Maximum entanglement and absolute separability of symmetric states under unitary transformations

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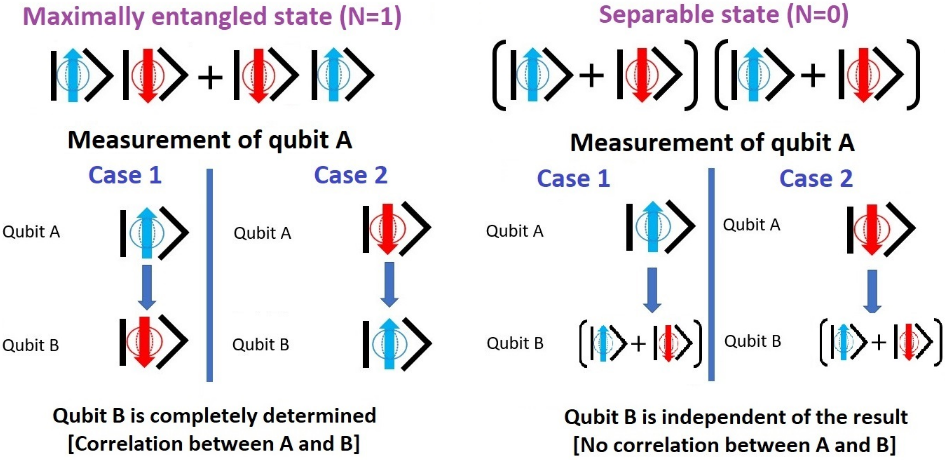
Outline of the talk

1. Statement of the problem
   a. Concepts
   b. Qubit-qubit system
   c. Symmetric case
2. Results
   a. Symmetric 2-qubit system (Main results)
   b. Symmetric 3-qubit system (Numerical results)
   c. SAS witnesses for symmetric N-qubit systems
3. Conclusions

Entanglement

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$



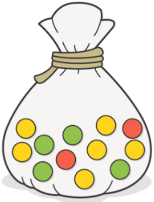
Entanglement of mixed states

Separable states

ρ is separable if

$$\rho = \int_{\mathcal{H}_1^{\otimes 2}} P(\boldsymbol{n}_1\,,\boldsymbol{n}_2)|\boldsymbol{n}_1\rangle|\boldsymbol{n}_2\rangle\langle\boldsymbol{n}_1|\langle\boldsymbol{n}_2|\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2\,.$$

with $P(\boldsymbol{n}_1, \boldsymbol{n}_2) \geq 0$. Otherwise is entangled.



Negativity: Measure of entanglement for qubit-qubit and qubit-qutrit, [Peres (1996)], [Horodecki et al (1996)]

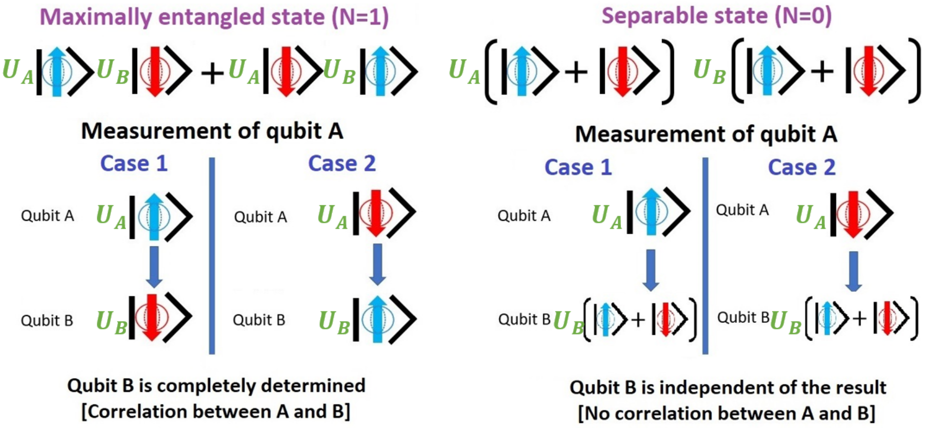
$\mathcal{N}$ is defined in terms of the negative eigenvalues Λ_k of ρ^{T_A}

$$\mathcal{N}(\rho) = -2\sum_{\Lambda_k<0}\Lambda_k\,,$$

- $\mathcal{N}(\rho_{sep}) = 0$.
- Invariant under local unitary transformations.

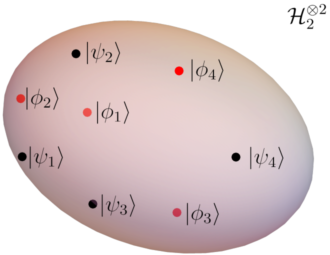
Entanglement

Invariant under local unitary transformations $U_A \otimes U_B \in SU(2) \otimes SU(2)$



Entanglement (Pure state case)

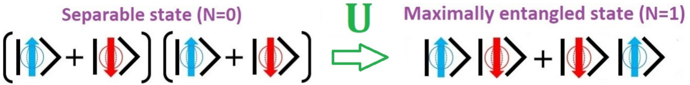
Not-invariant under **global** unitary transformations $SU(4)$



Global unitary transformation

$$\rho = \sum_{k=1}^{4} \lambda_k |\phi_k\rangle\langle\phi_k| \,,$$

$$U\rho U^\dagger = \sum_{k=1}^{4} \lambda_k |\psi_k\rangle\langle\psi_k| \,,$$

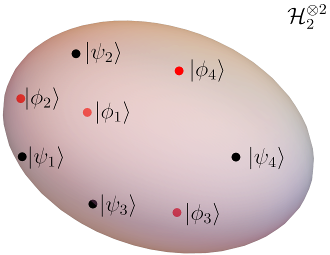


Pure state ρ_{pure}

$$\lambda_1 = 1 \,, \lambda_2 = \lambda_3 = \lambda_4 = 0 \,, \qquad \max_{U \in SU(4)} \mathcal{N}(U\rho_{pure}U^\dagger) = 1 \,,$$

Entanglement (Maximally mixed state case)

Not-invariant under **global** unitary transformations $SU(4)$



Global unitary transformation

$$\rho = \sum_{k=1}^{4} \lambda_k |\phi_k\rangle\langle\phi_k| \,,$$

$$U\rho U^\dagger = \sum_{k=1}^{4} \lambda_k |\psi_k\rangle\langle\psi_k| \,,$$

$$\rho_{MMS} = U\rho_{MMS}U^\dagger = \frac{1}{4}\mathbb{1} = \frac{1}{4}\int_{S^2\otimes S^2} |\boldsymbol{n}_1\rangle|\boldsymbol{n}_2\rangle\langle\boldsymbol{n}_1|\langle\boldsymbol{n}_2| \, \mathrm{d}^2\boldsymbol{n}_1 \, \mathrm{d}^2\boldsymbol{n}_2 \,.$$

MMS state ρ_{MMS}

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/4 \,, \qquad \max_{U\in SU(4)} \mathcal{N}(U\rho_{MMS}U^\dagger) = 0 \,,$$

Maximum entanglement in the unitary orbit of ρ

Question

For a state ρ with eigenspectrum $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$, what is the maximum entanglement attained in its $SU(4)$-orbit?

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Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$

$$\rho-\text{spectrum:} \quad (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

$$\max_{U \in SU(4)} \mathcal{N}\left(U\rho U^\dagger\right) = \max\left(0, \sqrt{(\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_4)^2} - \lambda_2 - \lambda_4\right),$$

Solved by Verstraete, Audenart & De Moor (2001).

Qubit-qutrit system $\mathcal{H}_2 \otimes \mathcal{H}_3$

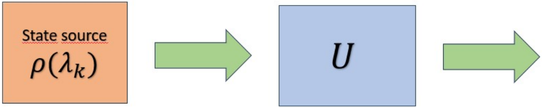
$$\rho-\text{spectrum:} \quad (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$$

$$\max_{U \in SU(6)} \mathcal{N}\left(U\rho U^\dagger\right)$$

Open question. Only partial results (Mendonça, Marchiolli, Herdemann (2017))

Physical implications

Unitary quantum gates as free operations



$\hat{H}$

Thermal state

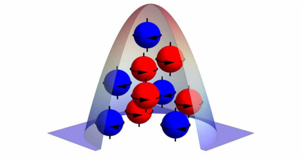
$$\max_{U \,\epsilon\, SU(d)} N(U\rho U^{\dagger})$$

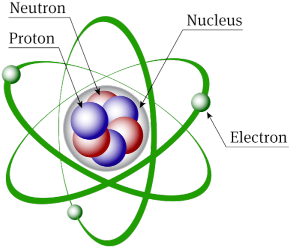
$N(U\rho U^{\dagger}) = 0$ for any U ?

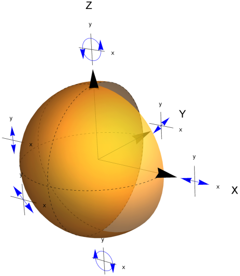
$N(U\rho U^{\dagger}) \neq 0$ for some U?

Statement of the problem

Bosons: BEC, spin-j system, multiphotons systems, etc.







New question

For a symmetric qubit-qubit state ρ_S, what is the maximum entanglement achievable under a global unitary transformation U_S restricted in the symmetric subspace ?

Symmetric bipartite systems

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$	Symmetric 2-qubit system $\mathcal{H}_2^{\vee 2}$
$\rho-$spectrum: $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	ρ_S-spectrum: $(\tau_1, \tau_2, \tau_3, 0)$
$\max_{U \in SU(4)} \mathcal{N}\left(U\rho U^\dagger\right)$	$\max_{U_S \in SU(3)} \mathcal{N}\left(U_S\rho_S U_S^\dagger\right)$
Qubit-qutrit system $\mathcal{H}_2 \otimes \mathcal{H}_3$	**Symmetric 3-qubit system $\mathcal{H}_2^{\vee 3}$**
$\rho-$spectrum: $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$	ρ_S-spectrum: $(\tau_1, \tau_2, \tau_3, \tau_4, 0, 0)$
$\max_{U \in SU(6)} \mathcal{N}\left(U\rho U^\dagger\right)$	$\max_{U_S \in SU(4)} \mathcal{N}\left(U_S\rho_S U_S^\dagger\right)$

Symmetric 2-qubit system

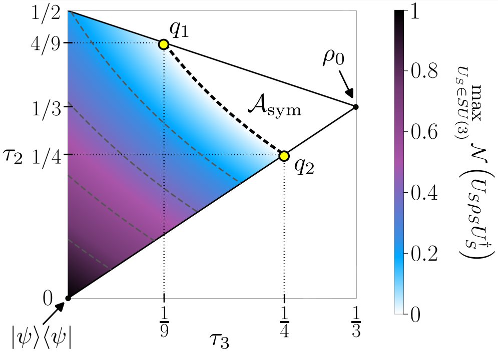
Symmetric qubit-qubit system

Main result

Theorem [ESE, Martin (2023)]

Let $\rho_S \in \mathcal{B}(\mathcal{H}_2^{\vee 2})$ with spectrum $\tau_1 \geq \tau_2 \geq \tau_3$. It holds that

$$\max_{U_S \in SU(3)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right) = \max\left(0, \sqrt{\tau_1^2 + (\tau_2 - \tau_3)^2} - \tau_2 - \tau_3\right).$$



Maximally entangled state

$$\rho_S = \begin{pmatrix} \tau_3 & 0 & 0 \\ 0 & \tau_1 & 0 \\ 0 & 0 & \tau_2 \end{pmatrix}$$

$$\tau_1 = 1 - \tau_2 - \tau_3$$

$\mathcal{A}$
Absolutely separable (AS) states [Życzkowski (1999)]

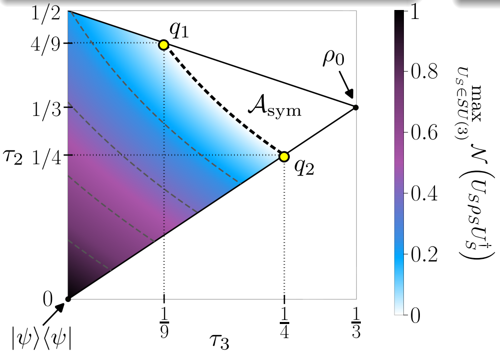
$$\max_{U\in SU(4)} \mathcal{N}(U\rho U^\dagger) = 0$$

$$\mathcal{A}(\mathcal{H}_2^{\vee 2}) = \{\rho_0\}$$

$\mathcal{A}_{\text{sym}}$
Symmetric absolutely separable (SAS) states [Giraud et al (2008)]

$$\max_{U\in SU(3)} \mathcal{N}(U_S\rho_S U_S^\dagger) = 0$$

$$\mathrm{d}(\mathcal{A}_{\text{sym}}(\mathcal{H}_2^{\vee 2})) = 2$$



Corollary [ESE, Martin (2023)]

$\rho_S \in \mathcal{A}_{\text{sym}}$ iff

$$\sqrt{\tau_2} + \sqrt{\tau_3} \geq 1\,.$$

- $\mathrm{Tr}(\rho_S^2) \leqslant \frac{3}{8} \Rightarrow \rho_S \in \mathcal{A}_{\text{sym}}$
- $\mathrm{Tr}(\rho_S^2) > \frac{11}{27} \Rightarrow \rho_S \notin \mathcal{A}_{\text{sym}}$

Applications

Symmetric qubit-qubit system at finite temperature

Hamiltonian: BEC [Ribeiro, Vidal, Mosseri (2007)], LMG model (1965)]

$$H = \gamma_x J_x^2 + \gamma_z J_z^2$$

Energy spectrum ϵ_j,

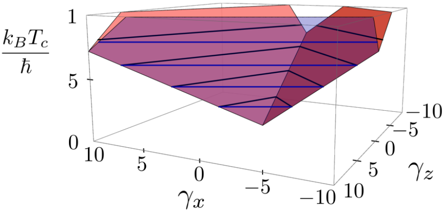
$$\left\{\gamma_x \, , \gamma_z \, , \gamma_x + \gamma_z\right\}$$

State at finite temperature T

$$\tau_k = \frac{e^{-\epsilon_{2s+2-k}/k_B T}}{Z}$$

Condition of SAS states

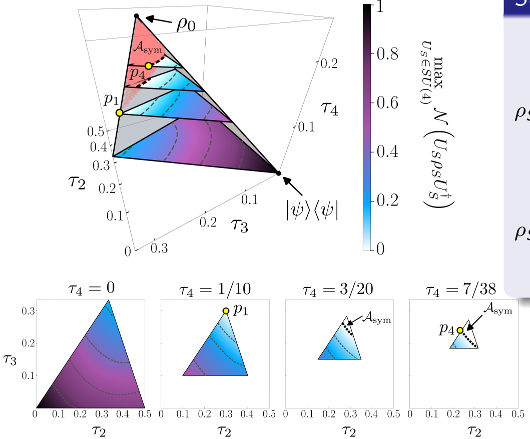
$$\rho \in \mathcal{A}_{\text{sym}} \Leftrightarrow T \geqslant \frac{\epsilon_1 + \epsilon_2 - 2\epsilon_3}{2k_B \ln 2} \, ,$$



Symmetric 3-qubit system

Symmetric 3-qubit system

$\mathcal{H}_2^{\vee 3} \subset \mathcal{H}_2 \otimes \mathcal{H}_3$, numerical results



Maximally entangled states in the $SU(4)$-orbit

$$\rho_S = \begin{pmatrix} \tau_4 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_2 \end{pmatrix},$$

$$\rho_S = \begin{pmatrix} \frac{\tau_1+\tau_4}{2} & 0 & 0 & \frac{\tau_1-\tau_4}{2} \\ 0 & \frac{\tau_2+\tau_3}{2} & \frac{\tau_2-\tau_3}{2} & 0 \\ 0 & \frac{\tau_2-\tau_3}{2} & \frac{\tau_2+\tau_3}{2} & 0 \\ \frac{\tau_1-\tau_4}{2} & 0 & 0 & \frac{\tau_1+\tau_4}{2} \end{pmatrix}$$

Symmetric 3-qubit system

Conjecture 1

$$\max_{U_S \in SU(4)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right) = \max_{\pi,\,\alpha_1,\,\alpha_2,\,\alpha_3} \mathcal{N}\left(\tilde{U}_S \rho_S^\pi \tilde{U}_S^\dagger\right),$$

$$\tilde{U}_S = \begin{pmatrix} 0 & -n_{1x}n_{2y} & n_{1y}n_{3x} - n_{1x}n_{2x}n_{3y} & -n_{1x}n_{2x}n_{3x} - n_{1y}n_{3y} \\ n_{1x} & n_{1y}n_{2x} & -n_{1y}n_{2y}n_{3y} & -n_{1y}n_{2y}n_{3x} \\ 0 & n_{1y}n_{2y} & n_{1x}n_{3x} + n_{1y}n_{2x}n_{3y} & n_{1y}n_{2x}n_{3x} - n_{1x}n_{3y} \\ n_{1y} & -n_{1x}n_{2x} & n_{1x}n_{2y}n_{3y} & n_{1x}n_{2y}n_{3x} \end{pmatrix},$$

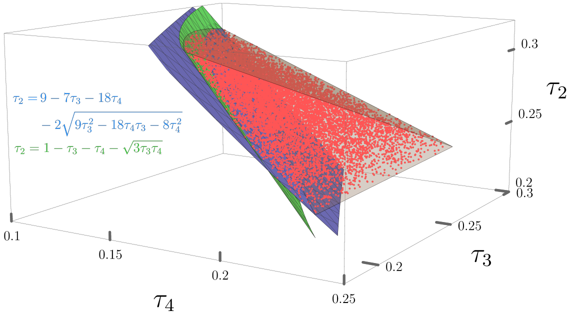
Conjecture 2

- $\mathrm{Tr}(\rho_S^2) \leqslant \frac{5}{19} \Rightarrow \rho_S \in \mathcal{A}_{\mathrm{sym}}$
- $\mathrm{Tr}(\rho_S^2) > \frac{7}{25} \Rightarrow \rho_S \notin \mathcal{A}_{\mathrm{sym}}$

Observation 1 (Almost tight)

If $\rho_S \in \mathcal{A}_{\text{sym}}$, then

$$\tau_1 > \sqrt{3\,\tau_3\tau_4} \quad \text{and} \quad (3\tau_1 - 2\tau_2)^2\tau_3 + 3(\tau_2^2 - \tau_3^2)\tau_4 > 9\tau_3\tau_4^2\,.$$



SAS witnesses for symmetric N-qubit states

SAS witnesses for symmetric N-qubit states

SAS-witness $\mathcal{W}$ [Bohnet-Waldraff, Giraud, Braun (2017)]

$$\rho \in \mathcal{A}_{\text{sym}} \quad \text{if} \quad \mathrm{Tr}(\rho^2) \leqslant \frac{1}{N+1}\left(1 + \frac{1}{2(2N+1)\binom{2N}{N} - (N+2)}\right),$$

SAS states

Let $\rho \in \mathcal{B}(\mathcal{H}_2^{\vee N})$, $\rho \in \mathcal{A}_{\text{sym}} \Leftrightarrow$ there exists $P(U\rho U^\dagger; \boldsymbol{n})$ such that

$$U\rho U^\dagger = \int_{S^2} P(U\rho U^\dagger; \boldsymbol{n}) |\boldsymbol{n}\rangle^{\otimes n}\langle \boldsymbol{n}|^{\otimes n} \, \mathrm{d}^2\boldsymbol{n},$$

and

$$\min_{\substack{U \in SU(N+1) \\ \boldsymbol{n} \in S^2}} P(U\rho U^\dagger; \boldsymbol{n}) \geqslant 0,$$

SAS witnesses for symmetric N-qubit states

Non-uniqueness of the P-function [Giraud, Braun, Braun (2008)]

$$P(\rho\,,\boldsymbol{n}) = \underbrace{\sum_{\sigma=0}^{N}\sum_{\mu=-\sigma}^{\sigma} a_{\sigma\mu}\,Y_{\sigma\mu}(\boldsymbol{n})}_{P_0\ ,\ \text{unique for }\rho} + \underbrace{\sum_{\sigma=N+1}^{\infty}\sum_{\mu=-\sigma}^{\sigma} a_{\sigma\mu}\,Y_{\sigma\mu}(\boldsymbol{n})}_{P'\ ,\ \text{arbitrary } a_{\sigma\mu}}\,,$$

Proposal

To consider $P(\rho,\boldsymbol{n})$ such that

i) They are covariant

$$P(U\rho U^\dagger,\boldsymbol{n}) = P(D(\mathrm{R})^\dagger\,U\rho U^\dagger\,D(\mathrm{R}),\boldsymbol{z}) = P(V\rho V^\dagger,\boldsymbol{z})\,.$$

ii) We built $P(U\rho U^\dagger,\boldsymbol{n})$ that their explicit expressions depend only on (or can be approximated) the unistochastic matrices

$$B_{ij} = |U_{ij}|^2\,,$$

SAS witnesses for symmetric N-qubit states

The two simplest cases

i) $P = P_0$ [Denis, Davis, Mann, Martin (2023)]

Observation 1

$$\min_{\substack{U\in SU(N+1)\\ \boldsymbol{n}\in S^2}} P_0(U\rho U^\dagger;\boldsymbol{n}) = \min_{V\in SU(4)} \mathrm{Tr}\left[\rho V D V^\dagger\right]$$

$$\left(\rho_{jk} = \tau_j \delta_{jk}\,, D_{jk} = \sigma_j(\alpha)\delta_{jk}\right) = \min_{B\in\mathcal{U}} \boldsymbol{\tau}^T B \boldsymbol{\Delta}\,,$$

B a unistochastic matrix, $B \in \mathcal{U} \subset \mathcal{B}$.

Observation 2 (Birkhoff's Theorem)

Permutations matrices achieve extremal values of a convex function $f(B)$

$$\min_{B\in\mathcal{U}} \boldsymbol{\tau}^T B \boldsymbol{\Delta} = \min_{\Pi\in S_3} \boldsymbol{\tau}^T B \boldsymbol{\Delta}\,,$$

SAS-witness $\mathcal{W}_1$ (Improvement of the SAS-witness $\mathcal{W}$)

$$\rho \in \mathcal{A}_{\text{sym}} \qquad \text{if} \qquad \boldsymbol{\tau}^{\uparrow T}\boldsymbol{\Delta}\downarrow \geqslant 0\,, \qquad \Delta_k = (-1)^{N-k}\binom{N+1}{k}\,,$$

ii) $P = P_0 + P'(\rho^{\otimes 2}, \boldsymbol{n})$

Observation 1

$$\min_{\substack{U\in SU(N+1)\\ \boldsymbol{n}\in S^2}} P(U\rho U^\dagger; \boldsymbol{n}) \geqslant \min_{B\in\mathcal{U}} \boldsymbol{\tau}^T B\boldsymbol{\Delta} + \sum_{L=1}^{N} a_L(\boldsymbol{\tau}^T B\boldsymbol{t}_L)^2 \,,$$

B a unistochastic matrix, $B \in \mathcal{U} \subset \mathcal{B}$.

Observation 2 (~~Birkhoff's Theorem~~ Minimize over the bistochastic matrices)

$$\min_{B\in\mathcal{U}} \boldsymbol{\tau}^T B\boldsymbol{\Delta} + \sum_{L=1}^{N} a_L(\boldsymbol{\tau}^T B\boldsymbol{t}_L)^2 \geqslant \min_{B\in\mathcal{B}} \boldsymbol{\tau}^T B\boldsymbol{\Delta} + \sum_{L=1}^{N} a_L(\boldsymbol{\tau}^T B\boldsymbol{t}_L)^2 \,,$$

SAS-witness $\mathcal{W}_2$ (Improvement of the SAS-witness $\mathcal{W}$)

$$\rho \in \mathcal{A}_{\mathrm{sym}} \qquad \text{if} \qquad \mathrm{Tr}(\rho^2) \leqslant \frac{1}{N+1}\left(1 - \frac{1}{A_0(N+1)\sum_{k=1}^{N}\frac{g_k^2}{A_k}}\right) .$$

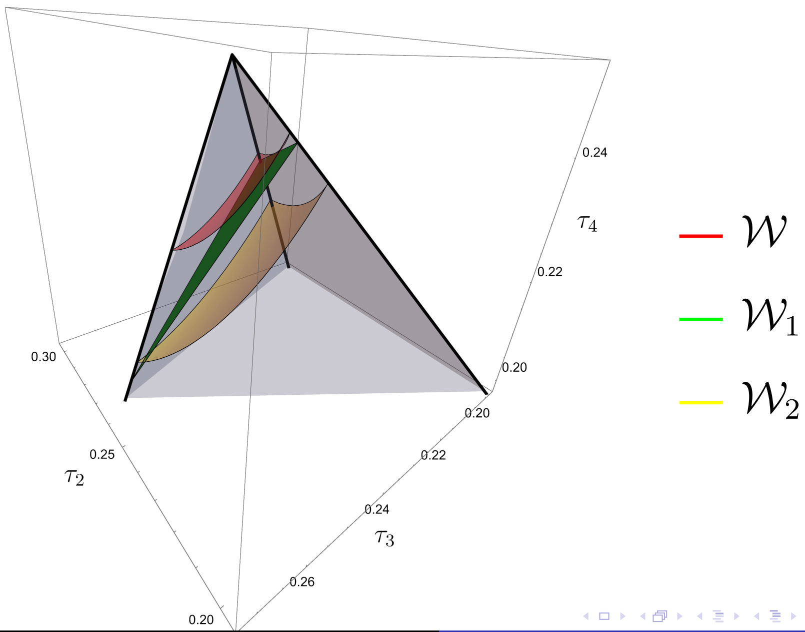
SAS witnesses for symmetric N-qubit states

Work on progress [ESE, Denis, Martin]

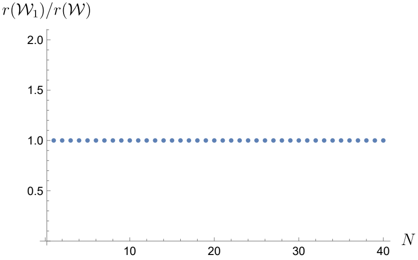
$\mathcal{W}_1$ and $\mathcal{W}_2$

$\rho \in \mathcal{A}_{\text{sym}}$ if $\tau_1 \geqslant \tau_2 \cdots \geqslant \tau_{N+1}$ fulfills

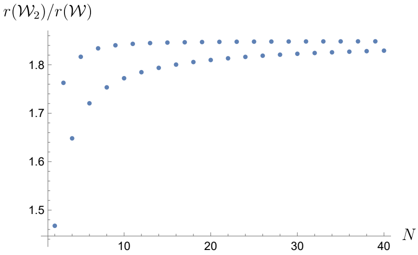
$$
\begin{aligned}
& N = 2\,, \\
& \qquad (\tau_1, \tau_2, \tau_3)^T(-3, 1, 3) \geqslant 0\,, \text{ or } \mathrm{Tr}(\rho^2) \leqslant \frac{9}{26} \\
& N = 3\,, \\
& \qquad (\tau_1, \tau_2, \tau_3, \tau_4)^T(-6, -1, 4, 4) \geqslant 0\,, \text{ or } \mathrm{Tr}(\rho^2) \leqslant \frac{179}{208} \\
& N = 4\,, \\
& \qquad (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)^T(-10, -5, 1, 5, 10) \geqslant 0\,, \text{ or } \mathrm{Tr}(\rho^2) \leqslant \frac{5089}{25390}\,, \\
& \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{aligned}
$$


0.24
0.22
0.20
τ_4
0.30
0.25
0.20
τ_2
0.20
0.22
0.24
0.26
τ_3
$\mathcal{W}$
$\mathcal{W}_1$
$\mathcal{W}_2$

SAS-witness

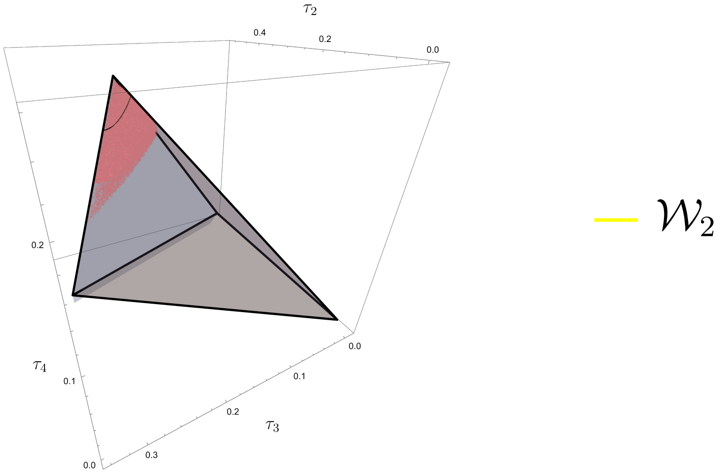


$r(\mathcal{W}_1)/r(\mathcal{W})$
2.0
1.5
1.0
0.5
10
20
30
40
N



$r(\mathcal{W}_2)/r(\mathcal{W})$
1.8
1.7
1.6
1.5
10
20
30
40
N

SAS-witness



Future work

- Exhaustive study of the SAS witnesses
- Extend to extra terms of higher order $P'(\rho^{\otimes k}, \boldsymbol{n})$

Conclusions

Maximum entanglement (negativity) over the unitary orbit Symmetric N-qubit system $\mathcal{H}_2^{\vee N}$ for $N = 2, 3$
ESE and Martin (2023)
SAS witnesses in symmetric N-qubit states
ESE, Denis and Martin (Coming soon)

Thank you very much for your attention!