

Analytic properties of different unitarization schemes

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Abstract. The analytic properties of the eikonal and U -matrix unitarization schemes are examined. It is shown that the basic properties of these schemes are identical. Both can fill the full circle of unitarity, and both can lead to standard and non-standard asymptotic relations for σ_{el}/σ_{tot} . The relation between the phases of the unitarised amplitudes in each scheme is examined, and it is shown that demanding equivalence of the two schemes leads to a bound on the phase in the U -matrix scheme.

1 Introduction

At the LHC, the investigation of diffraction processes will occupy an important place. However, the diffraction processes at very high energies does not simplify asymptotically, but can display complicated features [1,2]. This concerns especially the asymptotic unitarity bound connected with the so-called Black Disk Limit (BDL).

Summation of different sets of diagrams in the tree approximation can lead to different unitarization procedures of the Born scattering amplitude. In the partial-wave language, we need to sum many different waves with $l \rightarrow \infty$ and this leads to the impact parameter representation [3] converting the summation over l into an integration over b .

The different unitarization procedures are then naturally formulated in the impact-parameter representation. As they include different sets of inelastic states in the s channel, this leads to unitarization schemes with different coefficients in front of the multiple exchanges.

In the impact parameter representation, we shall write the Born term of the amplitude as

$$\chi(s, b) = \frac{1}{4\pi} \int_0^\infty e^{i\mathbf{b}\cdot\mathbf{q}} F_{Born}(s, q^2) d^2\mathbf{q}, \quad (1)$$

where we have dropped the kinematical factor $1/\sqrt{s(s-2m_p^2)}$ and a factor s in front of F . For proton-proton scattering, where five helicity amplitudes exist, the non-flip Born term in the near forward direction is $F_{Born}(s, t) = F_{Born}^+(s, t) = F_1(s, t) + F_3(s, t)$. and $\chi(s, b)$ is, in first approximation, given by that non-flip helicity amplitude only. After the unitarization procedure we get

$$\sigma_{tot}(s) = 4\pi \int_0^\infty G(\chi(s, b)) b db. \quad (2)$$

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Two unitarization schemes are more commonly used: one is the eikonal representation where

$$G(s, b) = [1 - e^{-\chi(s,b)}], \quad (3)$$

and the other is the U -matrix representation [4,5] where

$$G(s, b) = U_T(s, b) = 2 \frac{\hat{\chi}(s, b)}{1 + \hat{\chi}}, \quad (4)$$

with $\hat{\chi}(s, b) = \chi(s, b)/2$. Of course, the eikonal representation can be obtained in different ways starting from simple diagrams in the tree approximation. But many additional diagrams exist when inelastic states in the s -channel are taken into account. Other approaches exist, see for example [6,7,8], in which renormalized eikonal representations were obtained. No one really knows which are the leading diagrams and how to sum them. Therefore, all these approaches of the eikonal scheme remain phenomenological.

There are two most important conditions which must be satisfied by any unitarization scheme. Firstly, in the limit of small energies, every unitarization representation must reduce to the same scattering amplitude. Of course, the way this happens can depend on the representation of the Born amplitude in the model. The various unitarization schemes will give different results [9,10] only at high energies, when the number of diagrams and their forms are essentially different. Secondly, the unitarized amplitude cannot exceed the upper unitarity bound. In the different normalizations, this bound may equal to 1 or 2.

At LHC energies we will be in a regime close to the unitarity bound. Hence, it is very important to specify the possible domains of validity of the various unitarization schemes and to determine, from the analysis of the experimental data, which form really corresponds to the physical picture.

2 Properties of the standard U -matrix approach

The standard U matrix was intensively explored in [5], where it was written, in the partial-wave language, as $U_l(t)$ so that it is bounded by $U_l \leq 1$. In the impact parameter representation, the properties of the U matrix were explored in [11], where the U matrix was taken as pure imaginary. We will denote it by $iU_T(s, b)$ and it corresponds to the following definition of the Born amplitude, which varies from 0 to ∞ :

$$\hat{\chi}(s, b) = \frac{1}{2} \Im m(F^{Born}(s, b)). \quad (5)$$

In this case, the relation with the S matrix will be

$$S(s, b) = \frac{1 - \hat{\chi}(s, b)}{1 + \hat{\chi}(s, b)}. \quad (6)$$

We can see that when $\hat{\chi}$ varies in the domain $[0, \infty]$, the S matrix varies in the interval $[-1, +1]$. Hence the scattering amplitude in the impact parameter space $G(s, b) = T(s, b) = 1 - S(s, b)$ will span the domain $[0, 2]$. The upper value is the maximum of the unitarity bound. In this case, the unitarised amplitude can fill the full circle of unitarity, and remains in it for all energies.

This form of unitarization leads to unusual properties at super-high energies as was shown in [11]; with non-standard properties as $s \rightarrow \infty$: $\sigma_{inel}/\sigma_{tot} \rightarrow 0$, $\sigma_{el}/\sigma_{tot} \rightarrow 1$.

In a recent paper [12], it has been claimed that such properties arise from the difference between the rational representation (U matrix) and the exponential representation (eikonal). Let us show that it is not so. We can consider an extended eikonal in the form

$$\sigma_{tot}(s) = 8\pi \int_0^\infty E_R(s, b) b db = 8\pi \int_0^\infty [1 - e^{-\hat{\chi}(s,b)}] b db. \quad (7)$$

Formally, this is the same as the standard form of the eikonal (3), except for the coefficient in front of the overlap function and for the use of $\hat{\chi}(s, b)$ instead of $\chi(s, b)$. At small energy (i.e. small $\hat{\chi}$), we obtain the standard Born amplitude. But at high energy, we obtain the same properties as in the case of the U_T -matrix representation. Indeed, the inelastic cross section can be written

$$\eta(s, b) = \frac{1}{2} \exp(-\hat{\chi}(s, b)) [1 - \exp(-\hat{\chi}(s, b))], \quad (8)$$

which leads to the same result as the standard U_T matrix [11]: $\sigma_{inel}/\sigma_{tot} \rightarrow 0$ and $\sigma_{el}/\sigma_{tot} \rightarrow 1$ as $s \rightarrow \infty$.

Our calculations for the inelasticity corresponding to U_T and E_R are shown in Fig. 1. We see that both solutions have the same behavior in s and b but E_R has sharper anti-shadowing properties. Both solutions lie in the unitarity circle, reach the maximum of the unitarity bound as $s \rightarrow \infty$, and have the same analytic properties asymptotically.

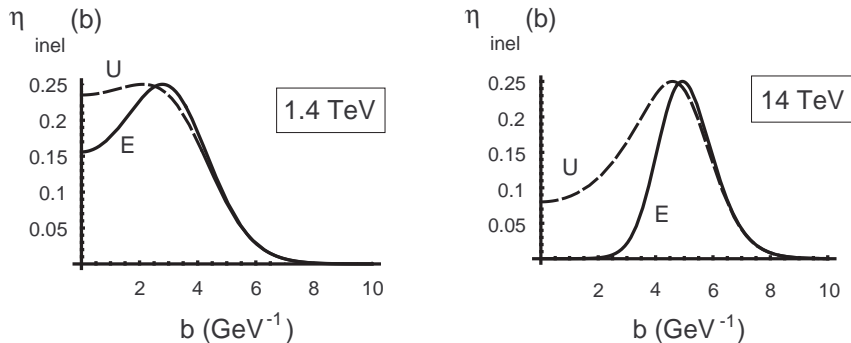


Fig. 1. Antishadowing effects with eikonal (full line) and U_T matrix (dashed line) $\sqrt{s} = 1.4$ TeV (left) and $\sqrt{s} = 14$ TeV (right) [$U_T(s, b) = \chi(\hat{s}, b) = 0.005 s^{0.4} \exp(-b^2/3^2)$]

Hence, non-standard analytic asymptotic properties are not unique to the U_T matrix with its special form (4), and can be reproduced by the exponential form (7) for E_R .

Most importantly, at LHC energies, these forms of unitarization do not have a Black-Disk limit. Indeed, the amplitude does not reach the saturation regime and the basic parameters of the scattering amplitude, such as $\rho(s, t)$ and the slope $B(s, t)$, do not change their behavior (see Fig. 2). It is only at still larger energies, when $\eta_{inel}(s, b)$ goes to zero and the scattering amplitude reaches the unitarity limit, that $\rho(s, t)$ and $B(s, t)$ exhibit properties similar to those in the saturation regime.

3 Properties of the standard eikonal and comparison with the U_T matrix

In order to extend the U matrix, and make it similar to the eikonal, we find it more transparent to work in a different normalisation, where $\sigma_{tot}(s) = 4\pi \text{Im}F(s, t)$. The standard form of eikonal does not change, but we take an extended U -matrix unitarization U_e with an additional coefficient $1/2$ in the denominator, as

$$\sigma_{tot}(s) = 4\pi \int_0^\infty b \left[\frac{\chi(s, b)}{1 + \chi(s, b)} \right] db. \quad (9)$$

This form of the U matrix then satisfies all the analytical properties of the standard eikonal representation: the inelastic overlap function is

$$\eta(s, b) = \frac{1}{2} \frac{\chi(s, b) + 2\chi^2(s, b)}{[1 + \chi(s, b)]^2},$$

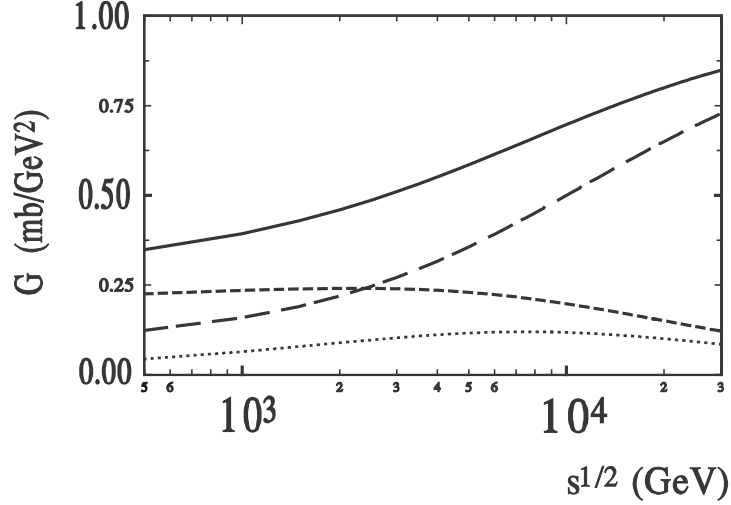


Fig. 2. The energy dependence of the amplitudes from $U_T(s, b=0)$ for a model with a soft and a hard pomerons (the full and dotted lines are the imaginary part and the modulus of the real part, the long dashed line gives $G_{el}(s, b=0)$, and the dotted line is $G_{inel}(s, b=0)$).

and it can easily be seen that in this case, when $s \rightarrow \infty$, we obtain: $\sigma_{inel} \rightarrow \sigma_{el}$ and $\sigma_{el}/\sigma_{tot} \rightarrow 1/2$. Differently stated, this U_e -matrix representation has the standard BDL. In Figs. 3, $\eta(s, b)_{inel}$ is shown for these cases.

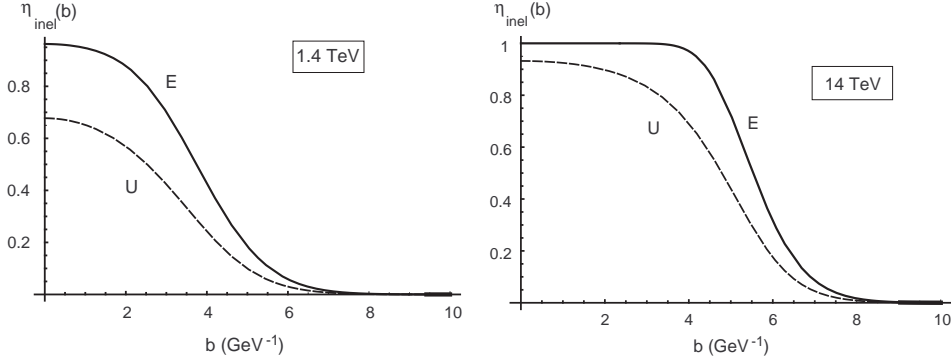


Fig. 3. Inelastic overlap function with eikonal (plane line) and U matrix (dashed line) unitarization, at $\sqrt{s} = 1.4$ TeV (left) and $\sqrt{s} = 14$ TeV (right), for a Born amplitude $\chi(s, b) = 0.005 s^{0.4} \exp(-b^2/3^2)$.

Now let us compare directly the different forms of unitarization schemes. In Ref. [4], the eikonal and U -matrix phases are compared from Eqs. (3) and (4). Assuming that the two unitarised amplitudes are equal, one obtains

$$\chi_u(s, b) = 2 \tanh \left[\frac{1}{2} \chi_e(s, b) \right]. \quad (10)$$

It is more convenient to analyze the inverse relation

$$\chi_e(s, b) = \log \left[\frac{1 + \chi_u(s, b)/2}{1 - \chi_u(s, b)/2} \right]. \quad (11)$$

Both dependencies are shown in Fig. 4. At low energy, we have approximately the same size for the two phases. But at high energy, when $\chi_e(s, b)$ goes to infinity, $\chi_u(s, b) \rightarrow 2$. Hence the energy dependences of these phases are very different.

At asymptotic energy, if $\chi_u(s, b)$ grows like the standard Born amplitude, $\chi_e(s, b)$ will tend to $i\pi$, as can be seen from Eq. (10). So the high-energy limit of U -matrix unitarisation corresponds to the low-energy of the eikonal amplitude. Indeed, for the standard eikonal at low energy we obtain the saturation of the maximum unitarity bound = 2 in our normalization (see Fig. 5a) in the case of a purely real scattering amplitude. Larger values of the real part of the scattering amplitude lead to a faster decrease of this bound with some oscillations (see Fig. 5b).

If we compare the phases of the extended and standard eikonal unitarization schemes, we obtain a similar result: the phase of the extended eikonal will also be bounded

$$\chi_{re}(s, b) = -\log \left[\frac{1}{2} (1 + e^{-\chi_e(s, b)}) \right]. \quad (12)$$

Hence at high energies, when $\chi(s, b) \rightarrow \infty$, $\chi_e(s, b)$ will be bounded by 0.693.

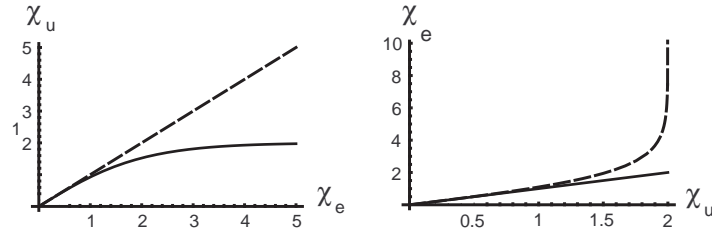


Fig. 4. The correlation of $\chi_e(s, b)$ and $\chi_u(s, b)$ obtained from the eqs. (3) and (4) (dashed line is $\chi_u(s, b)$, plane line is $\chi_e(s, b)$)

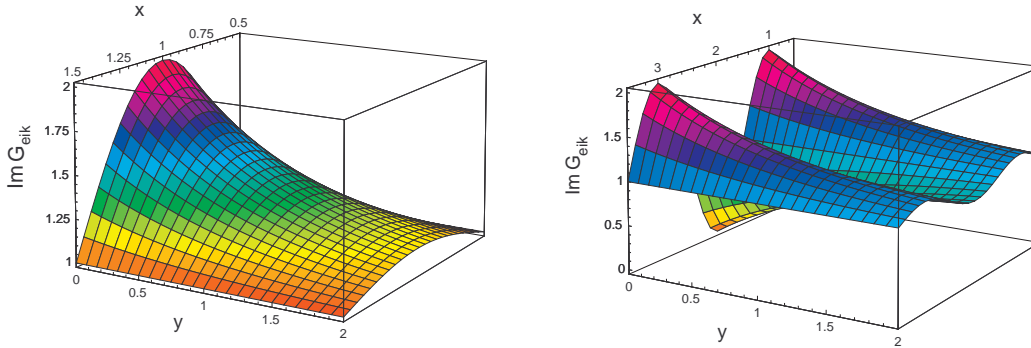


Fig. 5. The imaginary part of the unitarised eikonal in the region of small (a: left panel) and large (b: right panel) real part (x axis) of the scattering amplitude.

However, if we take the U matrix in the form (9), the relation between phases will be

$$\chi_e(s, b) = \log [1 + \chi_u(s, b)] \quad (13)$$

whose inverse relation is

$$\chi_u(s, b) = e^{\chi_e(s, b)} \left[1 - e^{\chi_e(s, b)} \right] \quad (14)$$

In this case both phases can vary from zero to infinity.

4 Conclusion

From the above analysis we are led to conclude that the unusual properties of the U_T -matrix unitarization are not connected with its specific form as a ratio of polynomials. The exponential form can have the same properties, including the antishadowing regime and the unusual asymptotic ratio of σ_{el} to σ_{tot} . These unitarizations do not have the BDL at high energies, and do not change significantly the behavior of the scattering amplitude at LHC energies, but they lead to a faster growth of σ_{tot} . Conversely, we have shown that an extended U -matrix unitarization has the same properties as the standard eikonal unitarization.

The analysis of the relation between the phases of the standard eikonal and of the U_T matrix shows that if the phase of the U_T matrix unitarization is bounded, both unitarization schemes will have the same properties. This would imply either that these unitarizations are valid only at low energies, or that their Born term must profoundly change at high energies and be bounded.

Up to now, we have not found any decisive argument to forbid unitarization approaches like the U_T matrix. But the predictions for the total cross sections in these two approaches of the unitarization have large differences (see, for example [10,13]) for the LHC energy region, so that we hope that future experimental data will give us the true answer.

A note on the real part

In a recent comment [14], Troshin has pointed out that the extended eikonal which we propose here unitarises only purely-imaginary amplitudes. We indeed considered here the eikonalised amplitude

$$f(s, b) = iG(s, b) = ik[1 - e^{i(\phi(s, b)/k)}], \quad (15)$$

with $k = 2$, and $\Im m\phi = \chi$. As is well known [8], this maps the complex half plane $\Im m\phi \geq 0$ to twice the unitarity circle, and maps $\phi = i\infty$ to $2i$. So it cannot be considered a generic unitarisation scheme.

However, we are not interested here to map the whole half-plane to the circle: indeed, we only want a scheme that unitarises at high energy. We see no reason why an arbitrary amplitude should always be unitarised: surely, the multiple exchanges depend on the underlying theory, and a scheme that would work in one theory has no reason to work in another. Hence, we are interested in unitarising high-energy hadronic amplitudes. These are dominated by their imaginary part at high energy, so that the scheme we propose here produces very small deviations from unitarity at high energy for amplitudes which fit the lower-energy data. The unitarity condition $|f - i|^2 \leq 1$ can indeed be rewritten

$$(k - 2) + k\rho^2 \leq 2(k - 1)\rho \cos(\Re e\phi/k) \quad (16)$$

with $\rho = e^{-\Im m\phi/k}$. When one includes a real part in the Born term, we see that eventually $k = 2$ will lead to a contradiction as the cosine term goes to zero. However, because the latter is suppressed by ρ , we also see that values of k very close to 2 will lead to a proper solution. In the case of the 2-pomeron model of [10], we find that values $k = 1.95$ do not lead to a violation of unitarity at high energy, and keep the properties of antishadowing described in this paper.

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