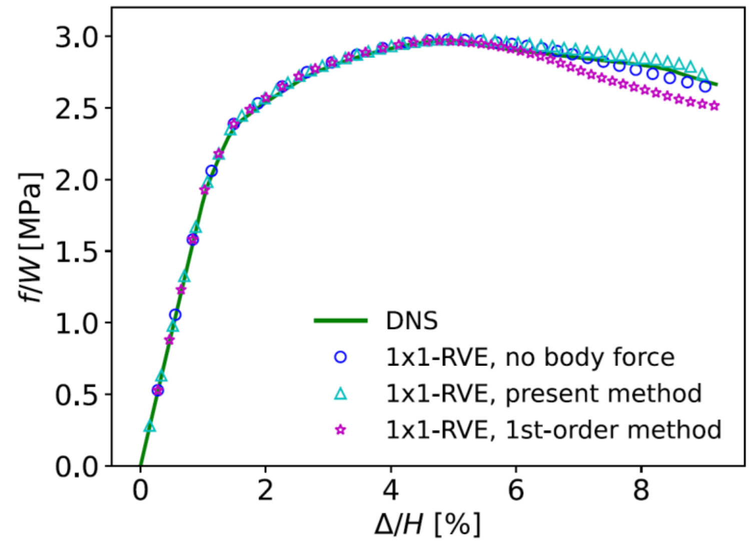


$$\mathbf{b}_\mathrm{m} = -\frac{\partial \mathbf{P}_\mathrm{m}}{\partial \mathbf{F}_\mathrm{m}} : \frac{\partial \mathbf{F}_\mathrm{m}}{\partial \mathbf{F}_\mathrm{M}} : \mathbf{G}_\mathrm{M} : \boldsymbol{I}$$

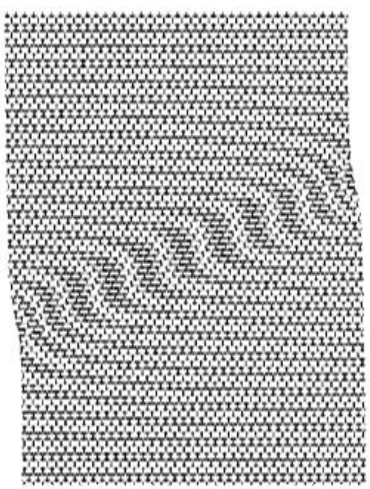
$$\mathbf{b}_\mathrm{m} = -\frac{\partial \mathbf{P}_\mathrm{M}}{\partial \mathbf{F}_\mathrm{M}} : \mathbf{G}_\mathrm{M} : \boldsymbol{I}$$

Second order homogenisation for cellular and metamaterials

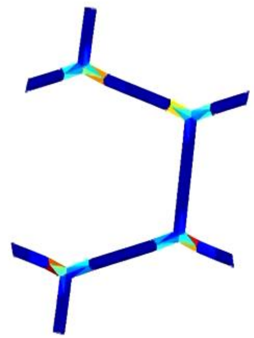
- Multiscale simulation on honeycomb structures
  - Limit of first-order homogenisation

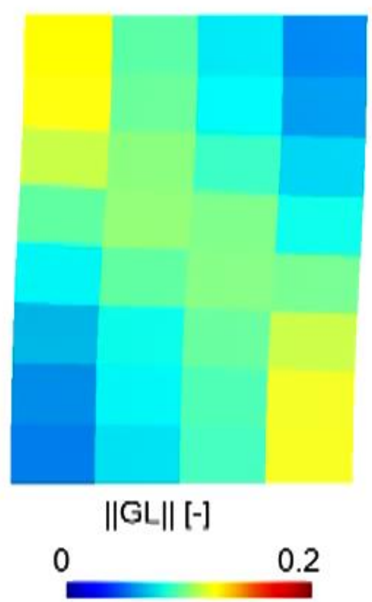


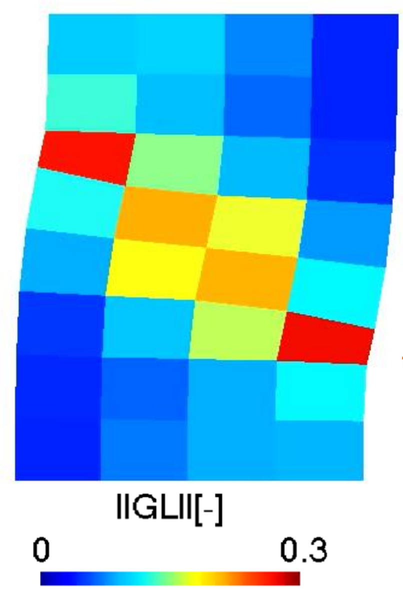
Direct simulation

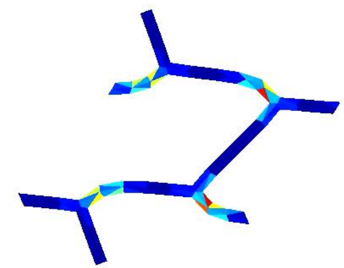


Body-force enhanced second-order homogenisation





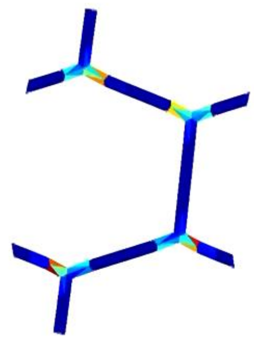


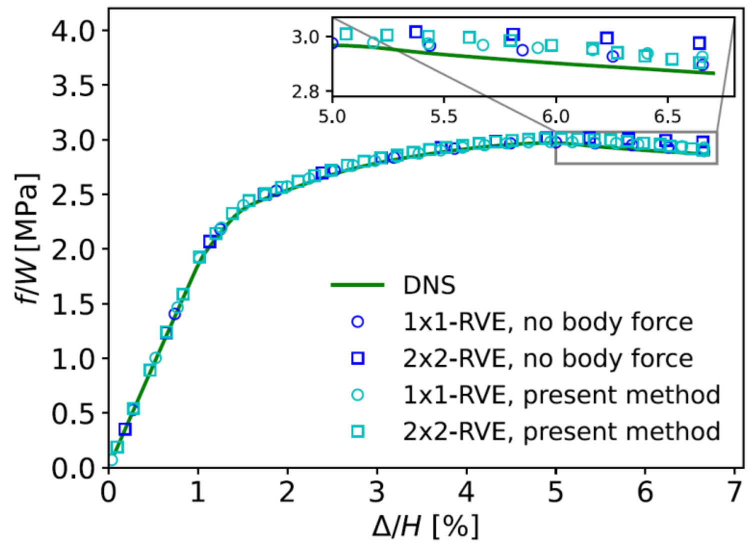


First-order homogenisation

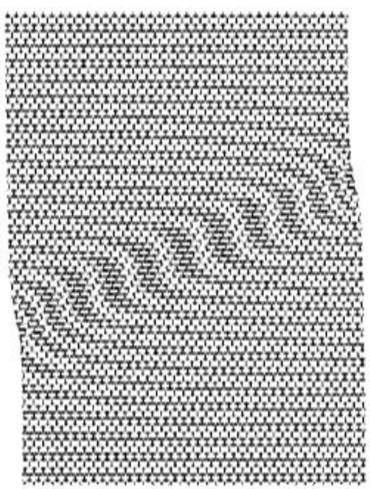
Second order homogenisation for cellular and metamaterials

- Multiscale simulation on honeycomb structures
  - Effect of RVE size disappears when considering body forces

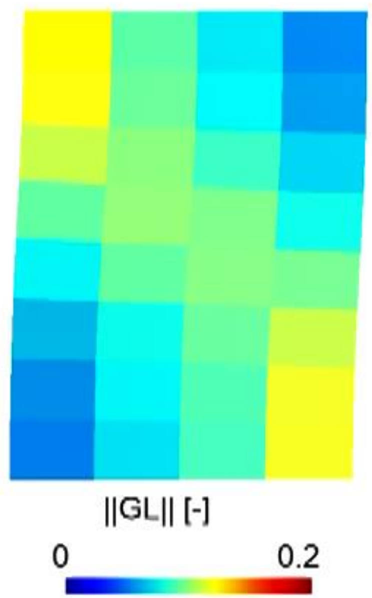


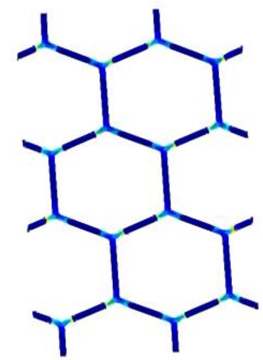


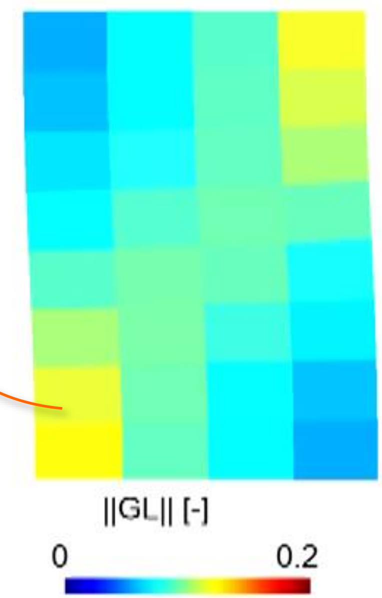
Direct simulation



Body-force enhanced second-order homogenisation



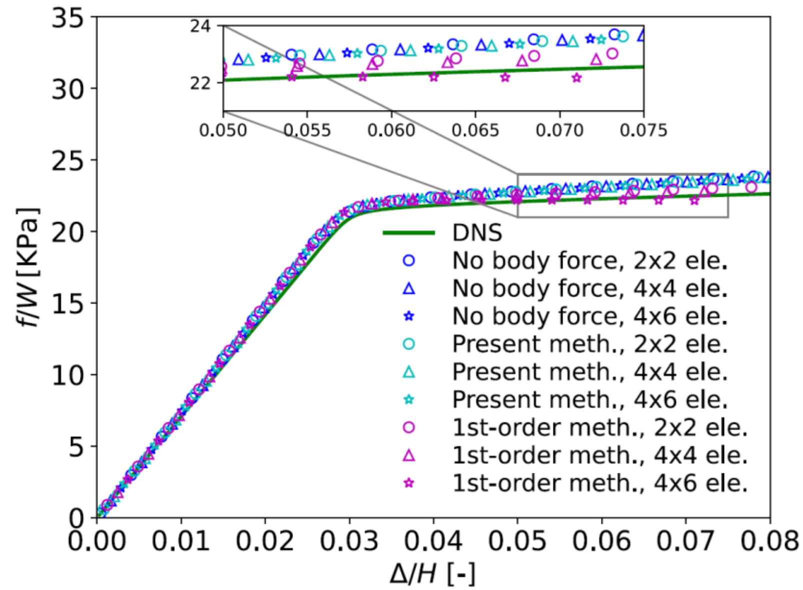




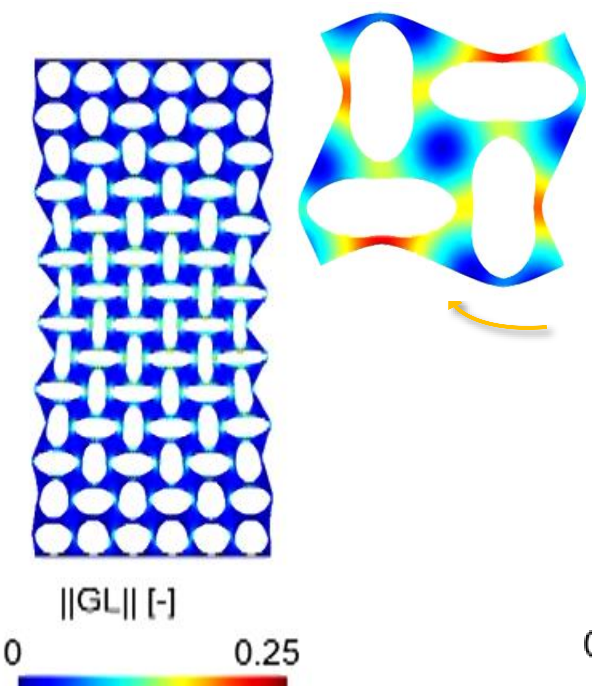
Body-force enhanced second-order homogenisation

Second order homogenisation for cellular and metamaterials

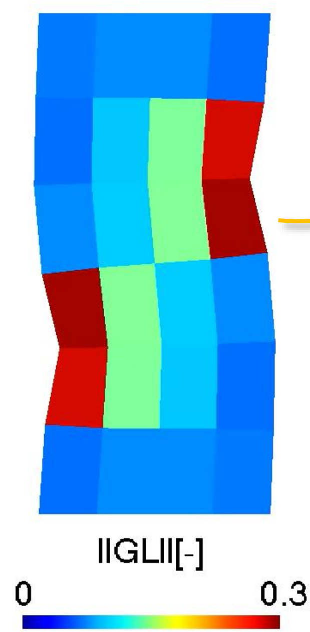
- Multiscale simulation on metastructures
  - Local instability under compression
    - 6x14 holes ➔ local instability
  - Limit of first-order homogenisation

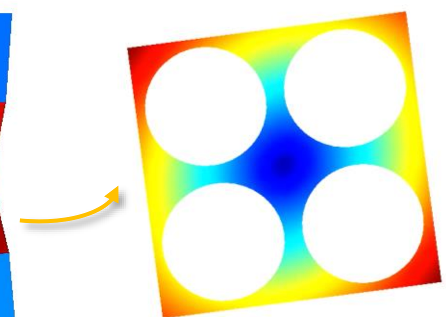


Direct simulation



Body-force enhanced second-order homogenisation

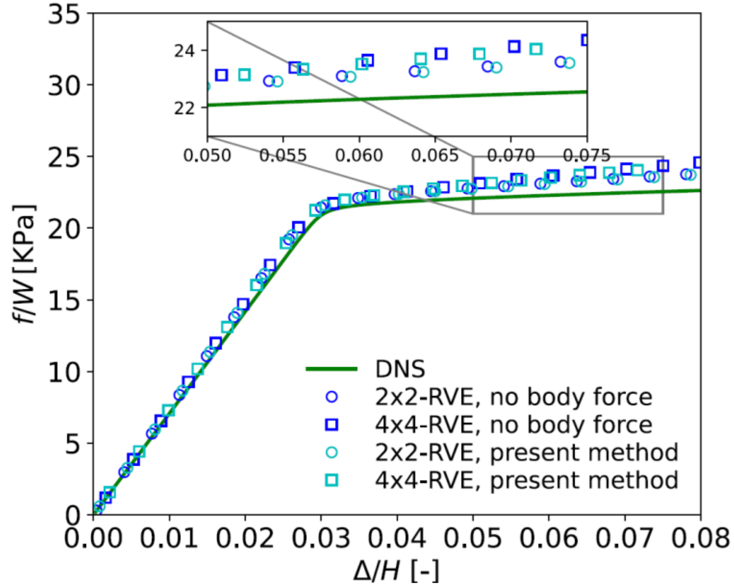




First-order homogenisation

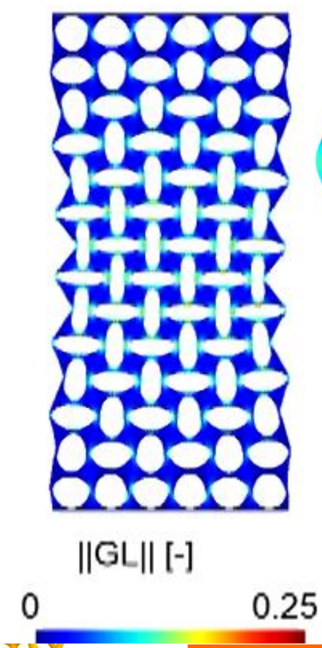
Second order homogenisation for cellular and metamaterials

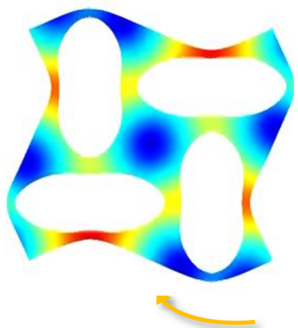
- Multiscale simulation on metastructures
  - Local instability under compression
    - 6x14 holes ➔ local instability
  - RVE size effect

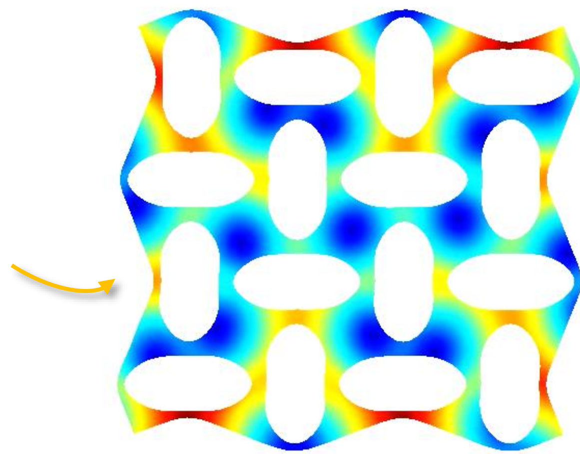


Direct simulation

Body-force enhanced second-order homogenisation



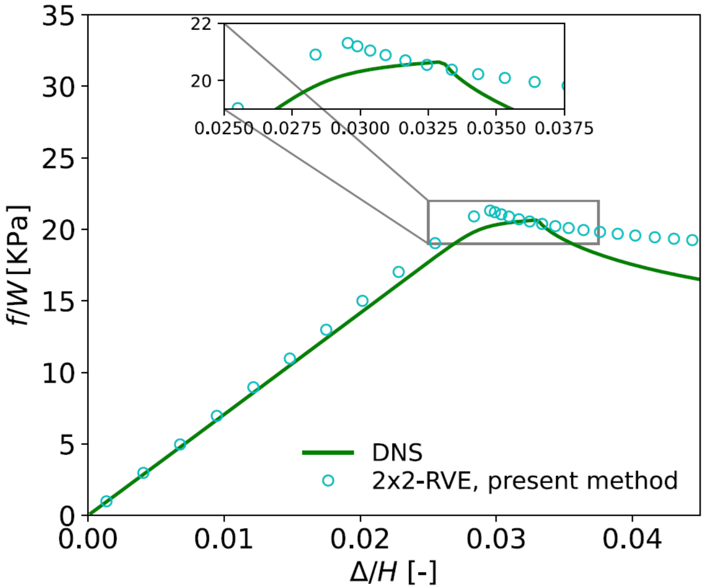


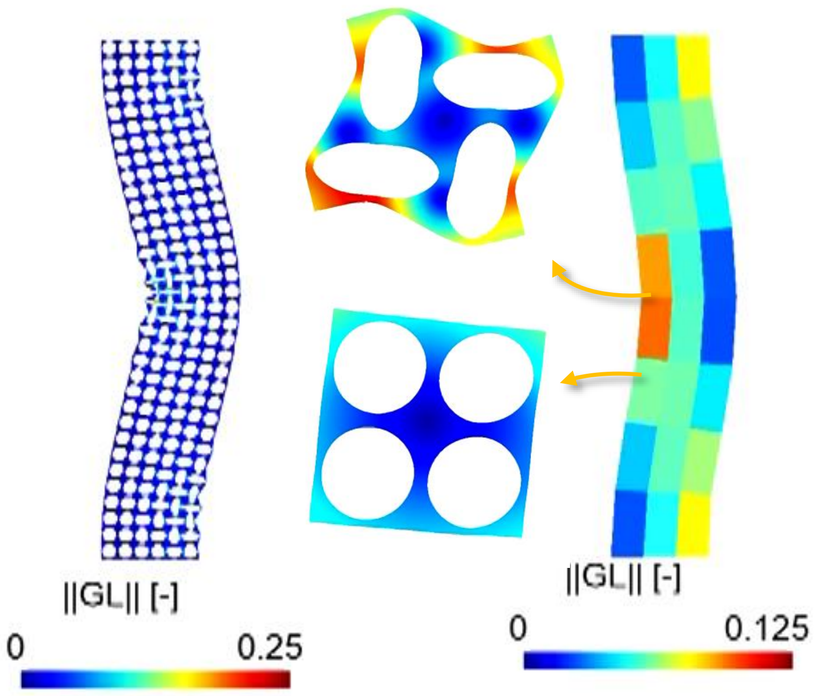


Body-force enhanced second-order homogenisation

Second order homogenisation for cellular and metamaterials

- Multiscale simulation on metastructures
  - Compression samples of different sizes
    - 6x34 holes ➡ global instability





Conclusions

- Effect of RVE size largely reduced
- Applicable to
  - Finite strain formulation
  - Elasto-plasticity
  - Local instabilities
  - Global instabilities
- More on
  - L. Wu, S. M. Mustafa, J. Segurado, and L. Noels. « Second-order computational homogenisation enhanced with non-uniform body forces for non-linear cellular materials and metamaterials. » Computer Methods in Applied Mechanics and Engineering, 407 (2023): 115931, doi: 10.1016/j.cma.2023.115931