Spectral Theory of Non-Markovian Dissipative Phase Transitions

B. Debecker, J. Martin and F. Damanet

Institut de Physique Nucléaire, Atomique et de Spectroscopie, CESAM, Université de Liège, 4000 Liège, Belgium

Motivations

- Dissipative Phase Transitions (DPTs) are mostly studied under the Markovian approximation.
- However, recent studies [1, 2] suggest that a Lindblad generator cannot always capture DPTs.
- Here, we propose a general theory to describe DPTs in non-Markovian systems.

Theoretical framework: HEOM

We consider a system S linearly coupled to a bosonic bath B at 0 K. The total Hamiltonian reads ($\hbar = 1$)

$$\begin{aligned} H &= H_S + H_B + H_{\text{int}} \\ &= H_S + \sum_k \omega_k a_k^\dagger a_k + \sum_k (g_k a_k L_k^\dagger + g_k^* a_k^\dagger L_k). \end{aligned}$$

Assuming $\rho(0) = \rho_S(0) \otimes \rho_B(0)$ and

$$\alpha(\tau) = \sum_k |g_k|^2 \langle a_k(t) a_k^\dagger(t+\tau)\rangle = \sum_{j=1}^{M} G_j e^{-i\omega_j \tau - \kappa_j|\tau|} = \int_0^{+\infty} e^{i\omega\tau} J(\omega) d\omega,$$

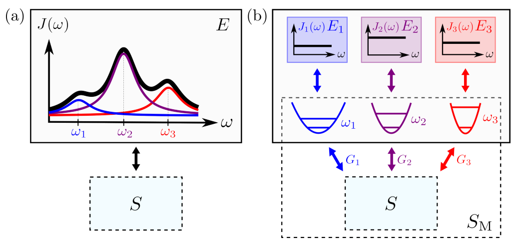
the dynamics of the system can be described by an exact method: the Hierarchical Equations Of Motion (HEOM)[3]

$$\begin{aligned} \frac{d\rho^{(\vec{n},\vec{m})}}{dt} &= -i[H_S, \rho^{(\vec{n},\vec{m})}] - (\vec{w}\cdot\vec{n} + \vec{w}^*\cdot\vec{m})\rho^{(\vec{n},\vec{m})} \\ &+ \sum_{j=1}^{M} \Big(G_j \left(n_j L_j \rho^{(\vec{n}-\vec{e_j},\vec{m})} + m_j \rho^{(\vec{n},\vec{m}-\vec{e_j})} L_j^\dagger \right) \\ &+ [\rho^{(\vec{n}+\vec{e_k},\vec{m})}, L_j^\dagger] + [L_j, \rho^{(\vec{n},\vec{m}+\vec{e_j})}] \Big), \end{aligned}$$

with $\vec{w} = (\kappa_j + i\omega_j) \in \mathbb{C}^M$, $\vec{n} = (n_j), \vec{m} = (m_j) \in \mathbb{N}^M$, $\rho^{(\vec{0},\vec{0})}$ the physical state.

$$\iff \frac{d|\rho\rangle\rangle}{dt} = \mathcal{L}_{\text{HEOM}}(k_{\max})|\rho\rangle\rangle,$$

where $|\rho\rangle\rangle$ is a stacked vector made of all the vectorized $\rho^{(n,m)} \to |\rho^{(\vec{n},\vec{m})}\rangle\rangle$ and $k_{\max}$ is s.t. $|\vec{n}| + |\vec{m}| \leq k_{\max}$. We focus on the spectral properties of the *HEOM Liouvillian* $\mathcal{L}_{\text{HEOM}}$.



Spectral properties of $\mathcal{L}_{\text{HEOM}}$ & DPTs

Properties

- The eigenvalues λ_i are symmetric w.r.t the real axis.
- The eigenvalue $\lambda = 0$ is always in the spectrum of $\mathcal{L}_{\text{HEOM}}$.
- For $k_{\max} \to +\infty$, $\text{Re}[\lambda_i] \leq 0 \ \forall \lambda_i$.

Definition of DPTs

A DPT of order M is a non-analytical change in a g-independent system observable O when the parameter g tends to a critical value g_c for $N \to \infty$ [4], *i.e.*

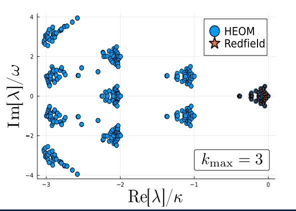
$$\lim_{g\to g_c} \left| \frac{\partial^M}{\partial g^M} \lim_{N\to+\infty} \langle O\rangle_{ss} \right| = +\infty.$$

Example

$H_S = \omega_0 S_z, \ L = 2S_x$

$\alpha(\tau) = G e^{(i\omega - \kappa)\tau}$

$(\omega_0 = 0.1\kappa, \ \omega = \kappa, \ G = 0.1\kappa)$



Symmetries

A weak symmetry is any unitary operator $\mathcal{U}$ *s.t.* $[\mathcal{L}_{\text{HEOM}}, \mathcal{U}] = 0$. The matrix reprenting $\mathcal{L}_{\text{HEOM}}$ is then block-diagonal

$$\mathcal{L}_{\text{HEOM}} = \bigoplus_{u_k} \mathcal{L}_{u_k},$$

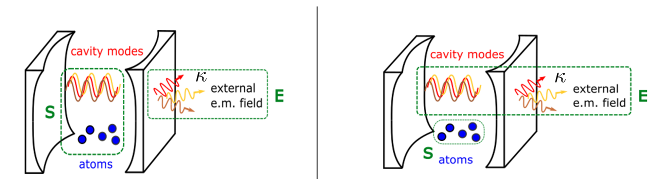
in the eigenvector basis of $\mathcal{U}$. In each block $\mathcal{L}_{u_k}$, we sort the eigenvalues by their real part $|\text{Re}[\lambda_0^{(k)}]| < |\text{Re}[\lambda_1^{(k)}]| < \dots$.

The unique steady state is always associated with the block $\mathcal{L}_{u_{k=0}}$ with $u_{k=0} = 1$. We have the implication $[(i,k) \neq (0,0)]$

$$\text{DPT} \implies \Delta \equiv \min(|\text{Re}[\lambda_i^{(k)}]|) \to 0 \text{ as } N \to +\infty.$$

Non-Markovian DPTs

Equivalent descriptions of atoms in a cavity



→ Lindblad master equation (system = atoms+cavity)

→ Non-Markovian description (system = atoms)

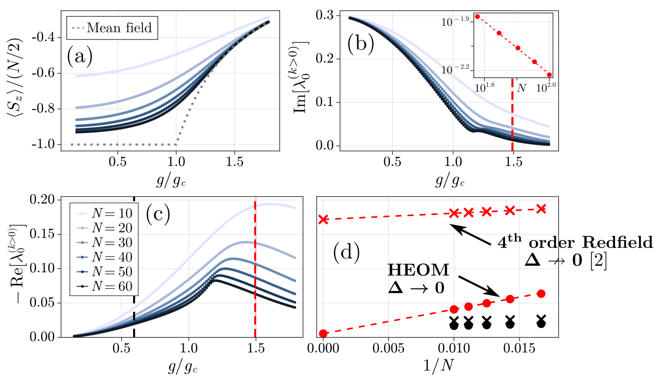
DPT in a U(1)-symmetric Dicke model [2]

Standard Markovian description

$$\begin{aligned} H &= \omega_0 S_z + \omega_A a^\dagger a + \omega_B b^\dagger b + \frac{g}{\sqrt{N}}(aS_+ + bS_- + \text{h.c.}), \\ \dot{\rho}_{\text{tot}} &= -i[H, \rho_{\text{tot}}] + \kappa(\mathcal{D}[a] + \mathcal{D}[b]). \end{aligned}$$

Equivalent non-Markovian description

$$H_S = \omega_0 S_z, \ \alpha(\tau) = \frac{g^2}{N}(e^{(i\omega_A - \kappa)\tau} + e^{(i\omega_B - \kappa)\tau}), \ L = S_-.$$



Parameters : $\omega_A = \omega_B = \kappa = 5\omega_0$.

Conclusion and outlook

Conclusion

We showed that a HEOM-based approach of DPTs successfully captures DPTs in non-Markovian quantum systems.

Outlook

- Study in general the impact of non-Markovianity on DPTs.
- Hybridization with advanced numerical techniques.
- Non-Markovian DPTs and symmetry breaking.

References

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[4] F. Minganti *et al.*, Phys. Rev. A **98**, 042118 (2018).
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