Fill-rate service level constrained Distribution Network design

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Abstract

The design of distribution networks that simultaneously consider location and inventory decisions seeks to balance costs and product availability. The most commonly observed measure of product availability in practical settings is the fill-rate service level. However, the optimal design of a distribution network that considers the fill-rate to control shortages of fast-moving consumer goods (FMCG) is considered intractable and has only been addressed by heuristic methods. This paper addresses the optimal design of a distribution network for FMCG able to provide high fill-rate service level under a continuous review (r, Q) policy. Considering the exact formulation for the provided fill-rate, we formulated a joint location-inventory model with fill-rate service level constraints as a convex MINLP for which a novel decomposition-based outer approximation algorithm is proposed. Numerical experiments have shown that our solution approach provides good-quality solutions that are on average 0.15% and, at worst, 2.2% from the optimal solution.

Keywords: Location; Inventory; Fill-rate; Network design; Conic integer programming; Outer approximation

1. Introduction

Fast-moving consumer goods (FMCG) are non-durable products that meet consumers' daily needs (Bala and Kumar (2011)). They include pre-packaged food and beverages, health and beauty items, household cleaning products, pet care items, and many others (Khan and Yu (2019)). For the manufacturer, FMCG

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are products with high volume demand, high inventory turnover, and low margins (Park (2015)). These characteristics of FMCGs are aligned with a low-cost strategy for distribution network design. However, the sales function of FMCG has been concentrated in large retail chains (Webster et al. (2006)) that require high service levels in terms of product availability at the supplier's cost. Consequently, the design of distribution networks that integrate location and inventory decisions simultaneously, defined as the *joint location-inventory problem* (JLIP) according to Daskin et al. (2002), is a relevant issue for FMCG suppliers to design distribution networks that balance cost and service level in terms of product availability.

The most commonly product availability measure observed in practical settings is the *fill-rate* service level (Babiloni and Guijarro, 2020; Teunter et al., 2017; Tan et al., 2017; Bijvank, 2014), defined as the fraction (β) of demand met directly from inventory on-hand in any period (Schneider (1981)). However, the most common used product availability measure in JLIP is the α_L service-level, defined as the probability (α_L) that stockouts do not occur during a replenishment cycle (Schneider (1981)) where L refers to the lead time. In practical settings, fill-rate controls the shortage size at an arbitrary time, and α_L service-level controls the frequency of replenishment cycle stockouts. Craig et al. (2016) found that retailers do not order more frequently in response to increased supplier service levels and infer that retail buyers place larger (and not more frequent) orders for products. Thus, the fill-rate service level is not only the most common measure of product availability in practical settings but also the most important in the supplier-retailer relationship. From the literature review, we observe that the fill-rate has been scarcely addressed in the design of distribution networks for FMCG, where it is usually more convenient and efficient to model demand over a period of time using a continuous distribution, e.g., normal or gamma distributions (Axsäter, 2015; Ramaekers et al., 2008; Peterson and Silver, 1979).

The objective of this paper is to determine the optimal configuration of a FMCG distribution network able to provide fill-rate service level. The optimal design of the distribution network should determine simultaneously the number and location of distribution centers (DCs), the allocation of customers to DCs, and the parameters of the inventory control system that ensure meeting the required fill-rate so that fixed installation costs, transportation costs, and inventory costs are minimized. We assume at each DC: (i) a continuous review (r, Q) policy with full-backorders, where Q is a fixed order quantity and r is the reorder point, (ii) normally distributed demand as an approximation for the FMCG demand, and (iii) fill-rate service level to control the shortage. We first model the JLIP with fill-rate service level constraints under a continuous review (r, Q) policy with full-backorders as a mixed integer nonlinear problem (MINLP), denoted as β-JLIP. Then, using conic integer programming, the β-JLIP model is reformulated as a convex MINLP with second-order cone constraints for which a novel decomposition-based outer approximation (OA) algorithm is proposed to solve it.

The main contributions are presented as: (1) We address for the first time the modeling and optimal solution of a FMCG distribution network design problem that takes into account providing and fulfilling high fill-rate service levels. Thus, we extend the design alternatives of FMCG distribution networks by considering the fill-rate service level to control shortages. (2) We show how to formulate the JLIP with fill-rate service level constraints as a convex MINLP. Using operations that preserve de convexity, we show that the inequality function of the fill-rate service level constraint is convex and, consequently, can be replaced by its outer-approximation. (3) We propose a novel decomposition-based OA algorithm to solve the JLIP with fill-rate service level constraints. Different from the OA approach described by Duran and Grossmann (1986), our decomposition approach uses a conic quadratic mixed-integer programming

(CQMIP) master problem and a convex nonlinear programming (NLP) primal problem (subproblem).

The paper is organized as follows. A review of related work is discussed in Section 2. Section 3 presents the JILP model under a continuous review (r, Q) policy with full-backorders, fill-rate service levels, and normally distributed demand. In Section 4, we present the JLIP with fill-rate service level constraints as a convex MINLP with second-order cone constraints. In Section 5, we describe the solution approach based on OA scheme. Numerical results are reported in Section 6. Finally, conclusions and suggested future extensions are described in Section 7.

2. Related literature

To the best of our knowledge, JLIP with fill-rate service level has been studied mainly for the design of slow-moving items distribution networks. These items are products with low demand and/or low inventory turnover. They include high-value or high-priority service parts or spare parts. For convenience, the term ”slow-moving item” is limited to items with 12 or fewer movements per year (Johnston et al. (2003)). A base stock inventory policy is typically chosen for slow-moving items (Wheatley et al. (2015)) and its demand is modeled over a period of time using an independent Poisson process (Sherbrooke (2006); muc (2005)), i.e., discrete distribution demand. In contrast, we study the design of distribution networks for fast-moving items using a continuous review (r, Q) policy with fill-rate and continuous distribution demand, which leads to a different optimization problem and solution approach.

Candas and Kutanoglu (2007) studied the design of a service parts distribution network using a multi-product JLIP under a continuous review base-stock policy with full-backorder and fill-rate. Customers follow a Poisson demand distribution to approximate slow-moving items. They developed a service level constraint that simultaneously ensures coverage and availability, denoted as a *time-based service coverage constraint*. The resulting model is a MINLP. They use piece-wise linear step functions to linearize the fill-rate, obtaining an integer linear problem (IP) approximating the original optimization model (neither relaxation nor constriction). A feasible solution is obtained using post processing. Jeet et al. (2009) studied the design of a service parts distribution network using a single-product JLIP under a continuous review base-stock policy with lost-sale and fill-rate. The model is formulated as a non-convex MINLP. They reformulate their model as a convex MINLP and solve it using an OA approach. Wheatley et al. (2015) addressed the same problem as Candas and Kutanoglu (2007). However, they present an exact methodology to solve the JLIP using Benders decomposition. All the above authors assume fill-rate defined as the fraction of demand met directly from inventory on-hand during the replenishment cycle (lead time). In contrast, the definition of fill-rate that we use in this paper is for any period according to Schneider (1981).

The JLIP under (r, Q) policy is a well-studied problem using α_L service-level. Daskin et al. (2002) studied the JLIP under EOQ quantity, and stochastic reorder point as approximations of the optimal parameters of a continuous review (r, Q) policy with α_L service-level. Customers follow normally distribution demand as an approximation of non-negative demand. The model is formulated as a nonlinear integer problem (INLP) because the EOQ quantity and stochastic reorder point are functions of the location-allocation variables. They solved the model by Lagrangian relaxation. Several solution approaches have been developed to address the formulation presented by Daskin et al. (2002), e.g., column generation (Shu et al., 2005; Shen et al., 2003), Lagrangean relaxation (You and Grossmann (2008)), and the CQMIP reformulation of Atamtürk et al. (2012) which can be solved directly using standard optimization solvers. Different generalizations have been made to the model of Daskin et al. (2002), e.g., the capacitated JLIP (Atamtürk et al., 2012; Ozsen et al., 2008; Miranda and Garrido, 2004), the multi-commodity JLIP (Atamtürk et al., 2012; Shen, 2005), the correlated demand JLIP (Atamtürk et al., 2012; Shahabi et al., 2014), the multi-echelon JLIP (Diabat et al. (2015)), the stochastic JLIP (Snyder et al. (2007)), the JLIP with differentiated service levels (Escalona et al., 2018, 2015), the stochastic capacity JLIP (Tapia-Ubeda et al., 2018; Miranda and Garrido, 2008, 2006), and the JLIP with supplier selection (Perez Loaiza et al. (2017)). All of the above authors approximate the optimal parameters of the continuous review (r, Q) policy by the EOQ quantity and stochastic reorder point, or approximate the inventory on-hand to the inventory level by considering the backorders negligible.

The JLIP with service level constraints (fill-rate or α_L) considers the inventories in single-echelon settings. More precisely, inventory policy is considered only at DCs. This approach is denoted *non-coordinated* in the sense that DCs observe uncertain demand from retailers, but do not observe retailers' inventory control policy. Recently, Darmawan et al. (2021) studied the coordinated JLIP where DCs and retailers operate under a continuous review (r, Q) policy with full-backorders and normally distributed demand. They use the *economic order quantity* (EOQ) approximating the replenishment lot size at DCs and retailers. The model is formulated as a MINLP with fill-rate service level constraints for the end-customers (retailers). Due to the complexity of the optimization problem, they used genetic algorithms to solve it without guaranteeing the quality-solution in terms of the optimality gap. Numerical experiments show that the inclusion of coordinated inventory control in DCs and retailers results in lower total cost. In contrast, we address a non-coordinated JLIP, we do not approximate the continuous review (r, Q) policy parameters, and we guarantee the quality-solution in terms of optimality gap.

On the other hand, in this paper we show how to formulate the JLIP with fill-rate service level constraints as a convex MINLP. Convex MINLP problems are an important subclass of MINLP problems, where their convex properties are exploited to derive efficient decomposition algorithms. Decomposition algorithms for convex MINLP are: Branch and Bound (B&B), extended cutting plane (ECP), extended supporting hyperplane (ESH), outer approximation (OA), and generalized Benders decomposition (GBD). In B&B methods (Leyffer (2001); Quesada and Grossmann (1992); Dakin (1965)) integer relaxed convex subproblems are solved in each node of a B&B tree. The decomposition methods ECP (Westerlund and Pettersson (1995)), ESH (Kronqvist et al. (2016)), and OA (Coey et al. (2020); Muts et al. (2020); De Mauri et al. (2020); Su et al. (2018); Duran and Grossmann (1986)), exploit the properties of convex MINLPs to derive linearizations of nonlinear constraints based on their gradients. These linearizations are equivalent to first-order Taylor expansions of nonlinear inequalities. The GBD (Floudas (1995); Geoffrion (1972)) exploits the nonlinear convex duality to derive linearizations of nonlinear constraints. Of the methods for solving convex MINLP, OA has been shown to be one of the most efficient algorithms (Bernal et al. (2022); Kronqvist et al. (2019); Su et al. (2015)).

In summary, the design of FMCG distribution networks able to provide fill-rate service level in terms of product availability under a continuous review (r, Q) policy with full-backorder has been scarcely studied. This paper contributes to extend the design alternatives of distribution networks for FMCG by considering the JLIP with fill-rate service level constraints and an efficient OA-based solution approach.

3. Problem description and formulation

The distribution network design problem we consider is as follows. A supplier ships a high demand volume product (FMCG) to a set of retailers (customers) through a set of DCs. Retailers require high fill-rate service levels in terms of product availability. Each retailer is supplied by a single DC (single source allocation) and the DCs are supplied by a single plant of unlimited capacity.

For the FMCG supplier, the distribution network design problem involves simultaneously determining: (i) Which DCs should be installed?, (ii) Which retailers should be supplied by each DC ?, and (c) What inventory policy parameters values should each DC use to satisfy the fill-rate service level required by the retailers?, such that the system-wide cost is minimized. The general structure of the concerned supply chain is shown in Figure 1.

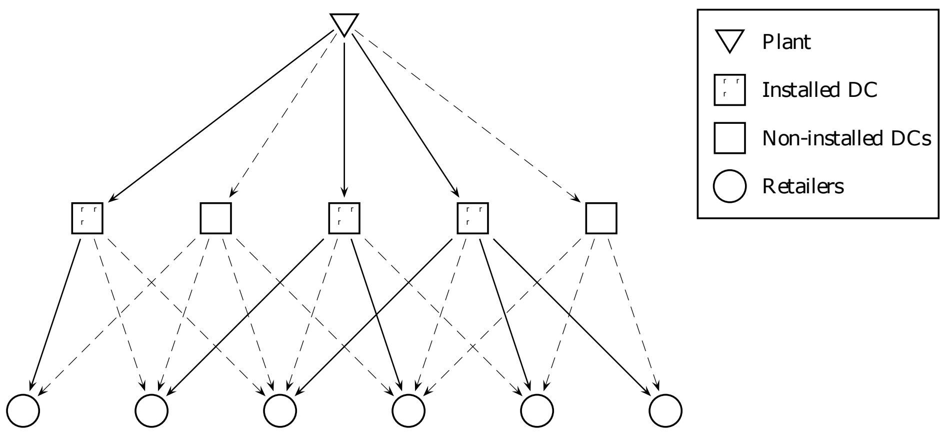

Fig. 1. Supply chain network structure (three echelons)

The plant location, site candidates to install the DCs, and retailers' location are known. Let J ($j = 1, ..., |J|$) be the set of site candidates to install DCs. Each DC follows a continuous review (r, Q) inventory policy with full-backorders and deterministic lead time, i.e., at each DC j, an order quantity of constant size, Q_j, is placed to the plant whenever the inventory position (i.e., inventory on-hand plus outstanding orders minus backorders) falls below a fixed reorder point, r_j, which arrives at a fixed $L_j > 0$ time unit later. Product shortages at each DC are controlled using the fill-rate service level defined as the fraction (β) of demand met directly from on-hand inventory in any period according to Schneider (1981).

The costs associated with the distribution network design are the fixed cost of installation, transportation costs, and inventory costs. Backorder costs are not considered because we have assumed that inventory shortages are controlled by a service level constraint.

Let I ($i = 1, ..., |I|$) be the set of retailers. We assume that the demand per time unit at each retailer is independent and normally distributed, as an approximation of FMCG demand, where μ_i and σ_i^2 are the mean and variance demand per time unit of retailer i, respectively. Let the location variable X_j be equal to 1 if site j is selected as a DC and 0 otherwise, and let the allocation variable Y_{ij} be equal to 1 if retail i is served by the DC located at j and 0 otherwise. The demand at candidate DC j is defined by the allocation variables. More precisely, under the assumption of independent and normally distributed demand, it is easy to show that $\mu_j = \sum_{i\in I} \mu_i Y_{ij} \geq 0$ and $\sigma_j^2 = \sum_{i\in I} \sigma_i^2 Y_{ij} \geq 0$, where μ_j and σ_j^2 are the mean and variance demand per time unit at DC j, respectively.

In what follows, the general cost function for each DC, the provided fill-rate, and the JLIP with fill-rate service level constraints are presented.

3.1. Cost function

The average cost per time unit at DC j considering fixed installation cost, linear transportation costs, and a continuous review (r, Q) policy is defined as:

$$AC_j(Q_j, r_j, X_j, \mathbf{Y}) = f_j X_j + a_j \sum_{i\in I} \mu_i Y_{ij} + \sum_{i\in I} d_{ij}\mu_i Y_{ij} + S_j \frac{\sum_{i\in I} \mu_i Y_{ij}}{Q_j} + h_j \left(\frac{Q_j}{2} + r_j - L_j \sum_{i\in I} \mu_i Y_{ij} \right), \tag{1}$$

where f_j is the fixed cost per time unit of installing a DC at candidate site j, a_j is the transportation cost per unit from the plant to DC j, d_{ij} is the transportation cost per unit from DC j to retailer i, S_j is the ordering cost at DC j, and h_j is the holding cost per unit and time unit at DC j.

The first term in (1) is the fixed cost per time unit, the second and third terms are the supply and distribution costs per time unit, respectively, and the fourth term is the ordering cost per time unit. The fifth term is approximated the holding cost per time unit because similar to Daskin et al. (2002), we approximate the inventory on-hand to the inventory level.

The cost function (1) has a time-base unit because it considers investment and operational costs. To make the five terms in (1) comparable, the fixed cost of installing a DC at candidate site j (one-time investment) must be converted to fixed installation cost per time unit using an appropriate discount rate.

3.2. The fill-rate service level constraint

The shortage in inventory systems is an inevitable random phenomenon under uncertain demand. In this paper, we assume that shortage events at each DC are controlled using the fill-rate service level defined by Schneider (1981). Under normally distributed demand, the fill-rate is equivalent to the *ready-rate* service level defined as the probability (α) of not being out of stock at an arbitrary time or equivalently as the fraction of time with a positive stock-on-hand (Escalona et al., 2021; Axsäter, 2015; Silver et al., 1998). Thus, the fill-rate under normally distributed demand explicitly controls the backorders size and stockouts frequency, both at an arbitrary time.

Let $\beta(r_j, Q_j, \mathbf{Y})$ be the provided fill-rate at DC j. Following Axsäter (2006), the provided fill-rate under a continuous review (r, Q) policy with full-backorders and normally distributed demand is defined as:

$$\beta(Q_j, r_j, \mathbf{Y}) = 1 - \frac{\sqrt{L_j \sum_{i\in I} \sigma_i^2 Y_{ij}}}{Q_j} \left[G\left(\frac{r_j - L_j \sum_{i\in I} \mu_i Y_{ij}}{\sqrt{L_j \sum_{i\in I} \sigma_i^2 Y_{ij}}} \right) - G\left(\frac{r_j + Q_j - L_j \sum_{i\in I} \mu_i Y_{ij}}{\sqrt{L_j \sum_{i\in I} \sigma_i^2 Y_{ij}}} \right) \right], \tag{2}$$

where $G(x) = \int_x^{\infty} (v - x)\varphi(v)\, dv = \varphi(x) - x(1 - \Phi(x))$ with $\Phi(x)$ and $\varphi(x)$ the distribution and density functions of the standard normal distribution, respectively.

Let $\bar{\beta} \in (0, 1)$ be the preset fill-rate for the network distribution, i.e., it is the minimum fraction of demand that the decision-maker at $DC\ j$ requires to meet directly from on-hand inventory at an arbitrary time. Thus, the fill-rate service level constraint is defined as $\beta(Q_j, r_j, \mathbf{Y}) \geq \bar{\beta}$ for any DC $j \in J$.

3.3. The JILP under fill-rate service level constraints

We can formulate a location-inventory model under continuous review (r, Q) policy with fill-rate service level constraints and normally distributed demand as a MINLP with nonlinear constraint and nonlinear objective function, denoted β-JLIP, as follows:

$$\beta\text{-JLIP}: \min_{\mathbf{X},\mathbf{Y},\mathbf{r},\mathbf{Q}} \quad \sum_{j\in J} \left\{ f_j X_j + \sum_{i\in I} \hat{d}_{ij} Y_{ij} + S_j \frac{\sum_{i\in I} \mu_i Y_{ij}}{Q_j} + h_j \left(\frac{Q_j}{2} + r_j - L_j \sum_{i\in I} \mu_i Y_{ij} \right) \right\} \tag{3}$$

$$\text{s.t:} \quad \sum_{j\in J} Y_{ij} = 1 \qquad \forall i \in I \tag{4}$$

$$Y_{ij} \leq X_j \qquad \forall i \in I, j \in J \tag{5}$$

$$\beta(Q_j, r_j, \mathbf{Y}) \geq \bar{\beta} \qquad \forall j \in J \tag{6}$$

$$r_j \geq L_j \sum_{i\in I} \mu_i Y_{ij} \qquad \forall i \in I, j \in J \tag{7}$$

$$Q_j \geq 0 \qquad \forall j \in J \tag{8}$$

$$X_j, Y_{ij} \in \{0, 1\} \qquad \forall i \in I, j \in J, \tag{9}$$

where $\hat{d}_{ij} = (a_j + d_{ij})\mu_i$, and $\beta(Q_j, r_j, \mathbf{Y})$ is the provided fill-rate at candidate DC j according to (2).

The objective function in (3) is the average total costs per time unit including fixed, supply and distribution, ordering, and holding costs. Constraints (4) establish that each retailer is served by only one DC. Constraints (5) ensure that the retailers are assigned to an installed DC. Constraint (6) ensures that the fill-rate provided by each DC is greater than or equal to the preset fill-rate defined for the distribution network. Constraint (7) ensures that the safety stock at each DC j is non-negative because, by definition, the safety stock in a continuous review (r, Q) policy is $r_j - \mathbb{E}\left(D_j(L_j)\right)$, where $\mathbb{E}(D_j(L_j)) = L_j \sum_{i\in I} \mu_i Y_{ij}$ is the expected demand during lead time at DC j. Constraint (8) is a non-negativity constraint of order quantity decision variables. Constraint (9) is an integrality constraint of location-allocation decision variables, respectively. A glossary of terms is presented in Appendix A.

4. An equivalent convex MINLP with second-order cone constraints of β-JLIP

The β-JLIP model is a non-convex MINLP in its objective function and constraint (6). In this section, we show how to reformulate the β-JLIP model as a convex MINLP using conic integer programming.

Similar to Atamtürk et al. (2012), we introduce a set of two non-negative continuous variables, $C1_j$ and $C2_j$, which exploits the fact that $Y_{ij} = Y_{ij}^2$. These variables are defined as follows:

$$C1_j^2 = \sum_{i \in I} \mu_i Y_{ij}^2 \qquad \forall j \in J \tag{10}$$

$$C2_j^2 = \sum_{i \in I} (\sigma_i Y_{ij})^2 \qquad \forall j \in J \tag{11}$$

$$C1_j \geq 0 \qquad \forall j \in J \tag{12}$$

$$C2_j \geq 0 \qquad \forall j \in J. \tag{13}$$

Furthermore, we define the safety stock at DC j as $v_j = r_j - L_j C1_j^2$ using (10). By implementing the change of variables in β-JLIP, we observe that the resulting objective function is increasing in $C1_j$, and $C2_j$ is not present in the objective function. Consequently, constraints (10) and (11) can be further relaxed as the following inequalities constraints:

$$C1_j^2 \geq \sum_{i \in I} \mu_i Y_{ij}^2 \qquad \forall j \in J \tag{14}$$

$$C2_j^2 \geq \sum_{i \in I} (\sigma_i Y_{ij})^2 \qquad \forall j \in J. \tag{15}$$

Thus, the β-JLIP model can be reformulated as the following MINLP with second-order cone constraints:

$$\beta\text{-CQLIP}: \quad \min_{\mathbf{X},\mathbf{Y},\mathbf{v},\mathbf{Q},\mathbf{C1},\mathbf{C2}} \quad \sum_{j \in J} \left\{ f_j X_j + \sum_{i \in I} \hat{d}_{ij} Y_{ij} + g_j(v_j, Q_j, C1_j) \right\} \tag{16}$$

$$\text{s.t:} \quad b_j(v_j, Q_j, C2_j) \leq 0 \qquad \forall j \in J \tag{17}$$

$$v_j \geq 0 \qquad \forall j \in J \tag{18}$$

$$(4), (5), (8), (9), (12), (13), (14), (15),$$

where

$$g_j(v_j, Q_j, C1_j) = S_j \frac{C1_j^2}{Q_j} + h_j \left(\frac{Q_j}{2} + v_j \right), \tag{19}$$

and

$$b_j(v_j, Q_j, C2_j) = -\sqrt{L_j} C2_j (1 - \bar{\beta}) + L_j \frac{C2_j^2}{Q_j} \left[G\left(\frac{v_j}{\sqrt{L_j} C2_j} \right) - G\left(\frac{v_j + Q_j}{\sqrt{L_j} C2_j} \right) \right]. \tag{20}$$

Constraints (14) and (15), together with constraints (12) and (13), define second-order cone constraints. Thus, β-CQLIP is a MINLP with second-order cone constraints.

Proposition 1. *The function g_j is jointly convex in v_j, Q_j, and $C1_j$, and the function of the inequality constraint (17), b_j, is jointly convex in v_j, Q_j, and $C2_j$.*

Proof. The proof is provided in Appendix B. □

Thus, under Proposition 1, it can be stated that β-CQLIP is a convex MINLP with second-order cone constraints. Furthermore, β-CQLIP is a reformulation of β-JLIP. Consequently, $Z^*_{\beta\text{-JLIP}} = Z^*_{\beta\text{-CQLIP}}$, where $Z^*_{\beta\text{-JLIP}}$ and $Z^*_{\beta\text{-CQLIP}}$ are the optimal objective function of β-JLIP and β-CQLIP, respectively.

5. An outer-approximation solution approach

The main consequence of Proposition 1 is that we can write an equivalent β-CQLIP model replacing g_j and b_j by their outer approximations, i.e., the intersection of an infinite set of supporting functions for each one. These supporting functions correspond to linearizations of g_j and b_j at all $(v_j^{(k)}, Q_j^{(k)}, C1_j^{(k)}, C2_j^{(k)}) \in \mathcal{G} = \{(v_j, Q_j, C1_j, C2_j) : v_j \geq 0, Q_j \geq 0, C1_j \geq 0, C2_j \geq 0\}$. We denoted fixed $(v_j, Q_j, C1_j, C2_j)$ variables as $(v_j^{(k)}, Q_j^{(k)}, C1_j^{(k)}, C2_j^{(k)})$. Thus, the following conditions are satisfied:

$$g_j(v_j, Q_j, C1_j) \geq g_j(v_j^{(k)}, Q_j^{(k)}, C1_j^{(k)}) + \frac{\partial g_j^{(k)}}{\partial v_j}\left(v_j - v_j^{(k)}\right) + \frac{\partial g_j^{(k)}}{\partial Q_j}\left(Q_j - Q_j^{(k)}\right) + \frac{\partial g_j^{(k)}}{\partial C1_j}\left(C1_j - C1_j^{(k)}\right), \quad (21)$$

$$b_j(v_j, Q_j, C2_j) \geq b_j(v_j^{(k)}, Q_j^{(k)}, C2_j^{(k)}) + \frac{\partial b_j^{(k)}}{\partial v_j}\left(v_j - v_j^{(k)}\right) + \frac{\partial b_j^{(k)}}{\partial Q_j}\left(Q_j - Q_j^{(k)}\right) + \frac{\partial b_j^{(k)}}{\partial C2_j}\left(C2_j - C2_j^{(k)}\right), \quad (22)$$

for any $(v_j^{(k)}, Q_j^{(k)}, C1_j^{(k)}, C2_j^{(k)}) \in \mathcal{G}$ due to the convexity of g_j and b_j, respectively.

Points $(v_j^{(k)}, Q_j^{(k)}, C1_j^{(k)}, C2_j^{(k)})$ for any $j \in J$ result from fixing $(\mathbf{X}, \mathbf{Y})$ variables in β-CQLIP to a 0-1 combination, which are denoted as $(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)})$. The formulation, denoted as *primal problem*, result in the following NLP:

$$P(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)}): \quad \min_{\mathbf{v},\mathbf{Q},\mathbf{C1},\mathbf{C2}} \quad \sum_{j \in J} \left\{ f_j X_j^{(k)} + \sum_{i \in I} \hat{d}_{ij} Y_{ij}^{(k)} + g_j(v_j, Q_j, C1_j) \right\} \tag{23}$$

$$\text{s.t:} \quad C1_j \geq \sqrt{\sum_{i \in I} \mu_i Y_{ij}^{(k)}} \qquad \forall j \in J \tag{24}$$

$$C2_j \geq \sqrt{\sum_{i \in I} \sigma_i^2 Y_{ij}^{(k)}} \qquad \forall j \in J \tag{25}$$

$$(8), (12), (13), (17), (18).$$

The primal problem, $P(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)})$, is a convex NLP because a convex function on a convex set is minimized. Furthermore, the primal problem is separable in $|J|$ subproblems, one for each DC $j \in J$.

Using (21), (22), and introducing a scalar W in β-CQLIP, we obtain the following equivalent CQMIP, denoted as the *master problem* (MP),

$$\text{MP}: \quad \min_{\mathbf{X},\mathbf{Y},\mathbf{v},\mathbf{Q},\mathbf{C1},\mathbf{C2},\mathbf{W}} \quad \sum_{j \in J} \left\{ f_j X_j + \sum_{i \in I} \hat{d_{ij}} Y_{ij} \right\} + W \tag{26}$$

$$\text{s.t:} \quad W \geq \sum_{j \in J} \left\{ g_j(v_j^{(k)}, Q_j^{(k)}, C1_j^{(k)}) + \frac{\partial g_j^{(k)}}{\partial v_j}\left(v_j - v_j^{(k)}\right) + \frac{\partial g_j^{(k)}}{\partial Q_j}\left(Q_j - Q_j^{(k)}\right) + \frac{\partial g_j^{(k)}}{\partial C1_j}\left(C1_j - C1_j^{(k)}\right) \right\} \qquad \forall k \in \mathcal{K} \tag{27}$$

$$0 \geq b_j(v_j^{(k)}, Q_j^{(k)}, C2_j^{(k)}) + \frac{\partial b_j^{(k)}}{\partial v_j}\left(v_j - v_j^{(k)}\right) + \frac{\partial b_j^{(k)}}{\partial Q_j}\left(Q_j - Q_j^{(k)}\right) + \frac{\partial b_j^{(k)}}{\partial C2_j}\left(C2_j - C2_j^{(k)}\right) \qquad \forall j \in J, k \in \mathcal{K} \tag{28}$$

$$(4), (5), (8), (9), (12), (13), (18), (14), (15),$$

where $\mathcal{K} = \{k : (\mathbf{v}^{(k)}, \mathbf{Q}^{(k)}, \mathbf{C1}^{(k)}, \mathbf{C2}^{(k)})$ is a feasible solution to $P(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)})\}$.

The MP model consist of valid linear supports, and hence relaxation of the nonlinear function g_j and b_j, for all points $(v_j^{(k)}, Q_j^{(k)}, C1_j^{(k)}, C2_j^{(k)})$ that result from fixing $(\mathbf{X}, \mathbf{Y}) = (\mathbf{X}^{(k)}, \mathbf{Y}^{(k)}) \in \mathcal{F} = \{(\mathbf{X}, \mathbf{Y}) : (4), (5), (9)\}$. As a result, it represents a relaxation of the β-CQLIP model, and hence it is a lower bound on its solution, and it is identical to its solution if all supports are included. Clearly, it is not efficient to solve the MP model directly since we need to know all feasible $(\mathbf{v}^{(k)}, \mathbf{Q}^{(k)}, \mathbf{C1}^{(k)}, \mathbf{C2}^{(k)})$ points which implies that we have to solve all the primal problems $P(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)})$, $(\mathbf{X}, \mathbf{Y}) \in \mathcal{F}$, i.e., exhaustive enumeration of 0-1 alternatives.

The natural approach to solving the master problem is relaxation, i.e., to consider an iterative approach where at each iteration, the linear supports around all previous linearization points are considered. Thus, at each iteration, a new set of linear support constraints are added that improve the relaxation and hence the lower bound.

The first step of the iterative procedure (iteration counter $k = 0$) considers solving the well-known Uncapacitated Facility Location Problem (UFLP) to obtain a feasible distribution network configuration in terms of location-allocation variables. Let $(\mathbf{X}^{(0)}, \mathbf{Y}^{(0)})$ and Z^*_{UFLP} be the optimal variables and objective function of the UFLP. Then, we set the lower bound $LB^{(0)} = Z^*_{UFLP}$, solve the primal problem $P(\mathbf{X}^{(0)}, \mathbf{Y}^{(0)})$, from which we obtain an optimal solution to $(\mathbf{v}^{(0)}, \mathbf{Q}^{(0)}, \mathbf{C1}^{(0)}, \mathbf{C2}^{(0)})$, set the best upper bound as $\inf UB = UB^{(0)} = Z_P^{(0)}$, where $Z_P^{(0)}$ is the optimal objective function of $P(\mathbf{X}^{(0)}, \mathbf{Y}^{(0)})$, and the iteration counter $k = k + 1$. The second step of the iterative procedure is solving the master problem with $\mathcal{K} = \{0, ..., k - 1\}$, denoted as the relaxed master problem (RMP). Let $(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)})$ be the optimal solution of the relaxed master problem, and set $LB^{(k)} = Z^*_{RMP}$ be the new current lower bound, where Z^*_{RMP} is the optimal objective function of the RMP model. Furthermore, $(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)})$ is the next point to be consider in the primal problem $P(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)})$, from which we obtain an optimal solution to $(\mathbf{v}^{(k)}, \mathbf{Q}^{(k)}, \mathbf{C1}^{(k)}, \mathbf{C2}^{(k)})$ and the current upper bound, i.e., $UB^{(k)} = Z_P^{(k)}$, where $Z_P^{(k)}$ is the optimal objective function of $P(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)})$. If the current upper bound is strictly lower than the best upper bound found so far, then $\inf UB = UB^{(k)}$. Thus, if $\frac{\inf UB - LB^{(k)}}{\inf UB} \leq \epsilon$, then terminate, and $\inf UB$ is an ϵ-optimal objective function to β-CQLIP, i.e., $\inf UB \leq Z^*_{\beta\text{-CQLIP}} = Z^*_{\beta\text{-JLIP}}$. Otherwise, $k = k + 1$ and return to step 2.

The pseudo-code of the OA approach to obtain an ϵ-optimal solution of β-JLIP, denoted CQ-OA Algorithm, is presented in Appendix C.

6. Numerical experiments and discussion

This section presents numerical results to evaluate the CQ-OA Algorithm performance to obtain an ϵ-optimal solution of β-JLIP, and then some managerial insights are discussed. Numerical experiments were performed with two data sets representing U.S. cities provided by Daskin (2011): (1) a 49-node set, and (2) an 88-node set. Each node is a retailer and a candidate site for installing a DC, i.e., there are as many candidate DC locations as there are retailer locations for each set.

Model RMP is solved using CPLEX 20.1. Model $P(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)})$ is solved using MINOS 5.4. The CQ-OA Algorithm stopping criterion is $\epsilon = 10^{-4}$ or 7200 seconds of CPU time. All tests were done on a PC with a 2.3 GHz Intel Core i9 CPU and 16 GB main memory.

6.1. Performance of CQ-OA Algorithm

The performance of the ϵ-optimal solution for β-JLIP resulting from the CQ-OA Algorithm is evaluated by CPU times and the optimality gap of those instances that reach the time limit. In order to cover a wide range of data, we generated test problems with heterogeneous parameters. In particular, 200 random instances were generated for each node set.

Test problems (test set) used the following common criteria and parameters. The time base defined for the test set is the month. The demand per time unit at each retailer is normally distributed with mean $\mu_i = U[3000, 12000]$ for any $i \in I$, and coefficient of variation $CV_i = U[0.1, 0.5]$ for any $i \in I$. It should be noted that the normal distribution is a good approximation of non-negative demand when $CV \leq 0.5$ (Peterson and Silver (1979)) in which case the probability of being less than zero is less than 0.0228. The preset service-level for the distribution network is $\bar{\beta} = U[0.75, 0.99]$. The fixed cost of installing a DC at candidate site j (per time unit) is $f_j = U[4000, 8000]$ for any $j \in J$. The transportation cost per unit from the plant to candidate DC site j is $a_j = a = U[0.3, 0.7]$ for any $j \in J$. The transportation cost per unit from DC j to retailer i, d_{ij}, is equal to the distance between DC j and retailer i multiplied by a transportation rate $c_{ij} = c = U[0.001, 0.01]$ for any $i \in I$ and $j \in J$. The holding cost per unit and time unit at DC j is $h_j = h = U[0.25, 1.25]$ for any $j \in J$. The ordering cost at DC j is $S_j = S = U[100, 500]$ for any $j \in J$, and the lead time is $L_j = U[0.2, 0.6]$ for any $j \in J$.

For all instances with 49 and 88 nodes, the CPU times of the CQ-OA Algorithm were computed and are shown in Figure 2.

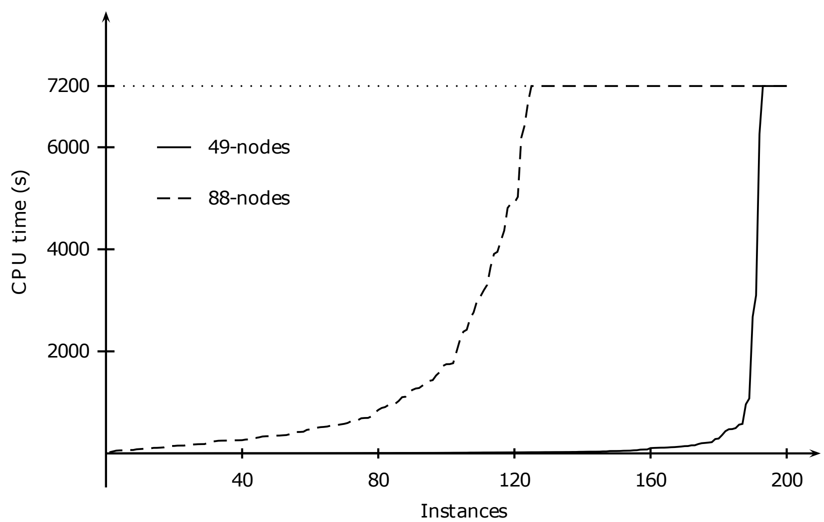

Fig. 2. CPU time using CQ-OA Algorithm

As expected, the CPU time increases with the number of nodes. Figure 2 shows that for the set of 49-nodes, 95% of the instances are solved in less than 7200 s, while for the set of 88-nodes, 63% of the instances are solved in less than 7200 s. We compute the relative optimality gap between the lower bound given by $LB^{(k)}$ and the upper bound given by $\inf UB$, i.e., $Gap(\%) = 100 \times (\inf UB - LB^{(k)}) / \inf UB$. Figure 3 shows the relative optimality gap for all instances and node sets.

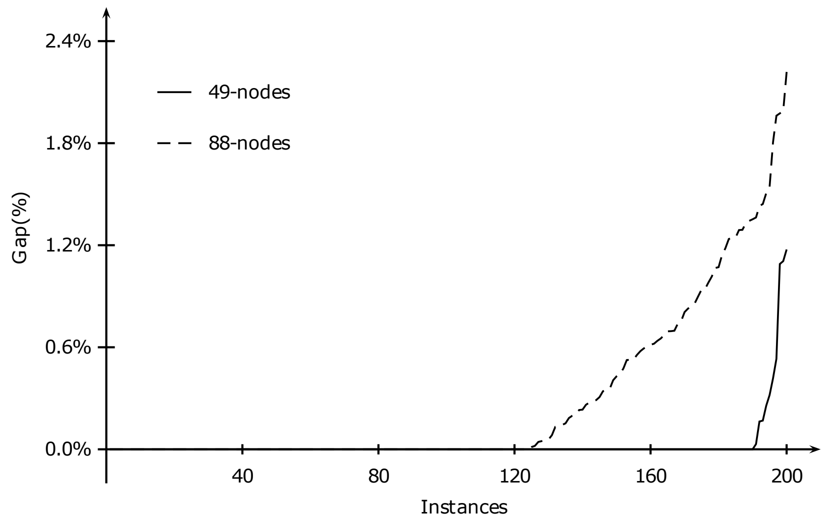

Fig. 3. Relative gap using CQ-OA Algorithm

From Figure 3, we observe that the optimality gap is increasing with respect to the number of nodes. The average optimality gap is 0.01% and 0.3% for 49 and 88 nodes, respectively. The maximum optimality gap is 1.2% and 2.2% for 49 and 88 nodes, respectively, which occurs in instances exceeding the maximum CPU time. Although instances of maximum optimality gap occur when the CQ-OA Algorithm exceeds the time limit, the results are still good-quality solutions for the β-JLIP.

6.2. Illustrative example

We use an instance of the 49-node test set as an illustrative example to show the network configuration in terms of the location-allocation variables and inventory policy parameters. Table D1, in Appendix D, shows the demand per time unit and the coefficient of variation for retailers, and the fixed cost of installation and lead time for DCs. Furthermore, $\bar{\beta} = 0.975$, $c = 0.001$, $a = 0.5$, $h = 0.75$, and $S = 250$.

The illustrative example is solved using the CQ-OA algorithm in 1028 seconds and optimality gap strictly smaller than ϵ. The network configuration is shown in Figure 4 where nine DCs are installed.

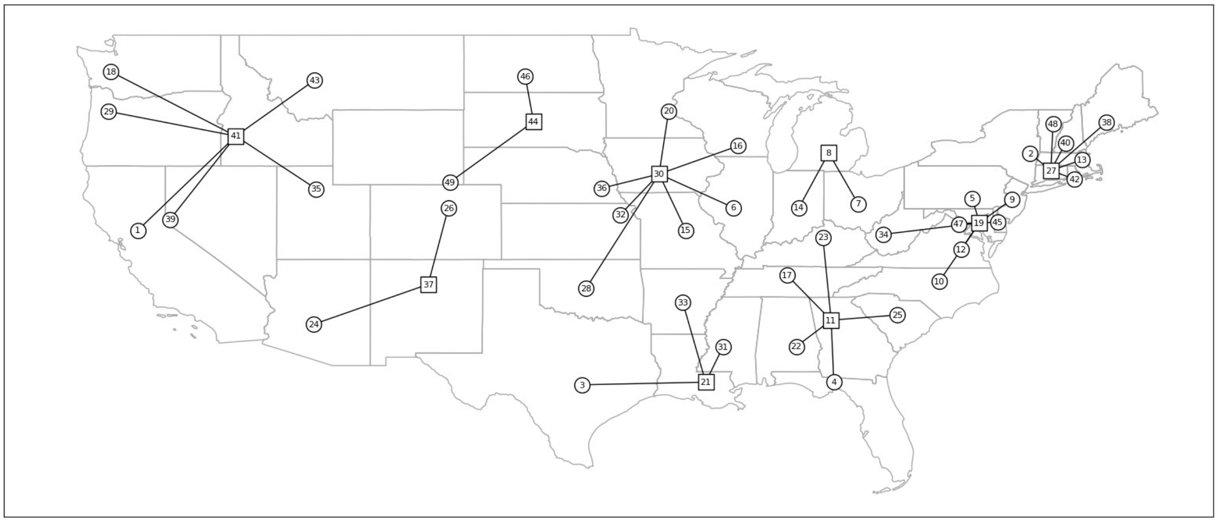

Fig. 4. Network configuration of the illustrative example

The total cost of the distribution network is $399\ 523$ $\$/$month, where 13% is fixed installation cost, 71% is transportation cost, 3% is ordering cost, and 13% is holding cost. The inventory policy parameters for each installed DC are shown in Table 1.

Table 1
Inventory policy parameters of the illustrative example

	$j = 8$	$j = 11$	$j = 19$	$j = 21$	$j = 27$	$j = 30$	$j = 37$	$j = 41$	$j = 44$
r_j	10598.5	14267.4	20618.7	21436.0	18103.1	26407.1	10452.6	36306.7	13254.0
Q_j	4580.9	7396.7	7951.7	6397.2	8168.9	8936.9	5038.0	8525.2	6092.3

6.3. Managerial insights

We derive management insights using a base case for each set of nodes. The fixed installation cost (f_j for any $j \in J$) and lead time (L_j for any $j \in J$) for the base case of 49 and 88 nodes are in Table D1 (Appendix D) and Table E1 (Appendix E), respectively. Furthermore, the base case for each set of nodes considers: $\bar{\beta} = 0.975$, $\mu_i = \mu = 7500$ for any $i \in I$, $CV_i = CV = 0.3$ for any $i \in I$, $c_{ij} = c = 0.001$ for any $i \in I$ and $j \in J$, $a_j = a = 0.5$ for any $j \in J$, $h_j = h = 0.75$ for any $j \in J$, and $S_j = S = 250$ for any $j \in J$.

Using the base case for each set of nodes, we observe how the variation of the preset fill-rate, the mean demand per time unit, the coefficient of variation, the transportation rate, and the holding cost per unit and time unit affect the number of installed DCs, costs, and performance of the CQ-OA Algorithm. We vary $\bar{\beta}$, μ, CV, c, and h between their lower and upper bounds defined in Section 6.1. Thus, we solved 44 problems, 22 for each node set. Table E2, in Appendix E, shows the base case for each node set, the disturbed parameters, results, cost components (FC: fixed cost, TC: transportation cost, OC: ordering cost, HC: holding cost), and the CQ-OA Algorithm performance.

We have the following comments derived from the numerical experiments in Table E2.

- Regarding the number of DCs installed:
  — The number of DCs is non-decreasing with the demand per time unit (μ) and transportation rate (c). This is because higher demand in the network or higher transportation rate induce higher transportation cost (TC) and, similar to the UFLP, the JLIP with fill-rate constraints increases the number of DCs to reduce the transportation cost. Figure 5 shows the effect of demand per time unit and transport rate on the number of DCs installed.

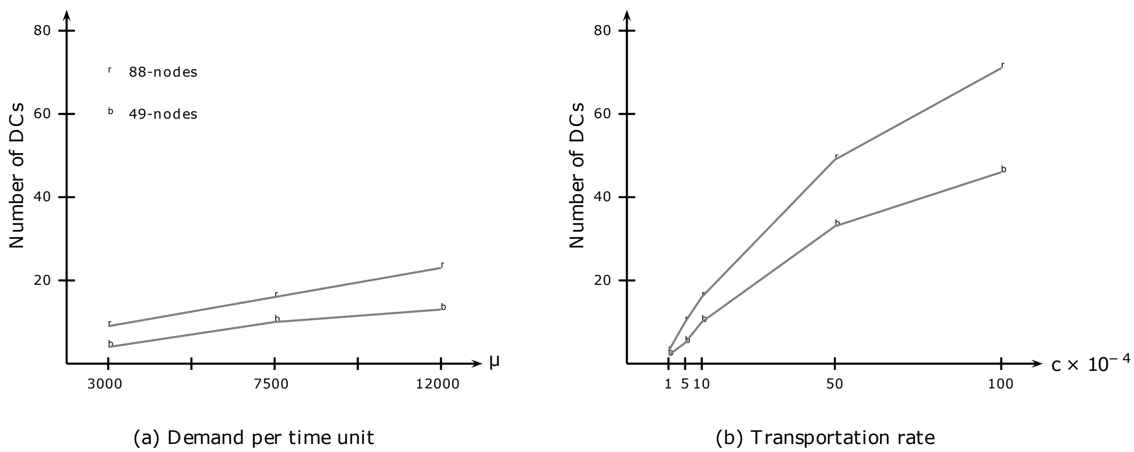

Fig. 5. Effect of demand and transportation rate with $\bar{\beta} = 0.975$, $CV = 0.3$, $h = 0.75$

  — The number of DCs is non-increasing with the preset fill-rate ($\bar{\beta}$), the coefficient of variation (CV), and the holding cost per unit and time unit (h). This is because a higher preset fill-rate, demand variability, or holding cost per unit and time unit, increases the holding cost (HC). Thus, JLIP with fill-rate constraints decrease the number of DCs by taking advantage of the pooling effect described by Eppen (1979). Figure 6 shows the effect of coefficient of variation and holding cost per unit and time unit on the number of DCs installed.

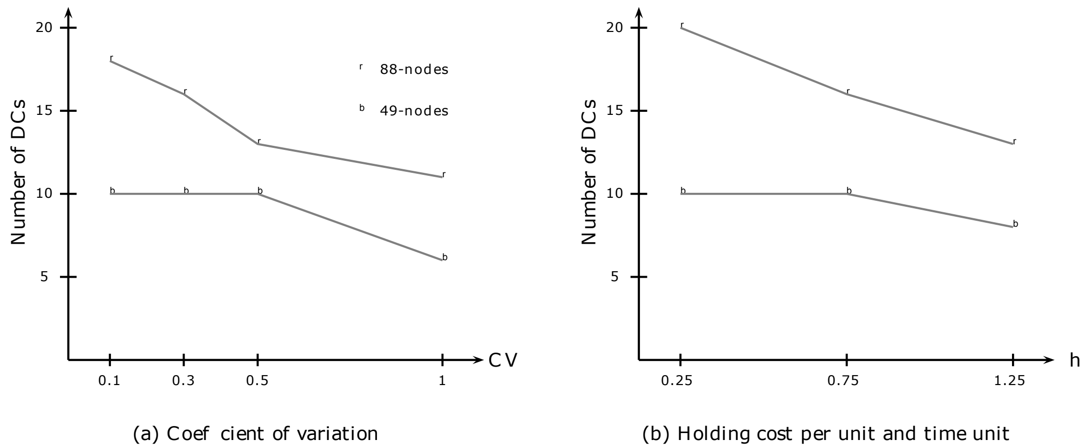

Fig. 6. Effect of coefficient of variation and holding cost with $\bar{\beta} = 0.975$, $\mu = 7500$, $c = 0.001$

- Regarding the cost components as percentages of the total cost:
  - The fixed installation cost (FC) increases its share of the total cost (as a percentage) when the transportation rate (c) increases. The fixed installation cost decreases its share of the total cost when the preset fill-rate ($\bar{\beta}$), the coefficient of variation (CV), or the holding cost per unit and time unit (h), increases.
  - The transportation cost (TC) decreases its share in the total cost when the demand per time unit (μ), the coefficient of variation (CV), the transportation rate (c), or the holding cost per unit and time unit (h), increases.
  - The ordering cost (OC) has a low share in the total cost in the tested instances. We only observe that the ordering cost increases its share in the total cost when the holding cost per unit and time unit (h), increase.
  - The holding cost (HC) increases its share in the total cost when the $\bar{\beta}$, μ, CV, c, or h, increases.

These results are consistent with the increase and decrease of the number of DCs. Furthermore, we observe that the transportation rate is the parameter that most affects the total cost and the number of installed DCs.

- Regarding the difficulty of solving the JLIP with fill-rate service level constraints:
  - As expected, as the size of the distribution network grows, it becomes more difficult to solve them.
  - From the 49-node network, where 70% of the instances in Table E2 were solved in less than the time limit, we observed that as the preset fill-rate, demand per time unit, coefficient of variation, or the holding cost per unit and time unit increases, the JLIP with fill-rate constraints becomes more difficult to solve.

It should be noted that we include in this section instances with $CV = 1$ to explore the effect of coefficients of variation larger than the limits defined in section 6.1. Although, under normal distribution the probability of negative demand increases for CVs strictly greater than 0.5 (Peterson and Silver (1979)), we observe that the results in terms of the number of installed DCs and cost components are consistent with the results obtained for CVs less than or equal to 0.5.

7. Conclusions and future work

In this study, we considered a joint location-inventory model for fast-moving consumer goods in which distribution centers observe uncertain demand from a set of retailers requiring a high fill-rate service level in terms of product availability. At each DC the inventory control system operates under a continuous review (r, Q) policy with full-backorders and shortage events are controlled using the fill-rate service level.

We formulate the joint location-inventory model with fill-rate constraints as a non-convex MINLP. Using conic integer programming, we show how to reformulate it as a convex MINLP with second-order conic constraints. Consequently, an outer-approximation approach is proposed to solve the convex MINLP with second-order conic constraints using a CQMIP master problem and a convex NLP primal problem.

Numerical results show that the OA solution approach can provide good-quality solutions since for the test set problems, the maximum optimality gap is 2.2%, a very good solution, which provides us with the distribution network configuration, including the DCs location, the allocation of the demands, and the inventory policy parameters that ensure to provide the fill-rate required by the retailers.

Several issues may be addressed in future research. The first is that our OA solution approach can be applied to a wide variety of JLIP problems. For example, the capacitated JLIP, the multi-commodity JLIP, the correlated demand JLIP, the stochastic capacity JLIP, and the coordinated JLIP are all under a continuous review (r, Q) policy with fill-rate service level. The second issue is related to the scalability of the OA solution approach. Scalability can be addressed by improving the performance of the OA scheme through valid inequalities, including extended polymatroid, and extended cover cuts similar to Atamtürk et al. (2012). The third issue is to consider the design of FMCG distribution networks with other product availability measures. For example, the fraction $(1-\gamma)$ of demand being on backorder and the fraction (δ_L) of the lead time without stockouts. The fourth issue is to relax the assumption of normally distributed demand by considering, for example, the gamma distribution.

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Appendix A. Glossary of terms.

Table A1
Sets, parameters, and variables

Sets	
I	Set of retailers, indexed by $i = 1, ..., \|I\|$
J	Set of possible DC locations, indexed by $j = 1, ..., \|J\|$
Parameters	
μ_i	Mean demand per time unit of retailer i
σ_i^2	Variance of demand per time unit of retailer i
f_j	Fixed cost per time unit to install a DC at candidate site j
d_{ij}	Transportation cost per unit from DC j to retailer i
c_{ij}	Transportation rate from DC j to retailer i
a_j	Transportation cost per unit from the plant to DC j
S_j	Ordering cost at DC j
h_j	Holding cost per unit and time unit at DC j
L_j	Fixed lead time in time unit from the supplier to DC j
$\bar{\beta}$	Preset fill-rate service level
Variables	
X_j	1 if a DC is located at candidate site j, and 0 otherwise
Y_{ij}	1 if DC j serves retailer i, and 0 otherwise
Q_j	Fixed order quantity at candidate DC j
r_j	Reorder point at candidate DC j

Appendix B. Convexity proof of g_j and b_j.

Proof. We first show that g_j is jointly convex in v_j, Q_j, and $C1_j$. The first term of g_j is convex in Q_j and $C1_j$ because it is a quadratic-over-linear function (Boyd et al. (2004), section 3.1.5, page 73). The second term of g_j is a linear function in Q_j and v_j, respectively. Thus, g_j is a convex function because it is a sum of convex functions.

The proof of the convexity of b_j is as follows. The first term in (20) is convex in $C2_j$ because it is a linear function. Thus, the convexity proof of b_j is reduced to proving the convexity of the second term of (20). Consider $\hat{b}_j$: $\mathbb{R}^2 \to \mathbb{R}$ defined as $\hat{b}_j(v_j, Q_j) = \frac{1}{Q_j}\left(G\left(v_j\right) - G\left(v_j + Q_j\right)\right)$, which is jointly convex in v_j and Q_j according to Zipkin (1986) and Axsäter (2015), and its *perspective* function $\bar{b_j} : \mathbb{R}^3 \to \mathbb{R}$ is defined as:

$$\bar{b}_j(v_j, Q_j, C2_j) = \sqrt{L_j}C2_j\hat{b}_j\left(\frac{v_j}{\sqrt{L_j}C2_j}, \frac{Q_j}{\sqrt{L_j}C2_j}\right) = L_j\frac{C2_j^2}{Q_j}\left[G\left(\frac{v_j}{\sqrt{L_j}C2_j}\right) - G\left(\frac{v_j + Q_j}{\sqrt{L_j}C2_j}\right)\right].$$

As the perspective operation preserves convexity (Boyd et al. (2004), section 3.2.6, page 89), we conclude that b_j is the sum of convex functions. □

Appendix C. CQ - Outer Approximation Algorithm for β-JILP.

Algorithm 1 CQ-OA Algorithm for β-JILP

$k = 0$
$(\mathbf{X}^{(0)}, \mathbf{Y}^{(0)}) \leftarrow$ solve UFLP
Set $LB^{(0)} = Z^*_{UFLP}$
$(\mathbf{v}^{(0)}, \mathbf{Q}^{(0)}, \mathbf{C1}^{(0)}, \mathbf{C2}^{(0)})$, $Z_P^{(0)} \leftarrow$ solve $P(\mathbf{X}^{(0)}, \mathbf{Y}^{(0)})$
Set $\inf UB = UB^{(0)} = Z_P^{(0)}$
while $\frac{\inf UB - LB^{(k)}}{\inf UB} > \epsilon$ **do**
    $k = k + 1$
    $(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)}), LB^{(k)} = Z^*_{RMP} \leftarrow$ solve RMP
    $(\mathbf{v}^{(k)}, \mathbf{Q}^{(k)}, \mathbf{C1}^{(k)}, \mathbf{C2}^{(k)}), UB^{(k)} = Z_P^{(k)} \leftarrow$ solve $P(\mathbf{X}^{(k)}, \mathbf{Y}^{(k)})$
    if $\inf UB > UB^{(k)}$ **then**
        $\inf UB = UB^{(k)}$
    end if
end while
while $j \leq |J|$ **do**
    if $X_j = 1$ **then**
        $r_j = v_j + L_j C1_j^2$
    else
        $r_j = 0$
    end if
end while
Return $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{Q})$, $\inf UB$

Appendix D. Illustrative example data.

Table D1
Illustrative example data

Node	μ_i	CV_i	f_j	L_j	Node	μ_i	CV_i	f_j	L_j	Node	μ_i	CV_i	f_j	L_j
1	8990	0.27	7276	0.27	18	3791	0.46	5145	0.53	35	5342	0.42	5159	0.6
2	9020	0.23	5455	0.29	19	3326	0.44	5244	0.3	36	8470	0.32	5435	0.43
3	3014	0.49	6145	0.41	20	6496	0.23	4611	0.31	37	7831	0.2	6209	0.36
4	6963	0.15	5866	0.2	21	11304	0.28	4577	0.57	38	8998	0.41	5605	0.58
5	7831	0.2	6209	0.36	22	8465	0.49	7028	0.4	39	5416	0.15	5326	0.51
6	9656	0.4	6413	0.39	23	5710	0.42	7277	0.52	40	5829	0.19	5011	0.33
7	4486	0.11	7639	0.4	24	8988	0.37	7411	0.25	41	11860	0.26	6797	0.57
8	10967	0.23	4987	0.45	25	8566	0.14	6213	0.43	42	6679	0.32	7199	0.36
9	4251	0.1	5741	0.2	26	3687	0.5	5790	0.52	43	6706	0.22	4894	0.22
10	8182	0.26	7397	0.42	27	10608	0.17	5431	0.24	44	11370	0.46	6123	0.32
11	10543	0.31	4073	0.22	28	4549	0.46	6024	0.44	45	3211	0.41	6040	0.34
12	10025	0.33	4289	0.47	29	9959	0.42	5940	0.35	46	7185	0.35	7048	0.53
13	7144	0.4	7754	0.56	30	11351	0.32	6750	0.33	47	11428	0.24	5409	0.43
14	3121	0.4	6174	0.38	31	7444	0.35	7443	0.45	48	9639	0.43	4953	0.39
15	3913	0.22	6819	0.27	32	9393	0.25	7737	0.47	49	7620	0.37	7875	0.54
16	8275	0.5	5737	0.3	33	6524	0.44	5872	0.36					
17	8025	0.35	4385	0.39	34	7192	0.41	4407	0.23					

Appendix E. Test set for managerial insights.

Table E1
Base case 88-nodes

Node	f_j	L_j	Node	f_j	L_j	Node	f_j	L_j	Node	f_j	L_j	Node	f_j	L_j
1	5482	0.2	19	5409	0.43	37	6666	0.31	55	7443	0.45	73	4407	0.23
2	5203	0.45	20	7754	0.56	38	5397	0.28	56	6750	0.33	74	6209	0.36
3	6797	0.36	21	6907	0.34	39	7075	0.34	57	5435	0.43	75	7591	0.55
4	4635	0.21	22	5811	0.45	40	7058	0.42	58	5737	0.3	76	7875	0.54
5	7659	0.3	23	7055	0.43	41	7276	0.27	59	7028	0.4	77	7048	0.53
6	7826	0.45	24	5863	0.32	42	5542	0.31	60	5872	0.36	78	5326	0.51
7	4767	0.25	25	4385	0.39	43	6465	0.5	61	7199	0.36	79	5011	0.33
8	7618	0.54	26	5790	0.52	44	5605	0.38	62	5159	0.6	80	6819	0.27
9	7411	0.25	27	6145	0.41	45	6096	0.58	63	5431	0.24	81	5145	0.53
10	5966	0.45	28	6841	0.55	46	4711	0.31	64	4987	0.45	82	5244	0.3
11	6597	0.27	29	6024	0.44	47	4016	0.38	65	6797	0.57	83	6040	0.34
12	4718	0.26	30	6970	0.54	48	6173	0.49	66	5866	0.2	84	7277	0.52
13	6174	0.38	31	4336	0.43	49	5075	0.46	67	7737	0.47	85	4894	0.22
14	4378	0.5	32	6877	0.25	50	5122	0.37	68	5940	0.35	86	5605	0.58
15	4015	0.44	33	7896	0.4	51	4611	0.31	69	7639	0.4	87	6123	0.32
16	6413	0.39	34	5831	0.39	52	4577	0.57	70	5455	0.29	88	4953	0.39
17	6606	0.26	35	7669	0.42	53	7397	0.42	71	6213	0.43			
18	7940	0.49	36	4073	0.22	54	4289	0.47	72	5741	0.2			

Table E2
Test set for managerial insights

		Parameters					Results		Cost Component				Performance		
Data set	Vary	β	μ	CV	c	h	$\sharp DC$	$\inf UB$	FC	TC	OC	HC	Iterations	$Gap(\%)$	CPU Time (s)
49-nodes	Case base	0.975	7500	0.3	0.001	0.75	10	398480	59855	279525	14372	44727	51	$< \epsilon$	2069
	$\bar{\beta}$	**0.75**	7500	0.3	0.001	0.75	10	369431	60874	277575	18647	12335	46	$< \epsilon$	367
		0.99	7500	0.3	0.001	0.75	10	407729	48125	294075	13649	51879	65	0.21%	7200[a]
	μ	0.975	**3000**	0.3	0.001	0.75	4	200438	20575	146190	5049	28624	31	$< \epsilon$	1128
		0.975	**12000**	0.3	0.001	0.75	13	570031	71799	428160	19514	50558	67	$< \epsilon$	3344
	CV	0.975	7500	**0.1**	0.001	0.75	10	375173	51293	287550	15326	21004	38	$< \epsilon$	508
		0.975	7500	**0.5**	0.001	0.75	10	423614	54943	287550	12467	68653	23	0.50%	7200[a]
		0.975	7500	**1**	0.001	0.75	6	474525	32810	329925	7982	103808	9	0.41%	7200[a]
	c	0.975	7500	0.3	**0.0001**	0.75	2	248083	9301	213600	6352	18831	24	$< \epsilon$	875
		0.975	7500	0.3	**0.0005**	0.75	5	334770	25678	266250	9711	33131	37	0.59%	7200[a]
		0.975	7500	0.3	**0.005**	0.75	33	583887	186046	281325	24599	91917	3	$< \epsilon$	6
		0.975	7500	0.3	**0.01**	0.75	46	616008	275139	198675	29572	112622	2	$< \epsilon$	2
	h	0.975	7500	0.3	0.001	**0.25**	10	362047	51669	281700	8619	20060	28	$< \epsilon$	169
		0.975	7500	0.3	0.001	**1.25**	8	430436	43252	303750	15103	68332	22	1.09%	7200[a]
	c, h	0.975	7500	0.3	**0.0001**	**1.25**	2	260217	8967	217875	7794	25581	25	$< \epsilon$	1729
		0.975	7500	0.3	**0.0005**	**1.25**	4	356688	25820	265650	12176	53042	25	1.11%	7200[a]
		0.975	7500	0.3	**0.005**	**1.25**	31	642570	173257	299175	28845	141293	6	$< \epsilon$	30
		0.975	7500	0.3	**0.01**	**1.25**	45	689130	269708	206550	35343	177530	2	$< \epsilon$	6
	μ, CV, c, h	0.975	**12000**	**0.5**	**0.0001**	**1.25**	2	407203	9301	341760	9188	46954	21	$< \epsilon$	877
		0.975	**12000**	**0.5**	**0.0005**	**1.25**	6	551520	29109	428520	14507	79384	17	1.17%	7200[a]
		0.975	**12000**	**0.5**	**0.005**	**1.25**	39	899027	232517	371040	38255	257215	2	$< \epsilon$	10
		0.975	**12000**	**0.5**	**0.01**	**1.25**	46	924143	275139	317880	42189	288935	2	$< \epsilon$	5
88-nodes	Case base	0.975	7500	0.3	0.001	0.75	16	679622	83508	493500	22820	79794	20	0.97%	7200[a]
	$\bar{\beta}$	**0.75**	7500	0.3	0.001	0.75	18	629774	99573	466500	32360	31341	63	0.79%	7200[a]
		0.99	7500	0.3	0.001	0.75	15	694137	90518	483000	24556	96063	20	1.03%	7200[a]
	μ	0.975	**3000**	0.3	0.001	0.75	9	344449	49007	231600	9963	53879	20	0.68%	7200[a]
		0.975	**12000**	0.3	0.001	0.75	23	972603	142855	682800	38734	108214	19	1.03%	7200[a]
	CV	0.975	7500	**0.1**	0.001	0.75	18	640797	100018	466500	29338	44941	30	1.15%	7200[a]
		0.975	7500	**0.5**	0.001	0.75	13	715544	75666	513000	18821	108057	12	0.73%	7200[a]
		0.975	7500	**1**	0.001	0.75	11	808305	60165	543000	13984	191157	11	0.63%	7200[a]
	c	0.975	7500	0.3	**0.0001**	0.75	3	441576	19822	377550	12129	32075	31	1.07%	7200[a]
		0.975	7500	0.3	**0.0005**	0.75	10	576598	58360	433500	19217	65521	19	0.88%	7200[a]
		0.975	7500	0.3	**0.005**	0.75	49	955426	280407	486000	40201	148818	62	0.01%	7200[a]
		0.975	7500	0.3	**0.01**	0.75	71	1056536	421779	402225	48478	184054	14	$< \epsilon$	211
	h	0.975	7500	0.3	0.001	**0.25**	20	616177	99666	465000	15706	35805	97	0.17%	7200[a]
		0.975	7500	0.3	0.001	**1.25**	13	729018	69277	517500	25250	116991	19	1.25%	7200[a]
	c, h	0.975	7500	0.3	**0.0001**	**1.25**	2	459383	15919	386400	12959	44105	32	1.01%	7200[a]
		0.975	7500	0.3	**0.0005**	**1.25**	9	615272	46106	456000	21331	91835	21	1.08%	7200[a]
		0.975	7500	0.3	**0.005**	**1.25**	44	1048073	253041	536250	46104	212678	31	0.24%	7200[a]
		0.975	7500	0.3	**0.01**	**1.25**	65	1173544	415796	410325	58375	289048	35	$< \epsilon$	1133
	μ, CV, c, h	0.975	**12000**	**0.5**	**0.0001**	**1.25**	3	728138	16294	617040	14969	79835	30	1.06%	7200[a]
		0.975	**12000**	**0.5**	**0.0005**	**1.25**	10	947215	53390	706800	26077	160948	16	0.98%	7200[a]
		0.975	**12000**	**0.5**	**0.005**	**1.25**	53	1493316	306716	738000	59473	389127	39	0.09%	7200[a]
		0.975	**12000**	**0.5**	**0.01**	**1.25**	74	1604897	441261	610560	70767	482309	7	$< \epsilon$	135

[a] Exceeds the time limit.