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The monomial functional equation for connected Lie groups

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Abstract. In this work, we present the Cauchy functional equation in the context of connected Lie groups. We consider two generalizations of this equation with higher orders of the finite difference.

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1. Introduction

Let G be a connected Lie group; if one considers the functions from G to $\mathbf{R}$, the Cauchy functional equation is defined by

$$f(xy) = f(x) + f(y).$$

This equation defines the homomorphism from G to $\mathbf{R}$. Cauchy considered this equation on $\mathbf{R}$ (thus using the additive notation) in [4] and showed that the continuous solutions are the monomials of degree 1: $f(x) = cx$ for some constant c. Darboux showed that only continuity at some point is required to reduce the set of solutions to these monomials [5]. Indeed, it is sufficient to ask for the solution to be bounded almost everywhere in a neighborhood to recover such a regularity [12]. This equation has been considered in various forms by many authors [1–3, 7–10, 13, 14, 16, 17].

Given $x \in G$, define left and right translations as maps defined on G by $L_x : y \mapsto xy$ and $R_x : y \mapsto yx$, respectively. We then define the difference operator as $\Delta_h = R_h^* - I$, where I is the identity operator and R_h^* denotes the pullback of R_h in the sense that $R_h^* f = f \circ R_h$. The Cauchy functional equation can be rewritten as

$$\Delta_y f(x) = f(y).$$

To generalize this equation, we will use higher order difference operators: for higher orders m, we would expect the solutions of such an equation on $\mathbf{R}$ to

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be the monomials of degree m. The monomial functional equation of order $m \in \mathbf{N}$ is thus simply defined as

$$\Delta_h^m f(x) = m! f(h). \tag{1}$$

Naturally, for the corresponding local equation, we will consider x and h in a neighborhood of the identity. Similar equations were considered in [6,12,15] for example.

In this paper we carry on the work initiated in [12] by considering general connected Lie groups instead of the Euclidean space $\mathbf{R}^n$. We first characterize the regular solutions of the monomial functional equation. Next, we give another version of this equation, where the steps h are not necessarily equal for the difference operators Δ_h and give more insight into the regular solutions of this equation. Such an equation is usually referred to as a mixed differences equation, while (1) is naturally called an unnmixed differences equation. We then show that this mixed differences equation admits no purely distributional solution. Finally, we give some simple examples.

Let us provide some notations used throughout this paper. We will consider an arbitrary connected Lie group G whose Lie algebra is $\mathfrak{g}$. The left-invariant vector field associated to $X \in \mathfrak{g}$ will be denoted by $\mathcal{L}_X$ and if $E_1, \ldots, E_n$ is a basis of $\mathfrak{g}$ (n will always stand for the dimension), $\mathcal{L}_{k_1,\ldots,k_m}$ will stand for the composed operator $\mathcal{L}_{E_{k_1}} \cdots \mathcal{L}_{E_{k_m}}$. We will use $X * Y$ as a short notation for $\log(\exp X \exp Y)$, with $X, Y \in \mathfrak{g}$ sufficiently close to 0 (log is only defined locally), where $\exp : \mathfrak{g} \to G$ is the exponential map.

2. The unmixed differences equation

Obviously, a solution f of the local monomial functional equation of order m also satisfies the Fréchet functional equation of order $m+1$,

$$\Delta_h^{m+1} f(x) = 0,$$

in a neighborhood of 1. We will advantageously use the structure of the local solutions of this equation obtained in [11] to collect information about the local solutions of Eq. (1).

Let us first recall the following result.

Proposition 2.1. *If f satisfies (1) in a neighborhood of the identity and is bounded almost everywhere in this neighborhood, then f is smooth in a neighborhood of the identity.*

Proof. As a local solution of Eq. (1) satisfies the local Fréchet functional equation of order $m+1$, this is then a consequence of Lemmata 3, 4 and 5 from [11]. $\square$

Proposition 2.2. *A function f that is bounded almost everywhere in a neighborhood of the identity satisfies (1) for x and h in a neighborhood of the identity if and only if there exists a homogeneous polynomial P on $\mathfrak{g}$ of degree m such that*

$$\begin{aligned} f(x \exp X) &= \sum_{j=0}^{m-1} \frac{\mathcal{L}_X^j f(x)}{j!} + P(x) \\ &= f(x) + \sum_{j=1}^{m-1} \sum_{1 \le k_1, \dots k_j \le m} \frac{\mathcal{L}_{k_1 \cdots k_j} f(x)}{j!} X_{k_1} \cdots X_{k_j} + P(x), \end{aligned}$$

with $\mathcal{L}_X^k f(1) = 0$ for all $k < m$, x in a neighborhood of 1 and X in a neighborhood of 0.

Proof. The condition is necessary.

We already know that f is smooth in a neighborhood of 1. Moreover as f is a local solution of the Fréchet functional equation of order $m+1$, Proposition 4 of [11] implies that

$$f(x \exp X) = \sum_{k=0}^{m} \frac{\mathcal{L}_X^k f(x)}{k!},$$

for x in a neighborhood U of 1 and $X \in \log U$. Using the exponential, (1) becomes

$$\Delta^m_{\exp tX} f(x) = m! f(\exp tX),$$

for x in a neighborhood V of 1, $X \in \log V$ and t in a neighborhood of 0, which gives

$$\mathcal{L}_X^m f(x) = m! \lim_{t \to 0} \frac{f(\exp tX)}{t^m}.$$

Let us show that $\mathcal{L}_X^k f(1) = 0$ for $k < m$. We trivially have $f(1) = 0$. If $\mathcal{L}_X f(1)$ does not vanish, we can write

$$\begin{aligned} 0 &= \lim_{t \to 0} \left| \frac{f(\exp tX)}{t^m} - \frac{\mathcal{L}_X^m f(1)}{m!} \right| \\ &= \lim_{t \to 0} \frac{1}{|t|^{m-1}} \left| \sum_{k=1}^{m-1} \frac{t^{k-1} \mathcal{L}_X^k f(1)}{k!} \right| = \infty, \end{aligned}$$

which implies $\mathcal{L}_X f(1) = 0$. Now, if $\mathcal{L}_X^k f(1) = 0$ for $k \in \{0, \dots, l\}$ with $l < m$, $\mathcal{L}_X^l f(1) \neq 1$ implies

$$0 = \lim_{t \to 0} \frac{1}{|t|^{m-l}} \left| \sum_{k=l}^{m-1} \frac{t^{k-l} \mathcal{L}_X^k f(1)}{k!} \right| = \infty.$$

As a consequence, we have $\mathcal{L}_X^k f(1) = 0$ for $k \in \{0, \dots, m-1\}$.

We thus have

$$f(\exp X) = \frac{\mathcal{L}_X^m f(1)}{m!} = \sum_{1\le k_1,\dots,k_m\le n} \frac{\mathcal{L}_{k_1\cdots k_m} f(1)}{m!} X_{k_1}\cdots X_{k_m}.$$

Moreover, the identity

$$\frac{\Delta^m_{\exp tX} f(x)}{t^m} = m!\frac{f(\exp tX)}{t^m}$$

implies $\mathcal{L}_X^m f(x) = \mathcal{L}_X^m f(1)$ in a neighborhood of the identity.

Let us show that the condition is sufficient. Let us write

$$Q(x,X) = \sum_{j=0}^{m-1} \frac{\mathcal{L}_X^j f(x)}{j!},$$

for any x in a neighborhood of 1 and X in a neighborhood of 0 in $\mathfrak{g}$. Since $\mathcal{L}_X^k f(1)$ vanishes for $k \in \{0,\dots,m-1\}$ and X close to 0, we have $Q(1,X)=0$ for such an X. Assume that $f(x\exp X) = Q(x,X) + P(X)$ for any x in a neighborhood U of 1 in G and X in a convex neighborhood V of 0 in $\mathfrak{g}$. For $x \in U$ and $X \in V/m$, we have

$$\begin{aligned}\Delta^m_{\exp X} f(x) &= \Delta^m_X[Q(x,\cdot)](0) + \Delta^m_X P(0)\\ &= m!P(X) = m!f(\exp X),\end{aligned}$$

where the difference operator in the right-hand side is the difference operator on $\mathfrak{g}$ with its vector space operation. □

3. The mixed differences equation

In order to consider the mixed differences equation, where the steps h are not necessarily equal for the difference operators Δ_h, the unknown function will be given by a map $f: G^m \to \mathbf{R}$; we can see G^m as a manifold with the group G acting on it diagonally:

$$(x_1,\dots,x_m)\cdot h = (x_1h,\dots,x_mh).$$

We will write $R_h(x_1,\dots,x_m)$ for $(x_1,\dots,x_m)\cdot h$, so that $\Delta_h = R_h^* - I$, where I is the identity operator acting on functions. We define the mixed differences version of the monomial functional equation of order m by

$$\Delta^m_{h_1,\dots,h_m} f(x_1,\dots,x_m) = m!f(h_1,\dots,h_m), \tag{2}$$

with

$$\Delta^m_{h_1,\dots,h_m} f(x_1,\dots,x_m) = \Delta_{h_m}(\Delta^{m-1}_{h_1,\dots,h_{m-1}} f)(x_1,\dots,x_m),$$

for $m>1$. If (2) holds for $x_1,\dots,x_m$ and $h_1,\dots,h_m$ in a neighborhood of the identity, we will refer to it as the local monomial functional equation of order m.

If we define $F = f \circ D_m$, where $D_m : G \to G^m$ is the diagonal map $x \mapsto (x, \dots, x)$, we have

$$\Delta_h^m F(x) = m! F(x),$$

for $h_1 = \cdots = h_m = h$, so that the restriction to the diagonal gives Equation (1).

A map $f : G^m \to h$ is said to be (locally) symmetric if

$$f(x_{\sigma_1}, \dots, x_{\sigma_m}) = f(x_1, \dots, x_m),$$

for any $\sigma \in \mathfrak{S}_m$, where $\mathfrak{S}_m$ is the set of permutations of m elements, and all $x_1, \dots, x_m$ in G (a neighborhood of 1 in G).

We already have the following result:

Proposition 3.1. *If f is a solution of (2) and $f \circ D_m$ is bounded almost everywhere in a neighborhood of 1, then $f \circ D_m$ is smooth in a neighborhood of 1.*

Indeed, we have more information about the solution.

Lemma 3.2. *If f is a local solution of (2), then f is locally multiadditive in a neighborhood of 1: it is a local homomorphism with respect to each of its variable if they vary in a neighborhood of 1.*

Moreover, if f and g are two (locally) symmetric solutions of (2), then $f = g$ (in a neighborhood of 1 in G^m) if and only if $f \circ D_m = g \circ D_m$ (in a neighborhood of 1 in G).

Proof. It is sufficient to prove that

$$f(h_1, \dots, h_j h_j', \dots, h_m) = f(h_1, \dots, h_j, \dots, h_m) + f(h_1, \dots, h_j', \dots, h_m),$$

for any $h_1, \dots, h_j, h_j', \dots, h_m$ in some neighborhood of 1. It is easy to check that

$$\Delta_{h_j h_j'} - \Delta_{h_j} - \Delta_{h_j'} = (R^*_{h_j} - I)(R^*_{h_j'} - I),$$

which implies

$$\begin{aligned} &f(h_1, \dots, h_j h_j', \dots, h_m) - f(h_1, \dots, h_j, \dots, h_m) - f(h_1, \dots, h_j', \dots, h_m) \\ &\quad = m! \Delta^{m+1}_{h_1, \dots, h_j, h_j', \dots, h_m} f(1, \dots, 1). \end{aligned}$$

Since the right-hand side vanishes for $h_1, \dots, h_j, h_j', \dots, h_m$ belonging to a sufficiently small neighborhood of 1, the first claim is proved.

Now, if f and g agree on the diagonal, from (2), we have

$$f(h_1, \dots, h_m) = \frac{1}{m!} \Delta^m_{h_1, \dots, h_m} (f \circ D_m)(1) = g(h_1, \dots, h_m).$$

The local version is naturally obtained by restricting these equalities on a neighborhood of the identity in G^m. $\square$

For the next result, we need the following notion.

Definition 3.3. A map $f : G \to \mathbf{R}$ is locally right-Abelian at $x_0 \in G$ if $f(xyz) = f(xzy)$, for any x in some neighborhood of x_0 and any $y, z \in G$ in some neighborhood of the identity.

Proposition 3.4. *If $f : G^m \to \mathbf{R}$ is a local solution of (2), then $f \circ D_m$ is locally right-Abelian. In particular, f is locally symmetric.*

Proof. Thanks to the previous result, we can write

$$(f \circ D_m)(xh_1h_2) = \sum_{(u_1,\ldots,u_m)\in\{x,h_1,h_2\}^m} f(u_1,\ldots,u_m) = (f \circ D_m)(xh_2h_1),$$

for any x, h_1 and h_2 in a neighborhood of 1 in G. This implies

$$\Delta^2_{h_1,h_2}(f \circ D_m)(x) = \Delta^2_{h_2,h_1}(f \circ D_m)(x),$$

still in a neighborhood of 1. More generally, we have

$$\Delta^m_{h_{\sigma_1},\ldots,h_{\sigma_m}}(f \circ D_m)(x) = \Delta^m_{h_1,\ldots,h_m}(f \circ D_m)(x),$$

for any x, h_1 and h_2 in a neighborhood of 1 in G and $\sigma \in \mathfrak{S}_m$. □

Proposition 3.5. *Let k be the dimension of* $\mathrm{Hom}(\mathfrak{g}, \mathbf{R})$ *in the category of Lie algebras. If f is a local solution of (2) such that $f \circ D_m$ is bounded almost everywhere in a neighborhood of* 1*, then there exist smooth local homomorphisms $f_1, \ldots, f_k$, real numbers a_α for $\alpha \in \mathbf{N}^k$ such that $|\alpha| = m$ and a neighborhood V of the identity such that*

$$(f \circ D_m)(h) = \sum_{|\alpha|=m} a_\alpha f_1(h)^{\alpha_1} \cdots f_k(h)^{\alpha_k}, \tag{3}$$

for all $h \in V$.

More precisely, we have

$$f = \sum_{|\alpha|=m} a_\alpha f_1^{\odot\alpha_1} \odot \cdots \odot f_k^{\odot\alpha_k}, \tag{4}$$

in a neighborhood of 1 *in G^m, with*

$$g_1 \odot \cdots \odot g_k = \frac{1}{k!} \sum_{\sigma\in\mathfrak{S}_k} g_{\sigma_1} \otimes \cdots \otimes g_{\sigma_k}.$$

If G is simply connected and f is a global solution, then the above equalities hold everywhere.

Proof. We just need to prove (4). Since f is symmetric, it is entirely determined by its restriction on the diagonal, so that we indeed have to prove that (3) holds. This identity follows from Proposition 2.2 above and Theorem 1 and Proposition 5 from [11]. The simply connected case is obtained using the isomorphism between $\mathrm{Hom}(\mathfrak{g}, \mathbf{R})$ and $\mathrm{Hom}(G, \mathbf{R})$. □

4. Distributional solutions of the mixed differences equation

Let $\delta : G \to (0,\infty)$ be the modular function of G, which we suppose to be equipped with a left Haar measure. It is easy to see that $\Delta_h^m f$ is well-defined as an element of $L^1_{\mathrm{loc}}(G)$, since it does not depend on the representation of f in $L^1_{\mathrm{loc}}(G)$. Let φ be a function of $D(G)$, the space of smooth functions on G with compact support; in a distributional sense, $\Delta_h^m f$ is given by

$$\begin{aligned}\langle \Delta_h^m f, \varphi\rangle &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\int_G f(xh^j)\,\varphi(x)\,dx\\ &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\int_G \delta(h^{-1})^j f(x)\varphi(xh^{-j})\,dx\\ &= \langle f, \Delta'^m_{h^{-1}}\varphi\rangle,\end{aligned}$$

with $\Delta'_h = \delta(h)R_h^* - I$. For the right Haar measure however, we get the expected identity $\langle \Delta_{h^{-1}}^m f, \varphi\rangle = \langle f, \Delta_h^m \varphi\rangle$. If G is unimodular, we recover a definition of the difference operator on the space of distributions $D'(G)$ that is consistent. We will thus work with such a group in this section.

Remark 4.1. If one wants to deal with groups G that are not unimodular, it suffices to consider a right Haar measure when working with right finite differences Δ_h.

In order to obtain an extension in the setting of distributions, we will consider each side of the equation as functions on $G^m \times G^m$. Equation (2) can be written as

$$\begin{aligned}&(-1)^m f(x_1,\ldots,x_m)\\ &\quad+\sum_{k=1}^{m}\sum_{1\le j_1\le\cdots\le j_k\le m}(-1)^{m-k}s^*_{j_1\cdots j_k}f(x_1,\ldots,x_m,h_1,\ldots,h_m)\\ &= \mathrm{pr}_2^* f(x_1,\ldots,x_m,h_1,\ldots,h_m),\end{aligned}$$

where

$$\mathrm{pr}_2(x_1,\ldots,x_m,h_1,\ldots,h_m) = (h_1,\ldots,h_m)$$

and

$$s_{j_1\cdots j_k}(x_1,\ldots,x_m,h_1,\ldots,h_m) = (x_1h_{j_1}\cdots h_{j_k},\ldots,x_mh_{j_1}\cdots h_{j_k}).$$

Since these two maps are submersions from $G^m\times G^m$ to G^m, we can extend their pullback to distributions. If Δ_h is the finite difference on G^m defined by the diagonal action, given a distribution T on G^m, we naturally set

$$\Delta^m_{h_1,\ldots,h_m}T(\varphi) = T(\Delta^m_{h_1^{-1},\ldots,h_m^{-1}}\varphi),$$

for $\varphi \in D(G^m)$. The difference on distributions is explicitly given by

$$\Delta^m T(\varphi) = \int_{G^m} \Delta^m_{h_1,\ldots,h_m} T_{(x_1,\ldots,x_m)}(\varphi(x_1,\ldots,x_m,h_1,\ldots,h_m))dh_1\cdots dh_m,$$

where $T_{(x_1,\ldots,x_m)}$ means that the distribution acts on φ seen as a function of $(x_1,\ldots,x_m)$, other variables being seen as constants. Moreover,

$$\mathrm{pr}_2^* T(\varphi) = \int_{G^m} T_{(h_1,\ldots,h_m)}(\varphi(x_1,\ldots,x_m,h_1,\ldots,h_m))dx_1\cdots dx_m.$$

The Cauchy functional equation of order m on $D'(G^m)$ is naturally given by

$$\Delta^m T = m!\mathrm{pr}_2^* T, \tag{5}$$

on $D(G^m \times G^m)$. Of course, one can consider the local version of this equation by working on $D'(V \times V)$, where V is some neighborhood of the identity.

Proposition 4.2. *If T is a distribution on G^m that satisfies (5) locally, then there exists a neighborhood V of 1 in G^m such that $T|_V \in C^\infty(V)$ is a distribution associated to some smooth function f on G^m defined locally by*

$$\sum_{|\alpha|=m} a_\alpha f_1^{\odot\alpha_1} \odot \cdots \odot f_1^{\odot\alpha_k},$$

in a neighborhood of 1 in G^m.

Moreover, if T satisfies (5) globally, then T is a smooth function in G^m.

Proof. Let $\varphi, \psi \in D(G^m)$ such that $\varphi \otimes \psi \in D(G^m \times G^m)$. For $\varphi \otimes \psi$, Equation (5) gives

$$\begin{aligned}\int_{G^m} &\Delta^m_{h_1,\ldots,h_m} T(\varphi)\psi(h_1\ldots,h_m)dh_1\cdots dh_m\\ &= m!T(\psi)\int_{G^m}\varphi(x_1,\ldots,x_m)dx_1\cdots dx_m.\end{aligned}$$

If $\delta_{(y_1,\ldots,y_m)}$ denotes the usual difference operator on G^m with multiplication by $(y_1,\ldots,y_m) \in G^m$ given componentwise, $\delta_{(y_1^{-1},\ldots,y_m^{-1})}\varphi$ belongs to $D(G^m)$ for any $(y_1,\ldots,y_m) \in G^m$. The equation at $\delta_{(y_1^{-1},\ldots,y_m^{-1})}\varphi \otimes \psi$ gives

$$\begin{aligned}\int_{G^m} &\delta_{(y_1,\ldots,y_m)}\Delta^m_{h_1,\ldots,h_m} T(\varphi)\psi(h_1,\ldots,h_m)dh_1\cdots dh_m\\ &= m!T(\psi)\int_{G^m}\varphi(x_1y_1^{-1},\ldots x_my_m^{-1}) - \varphi(x_1,\ldots,x_m)dx_1\cdots dx_m,\end{aligned}$$

which is equal to zero because of the invariance of the Haar measure. This implies that $\delta_{(y_1,\ldots,y_m)}\Delta^m_{h_1,\ldots,h_m}T(\varphi)$ vanishes for all $\varphi \in D(G^m)$ and almost all $h_1,\ldots,h_m$, $y_1,\ldots,y_m$ in G. By continuity, this quantity is equal to zero for all $h_1,\ldots,h_m$, $y_1,\ldots,y_m$ in G. As a consequence, $\Delta^m_{h_1,\ldots,h_m}T$ satisfies the Fréchet functional equation of order 1, so that it is equal to a constant function

$f(h_1,\ldots,h_m)$ in the sense of distributions, where f is smooth. Therefore, (5) implies

$$m!pr_2^*T(\varphi\otimes\psi)$$
$$=\int_{G^m\times G^m} f(h_1,\ldots,h_m)\varphi(x_1,\ldots,x_m)\psi(h_1,\ldots,h_m)dx_1\cdots dx_m dh_1\cdots dh_m,$$

for φ, ψ belonging to $D(G^m)$. As a consequence, we get

$$\int_{G^m}(m!T(\psi)-\int_{G^m} f(h_1,\ldots,h_m)\psi(h_1,\ldots,h_m)dh_1\cdots dh_m)$$
$$\varphi(x_1,\ldots,x_m)dx_1\cdots dx_m=0,$$

so that we have $f=m!T$ in $D(G^m)$, which proves that T is equal as a distribution to a smooth function on G^m. The local version can be obtained in the same way, by restricting the test-functions to those with a sufficiently small support. □

5. Some examples

Let us give some examples of solutions restricted to the diagonal for the mixed differences equation (2). They are easily obtained from the computation of $\mathrm{Hom}(G,\mathbf{R})$.

- If G is semisimple, any right-Abelian solution of the equation is zero. This results from the fact that $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ and $\mathrm{Hom}(\mathfrak{g},\mathbf{R})=\{0\}$. This includes the case of $\mathrm{SL}(n,\mathbf{R})$ or S^3 with quaternion multiplication as Lie group product.
- If G is the general linear group $\mathrm{GL}(n,\mathbf{R})$, then the right-Abelian solutions of the equation are given by
$$f(A)=a\ln(|\mathrm{det}A|)^m.$$
with some $a\in\mathbf{R}$.
- If G is solvable, then $[\mathfrak{g},\mathfrak{g}]$ is a proper non-trivial ideal of $\mathfrak{g}$ and it is also a nilpotent Lie subalgebra. As a consequence, $\mathrm{Hom}_{\mathrm{loc}}(G,\mathbf{R})$ is not trivial and the polynomials of degree m generated by elements of $\mathrm{Hom}_{\mathrm{loc}}(G,\mathbf{R})$ are non-trivial.
- If G is the Heisenberg group $\mathbf{H}_n=\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}$, then it is two-step nilpotent. The solutions of the equation are the polynomials
$$P(x,y,t)=\sum_{|\alpha|=m} a_\alpha x^{\alpha_1}y^{\alpha_2}$$
if they are looked on the Lie algebra of $\mathbf{H}_n$ because $[\mathfrak{g},\mathfrak{g}]=\{(0,0,t):t\in\mathbf{R}\}$.

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