A functional definition of polynomials for connected Lie groups

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Abstract

In this paper, we look at the solutions of the Fréchet functional equation for connected Lie groups. In particular, we give a characterization of theses solutions using the lower central series, providing a notion of polynomial for such groups.

1 Introduction

Let G be a connected Lie group and given $x \in G$, define the left and right translations as the maps defined on G by $L_x : y \mapsto xy$ and $R_x : y \mapsto yx$ respectively. We then set $\Delta_h = R_h^* - I$, where I is the identity operator and R_h^* denotes the pullback of R_h, $\Delta^2_{h_1,h_2} = \Delta_{h_1} \circ \Delta_{h_2}$ and

$$\Delta^{m+1}_{h_1,\ldots,h_{m+1}} = \Delta^m_{h_1,\ldots,h_m} \circ \Delta_{h_{m+1}},$$

for $m \in \mathbf{N}$. For the sake of simplicity, we will often write Δ_h^{m+1} instead of $\Delta^{m+1}_{h_1,\ldots,h_{m+1}}$; in this context, $h \in V$ must be understood as $h \in V^{m+1}$, that is $h_j \in V$ for any $j \in \{1, \ldots, m+1\}$. We will consider the Fréchet functional equation

$$\Delta_h^{m+1} f(x) = 0, \qquad (1)$$

where $m+1 \in \mathbf{N}$ will be called the order of the equation. We will be interested in local solutions at $x_0 \in G$, i.e. in maps $f : G \to \mathbf{R}$ for which (1) is satisfied for any $x \in G$ in some neighborhood of x_0 and $h \in G^{m+1}$ in some neighborhood of the identity (let us recall one last time that this means that $h_1, \ldots, h_{m+1}$ must be in some neighborhood of the identity) and global solutions, where (1) is satisfied for any $x \in G$ and any $h \in G^{m+1}$.

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These equations were first globally studied by M. Fréchet in the case $G = \mathbf{R}$ [11, 12]. He showed that all the continuous global solutions of (1) are given by the polynomials of degree at most m. Let us remark that a regularity condition is necessary to recover the polynomials from (1). For example, using a Hamel basis $\mathcal{H}$, it is easy to define a function on $\mathbf{R}$ which is affine on $x_j\mathbf{Q}$ for any $x_j \in \mathcal{H}$ so that we have

$$f(x+y) = f(x) + f(y). \tag{2}$$

Obviously, such a function satisfies (1) with $m = 1$. If f is not affine on $\mathbf{R}$, it is discontinuous at every point, not bounded on any Lebesgue-measurable set of positive measure (and thus not Borel-measurable) and its graph is dense in $\mathbf{R}^2$ [1]. Indeed, let f be a function satisfying (2); if f is continuous at some point, then it is continuous on $\mathbf{R}$. If it is bounded on some set of positive measure, then it is continuous at the origin and any Borel-measurable function is bounded on some set of positive measure. In other words, a function satisfying (2) is either affine or very irregular. The same phenomenon is observed with the Fréchet functional equation: it is sufficient to ask the solution to be continuous at some point or bounded almost everywhere on some interval to recover a polynomial [7, 20, 19].

Given $m \in \mathbf{N}$, for $h \in G$, let us also define $\underline{\Delta}_h = \Delta_h$ and $\underline{\Delta}_h^{m+1} = \Delta_h \circ \underline{\Delta}_h^m$. Instead of (1), one can consider its restriction to identical steps h,

$$\underline{\Delta}_h^{m+1} f(x) = 0, \tag{3}$$

with x in a neighborhood of $x_0 \in G$ and h in a neighborhood of the identity. On $\mathbf{R}^n$ (this is even true in the Abelian case), this equation is equivalent to the Fréchet functional equation (1). In a more general context however, solutions of (3) could not satisfy (1). Equation (1) is usually called Fréchet mixed differences equation, while (3) is called Fréchet unmixed differences equation.

Now, if $\mathfrak{g}$ is the Lie algebra of G, given $f : G \to \mathbf{R}$ and $x_0 \in G$, define $f_{x_0} : \mathfrak{g} \to \mathbf{R}$ by $f_{x_0}(X) = f(x_0 \exp X)$, where $\exp : \mathfrak{g} \to G$ is the exponential map. We may also consider the following local equation on the group $(\mathfrak{g}, +, 0)$:

$$\underline{\Delta}_Y^{m+1} f_{x_0}(X) = 0, \tag{4}$$

for $X, Y \in \mathfrak{g}$ in a neighborhood of 0. If G is Abelian, it is easy to check that $\underline{\Delta}_Y^{m+1} f_{x_0}(X) = \underline{\Delta}_{\exp(Y)}^{m+1} f(x_0 \exp(X))$. As G is connected, exp is surjective and equations (4) and (3) are (locally and globally) equivalent in the Abelian case. The same remark can be made in the unmixed case.

The global Fréchet functional equation has been studied on groups (with sometime a more restrictive definition of the equation), which were most of the time assumed to be Abelian (see for example [28, 24, 26, 13, 16, 15, 7, 22, 21, 18, 5, 9, 3, 6, 27, 14, 23, 2, 4] and references therein). There are also results on topological groups and semigroups. As far as we know, they only are of global nature [25, 8]. In [8], the equation was studied from a representation theory point of view with several notions of polynomials; the authors showed relations between these notions and existence results. In [25], the author extended the equation to general semigroups; he provided a general form for the right-Abelian solutions thanks to multiadditive mappings.

Fréchet considered Equation (1) as a functional definition of the polynomials (on the Euclidean space). In general, a (smooth) solution of (1) is called a (smooth) generalized polynomial of degree m, while an ordinary polynomial is a polynomial on a vector space in the usual sense. A (smooth) solution of (3) will be called a weak (smooth) generalized polynomial. The object of this work is to give more insight about such polynomials for general connected Lie groups and carry on the work of Fréchet. Roughly speaking, we show that these are generated by $(\mathfrak{g}^{(N)})^\perp$, where N is the number for which the lower central series stabilizes (see Definition 5). In practice, since there are groups G where the global solutions are necessarily trivial while there exists non-constant local solutions (such examples are easy to obtain in some cases of compact groups, see Proposition 9), we use germs at the identity to define such polynomials. As for non-Abelian Lie groups, Equation (3) has no trivial relation with Equation (4), we also study the solutions of this equation on the Lie algebra. To do so, we replace the latter with another one which, thanks to the Baker-Campbell-Hausdorff formula, gives a direct connection with the Fréchet functional equation. The new equation however, has only a local character.

In this paper, we start by gathering results from previous works [20, 19] to describe more explicitly the "regular" solutions in the case (3) and make some easy remarks. We show that these previous results could hardly be improved upon. We also give the explicit form of some specific solutions of the Fréchet functional equation (1), by enhancing a result from [19]. Next, we show that there is no distributional solution that is not associated to a smooth function. We then consider the explicit cases of the Abelian and nilpotent Lie groups, showing that there is no clear notion of degree that can be associated to the smooth solutions of (1). We then consider the structure of the (smooth) generalized polynomials on G by studying the set of (smooth) functions satisfying (1) for some m; we characterize these solutions thanks to the lower central series. Finally, we consider the Fréchet functional equation on homogeneous spaces.

2 Notations and previous results

Let use precise some notations used throughout this paper. We will consider an arbitrary connected Lie group G whose Lie algebra is $\mathfrak{g}$. The left-invariant vector field associated to $X \in \mathfrak{g}$ will be denoted by $\mathcal{L}_X$ and if $E_1, \dots, E_n$ is a basis of $\mathfrak{g}$ (n will always stand for the dimension), $\mathcal{L}_{k_1,\dots,k_m}$ will stand for the composed operator $\mathcal{L}_{E_{k_1}} \cdots \mathcal{L}_{E_{k_m}}$. We will use $X * Y$ as a short notation for $\log(\exp X \exp Y)$, with $X, Y \in \mathfrak{g}$ sufficiently close to 0 (log is only defined locally).

Let us recall two results from previous works [20, 19]. First, every solution of (1) or (3) that is "regular" is necessarily smooth.

Proposition 1. *Let G be a connected Lie group equipped with a left Haar mea-*

sure; if f is a solution of (3) at x_0 that is bounded almost everywhere in a neighborhood of x_0, then f is smooth in a neighborhood of x_0.

In particular, continuous solutions are smooth. We can describe the solutions of Equation (3).

Proposition 2. *A function $f : G \to \mathbf{R}$ that is locally bounded almost everywhere in a neighborhood of x_0 is a smooth solution of (3) at x_0 if and only if there exist a neighborhood V_{x_0} of x_0 and a neighborhood V_0 of 0 in $\mathfrak{g}$ such that*

$$f_x(X) = \sum_{j=0}^{m} \frac{1}{j!} \mathcal{L}_X^j f(x) = f(x) + \sum_{j=1}^{m} \sum_{1 \leq k_1, \ldots k_j \leq n} \frac{\mathcal{L}_{k_1 \cdots k_j} f(x)}{j!} X_{k_1} \cdots X_{k_j}, \quad (5)$$

for $X \in V_0$ and $x \in V_{x_0}$.

The following example will help us to underline the difference between the Fréchet mixed differences equation (1) and the unmixed differences equation (3).

Example 1. If G is a 2-nilpotent Lie group, there exists a linear functional f on $\mathfrak{g}$ that does not identically vanish on $[\mathfrak{g}, \mathfrak{g}]$. For $X, Y, Z \in \mathfrak{g}$, we have

$$\Delta^2_{Y,Z} f(X) = \frac{1}{2} f([Y, Z]),$$

which is not zero for some appropriate choice of Y and Z, while it vanishes for $Y = Z$. Let us also remark that we have

$$f(X * Y * Z) - f(X * Z * Y) = f([X, Z]);$$

this identity will be useful in Section 6.

3 The left Fréchet equation

Let us first show that the Fréchet functional equation is symmetric: the left local Fréchet equation is equivalent to (1). Given $h \in G$, let ${}_h\Delta = L_h^* - I$ in order to define the left difference operator ${}_h\Delta^m$ in the same way as Δ_h^m.

Proposition 3. *A function $f : G \to \mathbf{R}$ is a solution of the local Fréchet Equation (1) at x_0 if and only if it satisfies*

$${}_h\Delta^{m+1} f(x) = 0,$$

for any x in some neighborhood of x_0 and any h in some neighborhood of the identity.

Proof. Given $x \in G$ (in a neighborhood of x_0) and $h \in G^{m+1}$ (in a neighborhood of 1), let us define $h' \in G^{m+1}$ by setting $h'_j = x^{-1} h_j x$ for $j \in \{1, \ldots, m+1\}$, so that we have

$${}_h\Delta^{m+1} f(x) = \Delta_{h'}^{m+1} f(x).$$

The result follows from the fact that $(x, h) \mapsto x^{-1} h x$ is continuous at $(x_0, 1)$. □

Of course, this result can easily be adapted for the equation with unmixed differences (3).

4 A simple case where the unmixed differences equation is equivalent to the mixed differences equation

On $\mathbf{R}^n$, Equation (3) is equivalent to the Fréchet mixed differences equation (1). Indeed, this result is well-known for Abelian group in general and has been obtained for more general cases (see [10, 15] and references therein). For example, it is shown in [15] that if G and G' are Abelian groups such that, for every prime number $p \leq m+1$, either G' does not contain any element of order p or $|G/pG| \leq p$, then $f : G \to G'$ is a solution of (1) if and only if it is a solution of (3). If G is not Abelian however, solutions of (3) could not satisfy the Fréchet equation; Example 1 provides a function f that satisfies (3) in a neighborhood of 1 for $m = 1$ but not (1) for the same m. We consider here the local case and easily show that in the Abelian setting, polynomial germs are solutions of both (1) and (3). In particular, these equations admit the same smooth solutions. This equivalence is a particular case of Lemma 13 in [15].

Definition 1. A map $f : G \to \mathbf{R}$ is a polynomial germ at x_0 if the germ of f_{x_0} at the origin is the germ of a polynomial function P in $\mathfrak{g}$. In this case, the degree of f at x_0 is the degree of P.

Proposition 4. *If G is Abelian, polynomial germs at x_0 of degree at most m are solution of the local equations* (1) *and* (3) *at x_0.*

As a consequence, if G is Abelian, the Fréchet equation (1) *is equivalent to the unmixed version* (3) *for the smooth solutions.*

Proof. Since G is Abelian, the difference operator can be locally seen as the classical difference operator on $\mathfrak{g}$ with its vector space structure. For X and Y in a neighborhood of 0 in $\mathfrak{g}$, we have

$$\Delta_{\exp Y} f(x_0 \exp X) = \Delta_Y f_{x_0}(X),$$

where the operator in the right-hand side is the difference operator on a vector space. In this case, the smooth solutions are given by polynomials of degree at most m defined on a neighborhood of x_0. □

5 A relation with linear partial differential equations

Let us give a complementary result in the case of the unmixed differences equation (3), which shows that there is very little hope to be able to describe more explicitly all the solutions of (3) than what is proposed in Proposition 2: the coefficients of the operators are not tractable since they rely on a quite cumbersome formula. It suggests that the obstruction on the existence of solutions is, at least partially, of algebraic nature.

We will denote by ad the adjoint representation of a Lie algebra; recall that $\mathrm{ad}(X)(Y) = [X, Y]$. As usual, $\mathfrak{S}_{m+1}$ denotes the set of permutations of $m+1$ elements.

Proposition 5. *A smooth solution f of the local Fréchet equation at x_0 is a local solution of the system of $\binom{n+m}{m+1}$ homogeneous linear partial differential equations of order $m+1$ defined by*

$$\sum_{\sigma \in \mathfrak{S}_{m+1}} \mathcal{L}_{k_{\sigma_1}, \cdots, k_{\sigma_{m+1}}} f = 0 \quad \forall k_1, \ldots, k_{m+1} \in \{1, \ldots, n\},$$

in a neighborhood of x_0.

The system can be rewritten as

$$\sum_{1 \leq |\alpha| \leq m+1} b^{\alpha}_{k_1, \ldots, k_{m+1}}(\mathrm{ad}(X)) D^{\alpha} f_{x_0}(X) = 0 \quad \forall k_1, \ldots, k_{m+1} \in \{1, \ldots, n\},$$

where X is in a neighborhood of 0 in $\mathfrak{g}$, $b^{\alpha}_{k_1, \ldots, k_{m+1}}$ are some analytic functions in a neighborhood of 0 that are symmetric with respect to the indices $k_1, \ldots k_{m+1}$ and D^{α} represents the ordinary derivatives with respect to a fixed basis of $\mathfrak{g}$.

Proof. From (5) and the Taylor theorem, we have

$$\sum_{\sigma \in \mathfrak{S}_{m+1}} \mathcal{L}_{k_{\sigma_1}, \ldots, k_{\sigma_{m+1}}} f = 0$$

on a neighborhood of 1 for any $k_1, \ldots, k_{m+1} \in \{1, \ldots, n\}$. Since a solution of the equation of order $m+1$ is also a solution for the equation of order $m+2$, the terms of order $m+1$ in the polynomial solution $f_x(X)$ must be equal to zero. The coefficient of $X_{k_1}, \ldots, X_{k_{m+1}}$, given by

$$\frac{1}{(m+1)!} \sum_{\sigma \in \mathfrak{S}_{m+1}} \mathcal{L}_{k_{\sigma_1}, \ldots, k_{\sigma_{m+1}}} f(x)$$

is thus equal to zero for any x sufficiently close to x_0.

Let us examine the local expression of the equations in the chart $\exp : \mathfrak{g} \to G$. For X sufficiently small and $k \in \{1, \ldots, n\}$, we have

$$\mathcal{L}_k f(x_0 \exp X) = [D_t f_{x_0}(X * tE_k)]_{t=0} = \sum_{j=1}^{n} a^j_k(X) D_j f_{x_0}(X),$$

with $a^j_k(X) = [D_t(X * tE_k)_j]_{t=0}$. An application of the Baker-Campbell-Hausdorff formula shows that

$$a^j_k(X) = [\phi(e^{\mathrm{ad}(X)})(E_k)]_j,$$

where

$$\phi(T) = I + \sum_{j=1}^{\infty} \frac{(-1)^j}{j(j+1)} (T - I)^j,$$

for $T \in \mathcal{L}(\mathfrak{g},\mathfrak{g})$ sufficiently close to the identity (see [17]). Indeed, we have

$$\begin{aligned} X * tE_k &= X + \int_0^1 \phi(e^{\mathrm{ad}(X)} e^{u\mathrm{ad}(tE_k)})\, tE_k\, du \\ &= X + \int_0^t \phi(e^{\mathrm{ad}(X)} e^{v\mathrm{ad}(E_k)})\, E_k\, dv. \end{aligned}$$

So that differentiating at $t = 0$ gives the expression of $a_k^j(X)$. The coefficients $a^\alpha_{k_1,\dots,k_{m+1}}$ are thus simply a sum of products of the a_k^j and their derivatives up to order m. The analyticity of the coefficients is clear from the formula for the coefficients a_k^j. □

6 Right-Abelian solutions

Let us give a condition on the solutions under which they are characterized in terms of local homomorphisms; this section improves results from [19].

Definition 2. A map $f : G \to \mathbf{R}$ is locally right-Abelian at $x_0 \in G$ if $f(xyz) = f(xzy)$, for any x in some neighborhood of x_0 and any $y, z \in G$ in some neighborhood of the identity.

Here, given some distance on G, $B(r)$ denotes the ball centered at 1 with radius r.

Proposition 6. *If $f : G \to \mathbf{R}$ is smooth in a neighborhood of x_0 and locally right-Abelian at x_0, then there exists a neighborhood V_{x_0} of x_0 such that $\mathcal{L}_{k_1,\dots,k_m} f = \mathcal{L}_{\sigma_{k_1},\dots,\sigma_{k_m}} f$ on V_{x_0} for any $\sigma \in \mathfrak{S}_m$.*

Moreover, $\mathcal{L}_X^k f$ is also locally right-Abelian for any $X \in \mathfrak{g}$ and any $k \in \mathbf{N}$.

Proof. Let d be a left-invariant Riemannian distance on G and $r > 0$ be such that $f(xyz) = f(xzy)$ for any $x \in V_{x_0}$ and y, z in the ball $B(r)$. It is easy to show that we have

$$f(xy_{\sigma_1}\cdots y_{\sigma_k}) = f(xy_1\cdots y_k),$$

for any $\sigma \in \mathfrak{S}_k$ with $k \le m$, $x \in V_{x_0}$ and $y_1, \dots, y_k \in B(r/(k+1))$.

Since the dilation $t \mapsto \exp(tX)$ is continuous in a neighborhood of the origin for any $X \in \mathfrak{g}$, there exists a neighborhood V_0 of 0 in $\mathbf{R}$ such that

$$f(x\exp(t_{k_1}E_{k_1})\cdots\exp(t_{k_m}E_{k_m})) = f(x\exp(t_{\sigma_{k_1}}E_{\sigma_{k_1}})\cdots\exp(t_{\sigma_{k_m}}E_{\sigma_{k_m}})),$$

for any $x \in V_{x_0}$ and $t_{k_1},\dots,t_{k_m} \in V_0$. Both sides of this equality seen as functions of $t_{k_1},\dots,t_{k_m}$ are smooth in a neighborhood of the origin, which can be assumed to be V_0^m. The first part of the result is obtained by differentiating with respect to these variables at 0.

It remains to prove the second statement for the first order differentiation. For $X \in \mathfrak{g}$, we have $f_{xyz}(tX) = f_{xzy}(tX)$ for any $x \in V_{x_0}$, y, z in a neighborhood of 1 and t in a neighborhood of 0. Differentiating with respect to t at 0 allows to conclude. □

Combining results from [19], we have the following:

Proposition 7. *Let k be the dimension of* $\mathrm{Hom}_{\mathrm{loc}}(G, \mathbf{R})$*; a locally right-Abelian map $f : G \to \mathbf{R}$ at x_0 that is also bounded almost everywhere in a neighborhood of this point is a solution of the local Fréchet equation* (1) *at x_0 if and only if there exist $f_1, \dots f_k \in \mathrm{Hom}_{\mathrm{loc}}(G, \mathbf{R})$, real numbers a_α for $\alpha \in \mathbf{N}^k$ such that $|\alpha| \le m$ and a neighborhood V_1 of the identity such that*

$$f(x_0 h) = \sum_{|\alpha| \le m} a_\alpha f_1(h)^{\alpha_1} \cdots f_k(h)^{\alpha_k},$$

for any $h \in V_1$.

If moreover G is simply connected, we also have $f_1, \dots, f_k \in \mathrm{Hom}(G, \mathbf{R})$.

Unfortunately, not every solution of (1) is right-Abelian; once again, Example 1 gives a function f that is not right-Abelian but satisfies (1) for $m = 2$.

7 The Fréchet functional equation for distributions

In this section, we wonder whether or not there are distributional solutions that are not associated to a smooth function.

Let $\delta : G \to (0, \infty)$ be the modular function of G, which we suppose equipped with a left Haar measure. It is easy to see that $\Delta_h^m f$ is well-defined as an element of $L^1_{\mathrm{loc}}(G)$, since it does not depend on the representation of f in $L^1_{\mathrm{loc}}(G)$. Let φ be a function of $D(G)$, the space of smooth functions on G with compact support; in a distributional sense, $\underline{\Delta}_h^m f$ (we use unmixed differences for the sake of clarity) is given by

$$\begin{aligned}\langle \underline{\Delta}_h^m f, \varphi \rangle &= \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \int_G f(xh^j)\, \varphi(x)\, dx \\ &= \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \int_G \delta(h^{-1})^j f(x) \varphi(xh^{-j})\, dx \\ &= \langle f, \underline{\Delta}'^m_{h^{-1}} \varphi \rangle,\end{aligned}$$

with $\Delta'_h = \delta(h) R_h^* - I$. For the right Haar measure however, we get the expected identity $\langle \Delta^m_{h^{-1}} f, \varphi \rangle = \langle f, \Delta^m_{\tilde{h}} \varphi \rangle$, where, if $h = (h_1, \dots, h_m)$, $\tilde{h} = (h_m, \dots, h_1)$. If G is unimodular, we recover a definition of the difference operator on the space of distributions $D'(G)$ that is consistent. We will thus work with such a group in this section.

Remark 1. If one wants to deal with groups G that are not unimodular, it suffices to consider a right Haar measure when working with right finite differences Δ_h.

Definition 3. Given $T \in D'(G)$ and $h \in G^m$, $\Delta_h^m T$ is the distribution on G defined by $\Delta_h^m T(\varphi) = T(\Delta^m_{h^{-1}} \varphi)$, for any $\varphi \in D(G)$. Its restriction to $L^1_{\mathrm{loc}}(G)$

is the usual difference operator of function (up to a vanishing set for the Haar measure).

Naturally, from the point of view of distributions, the local Fréchet functional equation at x_0 of order $m+1$ is given by

$$\Delta_h^{m+1} T = 0 \quad \forall \varphi \in D(V_{x_0}), \tag{6}$$

where V_{x_0} is an open neighborhood of x_0, with h in a neighborhood of 1. If $f \star g$ denotes the convolution of functions f and g on G, let us show that there is no purely distributional solution to (6). Indeed, we show a slightly stronger result.

Proposition 8. *If $T \in D'(G)$ is a solution of the local equation*

$$\underline{\Delta}_h^{m+1} T = 0 \quad \forall \varphi \in D(V_{x_0}),$$

where V_{x_0} is an open neighborhood of x_0, with h in a neighborhood of 1, then the restriction of T to some neighborhood of x_0 as a function is smooth.

Proof. Let ρ be a smooth function on $\mathfrak{g}$ that is compactly supported on the ball $B(r/(m+1))$ centered at the origin for some $r > 0$ such that the exponential is a diffeomorphism between $B(r)$ and its image. Moreover, we assume that $C = \int_{\mathfrak{g}} \rho(X) dX \neq 0$. Next, define

$$\tilde{\Phi}(X) = \sum_{j=1}^{m+1} (-1)^{m-j+1} \binom{m+1}{j} \frac{1}{j^n} \rho(\frac{X}{j}).$$

This function is supported on $B(r)$. We then set

$$\Phi(X) = \frac{(-1)^m}{C} \tilde{\phi}(X)$$

and

$$\varphi(x) = \Phi(\log x) \vartheta(\log x),$$

where $\vartheta(X)$ is the inverse of the Jacobian determinant at X of the exponential map (with respect to a fixed basis of left-invariant vector fields).

Let us consider $T \star \tilde{\varphi}$, where $\tilde{\varphi}(x) = \varphi(x^{-1})$. We have

$$(\psi \star \varphi)(x) - \psi(x) = \frac{(-1)^{m+1}}{C} \int_{B(r)} \Delta_{\exp(-X)}^{m+1} \psi(x) \rho(X)\, dX,$$

for any test function ψ. Let $V = x_0 \exp(B(r))$ in such a way that the exponential is a diffeomorphism from $B(r)$ into its image. For $\psi \in D(V)$, we have

$$\begin{aligned} \langle T \star \tilde{\varphi}, \psi \rangle - T(\psi) &= \int_G T(L_{x^{-1}}^* \varphi) \psi(x)\, dx - T(\psi) = T(\psi \star \varphi - \psi) \\ &= \frac{(-1)^{m+1}}{C} \int_{B(r)} \Delta_{\exp(X)}^{m+1} T(\psi) \rho(X)\, dX = 0. \end{aligned}$$

Therefore T is equal as a distribution to a smooth function on V. As the restriction of T to V as a function is bounded almost everywhere, it is smooth on some neighborhood of x_0. $\square$

8 The case of the Abelian Lie groups

Let us consider the example of the Abelian Lie groups.

Since the solutions of the Fréchet functional equation for an Abelian group are locally right-Abelian, the solutions are given by polynomials with degree at most m when the exponential chart is applied. For example, in $\mathbf{R}^n$, the solutions of the local Fréchet functional equation at x_0 are given by

$$f(x) = \sum_{|\alpha| \leq m} a_\alpha (x - x_0)^\alpha,$$

in a neighborhood of x_0.

In the case of the one-dimensional torus S^1, the solution of the local equation at x_0 are given by

$$f(x) = \sum_{j=0}^{m} a_j \arg(x x_0^{-1})^j,$$

in a neighborhood of x_0 (here, S^1 is interpreted as the unit circle of the complex plane). If we consider the local Fréchet functional equation on $\mathfrak{g}$ at x_0:

$$\underline{\Delta}_Y^{m+1} f_{x_0}(X) = 0,$$

with $X, Y \in \mathfrak{g}$ in a neighborhood of 0, we get

$$f(\exp_{x_0} X) = \sum_{|\alpha| \leq m} a_\alpha X^\alpha,$$

with respect to some basis in the tangent space, X being in a neighborhood of the origin. For the global solution, we have the following result:

Proposition 9. *If G is Abelian, let $k, n \in \mathbf{N}_0$ be such that $G = \mathbf{R}^n \times (S^1)^k$; the global solutions of* (1) *that are bounded almost everywhere are of the form*

$$f(x, y) = \sum_{|\alpha| \leq m} a_\alpha x^\alpha,$$

for $x \in \mathbf{R}^n$ and $y \in (S^1)^k$.

Proof. We just need to show that if G is a torus, the only global solutions that are bounded almost everywhere are constant. Let us assume that $G = (S^1)^k$ and develop f as a Fourier series. Let $\hat{f}(l)$ be the Fourier coefficient relative to $x \mapsto e^{\langle l,x \rangle}/(2\pi)^{k/2}$, so that we can write

$$f(e^{ix_1}, \ldots, e^{ix_k}) = \sum_{l \in \mathbf{Z}^k} \frac{1}{(2\pi)^{k/2}} \hat{f}(l)\, e^{i\langle l,x \rangle}.$$

With the Fourier decomposition of $(x_1,\dots,x_k)\mapsto \underline{\Delta}_h^m f(e^{ix_1},\dots,e^{ix_k})$, we get

$$\begin{aligned}
0 &= \int_{[0,2\pi)^k} \underline{\Delta}_h^m f(e^{ix_1},\dots,e^{ix_k})\, e^{i\langle l,x\rangle}\, dx_1\cdots dx_k \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\int_{[0,2\pi)^k} f(e^{ix_1}h_1^j,\dots,e^{ix_k}h_k^j)\, e^{i\langle l,x\rangle}\, dx_1\cdots dx_k \\
&= \frac{1}{(2\pi)^k}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\int_{(S^1)^k} f(z_1h_1^j,\dots,z_kh_k^j)z_1^{l_1}\cdots z_k^{l_k}\, dz_1\cdots dz_k \\
&= \frac{1}{(2\pi)^k}\int_{(S^1)^k} f(z_1,\dots,z_k)z_1^{l_1}\cdots z_k^{l_k}(h_1^{-l_1}\cdots h_k^{-l_k}-1)^m\, dz_1\cdots dz_k,
\end{aligned}$$

where the integrals on $(S^1)^k$ are Haar integrals. For $l\neq 0$, by choosing $h_1,\dots,h_k$ close to 1 and such that $h_1^{l_1}\cdots h_k^{l_k}\neq 1$, we get $\hat f(l)=0$. We thus have $f=\hat f(0)$. □

9 Weak generalized polynomials for nilpotent Lie groups

In this section, we assume that G is nilpotent with order of nilpotency N. Let us recall that in this case, the lower central series is given by the sequence of subspaces

$$\mathfrak{g}\supset\mathfrak{g}^{(1)}\supset\cdots\supset\mathfrak{g}^{(N-1)}\supset\mathfrak{g}^{(N)}=\{0\},$$

with $\mathfrak{g}^{(1)}=[\mathfrak{g},\mathfrak{g}]$ and $\mathfrak{g}^{(j+1)}=[\mathfrak{g},\mathfrak{g}^{(j)}]$ for $j\in\mathbf{N}$. The group law of G is a polynomial of degree N if we look in the Lie algebra. In the sequel, we denote the group law on the Lie algebra by two reorderings of the terms of the Baker-Campbell-Hausdorff formula:

$$X*Y=\sum_{j=0}^{N-1}H_j(X,Y)=\sum_{j=1}^{N-1}\mathcal{H}_j(X,Y),$$

where H_j is a $\mathfrak{g}$-valued homogeneous ordinary polynomial of degree j with respect to X and $\mathcal{H}_j$ is a homogeneous polynomial of degree j with respect to X,Y. We denote by $M_{Y,j}$ the j-multilinear map in the variables $X_1,\dots,X_j$ on $\mathfrak{g}$ that is symmetric and such that $M_{Y,j}(X,\dots,X)=H_j(X,Y)$. By doing so, we have a family of multilinear maps $M_{Y,j}:\mathfrak{g}^j\to\mathfrak{g}$ that locally defines the group structure when restricted to the diagonal.

Before proceeding further, let us restate Equation (3). In this section, we are interested in the unmixed version. If f is the unknown function and if the equation is studied in a neighborhood of a point $x_0\in G$, we consider the function $f_{x_0}:\mathfrak{g}\to\mathbf{R}$ defined by $f_{x_0}(X)=f(x_0\exp(X))$. It is well-known that $\exp(X)\exp(Y) = \exp(X+Y)$ when $[X,Y]=0$ while $\exp(X)\exp(Y) = \exp(X*Y)$ in full generality. This implies that

$$\begin{aligned} f(x_0\exp(X)\exp(Y)^j) &= f(x_0\exp(X)\exp(jY)) \\ &= f(x_0\exp(X*jY)). \end{aligned}$$

So, for X and Y sufficiently close to 0 in $\mathfrak{g}$, Equation (3) becomes

$$\begin{aligned} 0 &= \underline{\Delta}^{m+1}_{\exp Y} f(x_0\exp(X)) \\ &= \sum_{j=0}^{m+1}(-1)^{m+1-j}\binom{m+1}{j} f(x_0\exp(X)\exp(Y)^j) \\ &= \sum_{j=0}^{m+1}(-1)^{m+1-j}\binom{m+1}{j} f(x_0\exp(X*jY)) \\ &= \sum_{j=0}^{m+1}(-1)^{m+1-j}\binom{m+1}{j} f_{x_0}(X*jY) \\ &= {}^{*}\underline{\Delta}^{m+1}_{Y} f_{x_0}(X), \end{aligned}$$

where ${}^{*}\underline{\Delta}^{m+1}_{Y}$ is the unmixed finite difference of order $m+1$ on the group $(\mathfrak{g},*,0)$. From now on, we will simply check the local equation

$$ {}^{*}\underline{\Delta}^{m+1}_{Y} f(X) = 0, \tag{7}$$

for functions f defined in a neighborhood of 0 in $\mathfrak{g}$. Note that (7) (which uses the operation $*$) is not equivalent to (3) (operation $+$ is used).

Let us first show that polynomials on $\mathfrak{g}$ provide solutions of the Fréchet functional equation (3) of sufficiently high order. We denote by R_Y the right translation by Y on $(\mathfrak{g},*,0)$.

Lemma 10. *For $k\in\{1,\dots,N\}$, we have ${}^{*}\underline{\Delta}^{k}_{Z} R_Y(X)\in\mathfrak{g}^{(k)}$, for any $X,Y,Z\in\mathfrak{g}$.*

Proof. We have $R_Y(X)=\sum_{j=1}^{N-1}H_j(X,Y)$ and thus

$$\begin{aligned} &{}^{*}\underline{\Delta}^{k}_{Z} R_Y(X) \\ &= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{m=0}^{N-1}H_m\Big(\sum_{l=0}^{N-1}H_l(X,jZ),Y\Big) \\ &= \sum_{m=0}^{N-1}\sum_{l_1,\dots,l_m\le N-1}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} M_{Y,m}(H_{l_1}(X,jZ),\dots,H_{l_m}(X,jZ)). \end{aligned}$$

We can write

$$\begin{aligned} &M_{Y,m}(H_{l_1}(X,jZ),\dots,H_{l_m}(X,jZ)) \\ &\quad = P_{Y,m,X,l_1,\dots,l_m}(jZ) + Q_{Y,m,X,l_1,\dots,l_m}(jZ), \end{aligned}$$

where $P_{Y,m,X,l_1,\dots,l_m}(jZ)$ represents the terms up to order $k-1$ with respect to jZ and $Q_{Y,m,X,l_1,\dots,l_m}(jZ)$ are the remaining terms.

Since $Q_{Y,m,X,l_1,\dots,l_m}(jZ)$ is obtained from a sum of iterated commutators of at least k times the factor jZ and one time the factor X, it belongs to $\mathfrak{g}^{(k)}$. As for $P_{Y,m,X,l_1,\dots,l_m}(jZ)$, we have

$$\underline{\Delta}_Z^k P_{Y,m,X,l_1,\dots,l_m}(0) = 0$$

where the finite difference here is the one for the group $(\mathfrak{g},+,0)$. It remains to notice that

$${}^{*}\underline{\Delta}_Z^k R_Y(X) = \sum_{m=0}^{N-1} \sum_{l_1,\dots,l_m \le N-1} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} Q_{Y,m,X,l_1,\dots,l_m}(jZ)$$

belongs to $\mathfrak{g}^{(k)}$. □

Lemma 11. *If P is additive and vanishes on $\mathfrak{g}^{(k)}$ for $k \le N$, then ${}^{*}\underline{\Delta}_Y^k P(X) = 0$. In particular, we have ${}^{*}\underline{\Delta}_Y^{N+1} P(X) = 0$ for any additive function P.*

Proof. This is trivial as we have

$${}^{*}\underline{\Delta}_Y^k P(X) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} P(X * jY) = P({}^{*}\underline{\Delta}_Y^k R_0(X)) = 0,$$

since ${}^{*}\underline{\Delta}_Y^k R_0(X)$ belongs to $\mathfrak{g}^{(k)}$. □

The previous result shows that the notion of degree for ordinary polynomials is not linked to the order of the Fréchet functional equation in a trivial way. A linear form vanishing on $\mathfrak{g}^{(k)}$ should be seen as a polynomial of degree at most $k-1$. This motivates the following notion of degree.

Definition 4. Let P be a diagonalization of a non-zero symmetric r-multiadditive mapping $T : \mathfrak{g}^r \to \mathbf{R}$. The G-degree of P is the number $r(k-1)$ where $k \in \{2,\dots,N\}$ is the lowest integer such that $T(X_1,\dots,X_{r-1},\cdot)$ vanishes on $\mathfrak{g}^{(k)}$ for all $X_1,\dots,X_{r-1} \in \mathfrak{g}$.

If P is any solution of (3), we may decompose it into a sum of functions P_j that are diagonalizations of j-multiadditive symmetric mapping for $j \in \{0,\dots,r\}$. The G-degree of P is the largest G-degree among those of $P_0,\dots,P_r$.

Remark 2. If P is a homogeneous ordinary polynomial on $\mathfrak{g}$ of degree r, then there is a unique r-multilinear symmetric mapping T on $\mathfrak{g}$ such that P is the diagonalization of T. Hence, the above definition also applies to ordinary polynomials on some Lie algebra.

Let us show that if P is a polynomial on $\mathfrak{g}$ of G-degree m, then $P \circ \log \circ L_{x_0^{-1}}$ satisfies (3) at x_0.

Proposition 12. *If P is an ordinary polynomial on $\mathfrak{g}$ of G-degree m, then we have ${}^{*}\underline{\Delta}_Y^{m+1}P(X) = 0$ for $X, Y \in \mathfrak{g}$ in a neighborhood of zero.*

Proof. By linearity, it suffices to consider homogeneous polynomials. Let P be a homogeneous ordinary polynomial of degree r with G-degree m and denote by T the associated symmetric r-form. We know that $T(X_1, \ldots, X_{r-1}, \cdot)$ vanishes on $\mathfrak{g}^{(k)}$ for all $X_1, \ldots, X_{r-1} \in \mathfrak{g}$ with $k = 1 + m/r$. We can write

$$
\begin{aligned}
&{}^{*}\underline{\Delta}_Y^{m+1}P(X) \\
&= \sum_{j=0}^{m+1} (-1)^{m-j+1} \binom{m+1}{j} T\Big(\sum_{l_1=1}^{N} \mathcal{H}_{l_1}(X, jY), \ldots, \sum_{l_r=1}^{N} \mathcal{H}_{l_r}(X, jY)\Big) \\
&= \sum_{l_1=1}^{k} \cdots \sum_{l_r=1}^{k} \sum_{j=0}^{m+1} (-1)^{m-j+1} \binom{m+1}{j} T(\mathcal{H}_{l_1}(X, jY), \ldots, \mathcal{H}_{l_r}(X, jY)).
\end{aligned}
$$

Since $T(\mathcal{H}_{l_1}(X, jY), \ldots, \mathcal{H}_{l_r}(X, jY))$ is an ordinary polynomial with degree at most $r(k-1)$ with respect to jY and

$$
\begin{aligned}
&\sum_{l_1=1}^{k} \cdots \sum_{l_r=1}^{k} \sum_{j=0}^{m+1} (-1)^{m-j+1} \binom{m+1}{j} T(\mathcal{H}_{l_1}(X, jY), \ldots, \mathcal{H}_{l_r}(X, jY)) \\
&\quad = \sum_{l_1=1}^{k} \cdots \sum_{l_r=1}^{k} \underline{\Delta}_Y^{m+1}(T(\mathcal{H}_{l_1}(X, \cdot), \ldots, \mathcal{H}_{l_r}(X, \cdot)))(0),
\end{aligned}
$$

the conclusion follows. □

In the case $N = 2$, i.e. for two-step nilpotent Lie groups, we can be more precise: the solutions are similar to the Abelian case. For example, Heisenberg groups are two-step nilpotent.

Proposition 13. *Let G be a two-step nilpotent Lie group; A function $f : G \to \mathbf{R}$ that is bounded almost everywhere in a neighborhood of x_0 is a solution to* (3) *at x_0 if and only if f_{x_0} is a polynomial of degree at most m in a neighborhood of 0. If, moreover, G is connected and simply connected, then the global result also holds.*

Proof. As the problem is local, we can assume that $G = \mathfrak{g}$ and $x_0 = 1$. It is obvious that the G-degree of a polynomial is equal to its usual degree, since any linear form vanishes on $\mathfrak{g}^{(2)} = \{0\}$. As for the global case, it is well-known that the exponential map is a diffeomorphism between G and $\mathfrak{g}$ when G is connected and simply connected so the previous argument can be used in a global fashion. □

The previous result does not remain true when replacing Equation (3) with (1), as evidenced by Example 1.

Finally, let us point out the following result that is proved the same way as above. It shows that even non-regular solutions of (4) also satisfy (3).

Proposition 14. *If $P : \mathfrak{g} \to \mathbf{R}$ is a function satisfying $\underline{\Delta}_Y^{r+1} P(X) = 0$ with G-degree m, then it is a solution of* (7).

Proof. We can follow the same proofs above since we only exploit additivity (linearity over integers). □

10 The set of generalized polynomials

We show here that the generalized polynomials, when composed with the exponential, are generated by additive functions on $\mathfrak{g}$ vanishing on $\mathfrak{g}^{(N)}$, where N is the number for which the lower central series stabilizes.

Once again, we change our perspective by looking (1) at the level of $\mathfrak{g}$. First, let us introduce the mixed difference ${}^{*}\!\Delta_Y^{m+1}$ of order $m+1$ which is simply the operator in (1) in $\mathfrak{g}$ using the operation $*$ given by $X * Y = \log(\exp(X)\exp(Y))$ for $X, Y \in \mathfrak{g}$ defined only in a neighborhood of 0. If f is some solution of (1) in a neighborhood of, say, the identity, then the function $f_{\mathfrak{g}} : \mathfrak{g} \to \mathbf{R}$ defined by $f_{\mathfrak{g}}(X) = f(\exp(X))$ satisfies

$$\begin{aligned}
0 &= \Delta_{\exp(Y)}^{m+1} f(\exp(X)) \\
&= \sum_{l=0}^{m+1} (-1)^{m+1-l} \sum_{i_1 < \cdots < i_l} f(\exp(X)\exp(Y_{i_1}) \cdots \exp(Y_{i_l})) \\
&= \sum_{l=0}^{m+1} (-1)^{m+1-l} \sum_{i_1 < \cdots < i_l} f(\exp(X * Y_{i_1} * \cdots * Y_{i_l})) \\
&= \sum_{l=0}^{m+1} (-1)^{m+1-l} \sum_{i_1 < \cdots < i_l} f_{\mathfrak{g}}(X * Y_{i_1} * \cdots * Y_{i_l}) \\
&= {}^{*}\!\Delta_Y^{m+1} f_{\mathfrak{g}}(X).
\end{aligned}$$

So, obviously, f is a local solution of (1) in a neighborhood of the identity if and only if $f_{\mathfrak{g}}$ locally satisfies

$$ {}^{*}\!\Delta_Y^{m+1} f_{\mathfrak{g}}(X) = 0, \tag{8}$$

in a neighborhood of 0 in $\mathfrak{g}$. The local solutions of (8) will be called Fréchet polynomials.

If G is not nilpotent, the lower central series is ultimately stable: there exists an order $N \in \mathbf{N}_0$ for which we have

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots \supset \mathfrak{g}^{(N)} = \mathfrak{g}^{(N+1)} = \cdots .$$

We thus have the following chain concerning their annihilator spaces:

$$\{0\} \subset (\mathfrak{g}^{(1)})^{\perp} \subset (\mathfrak{g}^{(2)})^{\perp} \subset \cdots \subset (\mathfrak{g}^{(N)})^{\perp} \subset \mathfrak{g}^{*}.$$

We adopt the following definition.

Definition 5. Given G, let $N \in \mathbf{N}_0$ be the smallest integer for which the lower central series stabilizes, i.e. such that $\mathfrak{g}^{(N)} = \mathfrak{g}^{(N+1)}$; $M(\mathfrak{g})$ (resp. $M^\infty(\mathfrak{g})$) is the space of additive (resp. linear) functions $f : \mathfrak{g} \to \mathbf{R}$ such that $\mathfrak{g}^{(N)} \subset \ker(f)$ and an element of $M(\mathfrak{g})$ (resp. $M^\infty(\mathfrak{g})$) is called a (resp. regular) fundamental monomial. The space of (resp. regular) fundamental polynomials $P(G)$ (resp. $P^\infty(G)$) is the space of germs at 1 of functions $f : G \to \mathbf{R}$ such that $f \circ \exp$ is generated by a basis of (resp. regular) fundamental monomials in a neighborhood of 0.

Obviously, since linear functions are additive, we have $M^\infty(\mathfrak{g}) \subset M(\mathfrak{g})$. Let us first show that the elements of $P(G)$ are generalized polynomials at the level of germs at the identity.

Theorem 15. *If $f \in P(G)$, then there exists $m \in \mathbf{N}$ such that f satisfies* (1) *for all x and h in a neighborhood of* 1.

Proof. Let $f \in P(G)$; without loss of generality, we may assume that $f_\mathfrak{g} = f \circ \exp$ has the form $f_1^{\alpha_1} \cdots f_k^{\alpha_k}$, with $f_1, \dots, f_k$ additive such that $\mathfrak{g}^{(N)} \subset \ker(f_i)$ ($i \in \{1, ..., k\}$) and $\alpha_1, \dots, \alpha_k \in \mathbf{N}$ with $\alpha_1 + \dots + \alpha_k = d$. Obviously, $f_\mathfrak{g}$ can be seen as the diagonalization of some multiadditive mapping

$$T : \mathfrak{g}^d \to \mathbf{R}$$

and this mapping is such that $T(X_1, \dots, X_d) = 0$ as soon as one of the entries $X_1, \dots, X_d$ belongs to $\mathfrak{g}^{(N)}$, we simply have to consider

$$T = f_1^{\otimes \alpha_1} \otimes \cdots \otimes f_k^{\otimes \alpha_k}.$$

Note that we do not necessarily require symmetry for T here, nor in what follows. We then have $f_\mathfrak{g}(X) = (T \circ \delta_d)(X)$, where $\delta_d : \mathfrak{g} \to \mathfrak{g}^d$ is the diagonal mapping. We will apply an algorithmic procedure to prove that ${}^*\!\Delta_H^{m+1} f_\mathfrak{g}(X)$ vanishes for $X, H_1, \dots, H_{m+1}$ in some neighborhood of 0 in $\mathfrak{g}$ for the appropriate choice of m. The finite difference here uses the Baker-Campbell-Hausdorff series as the group operation; we restrict ourselves to a small neighborhood of 0 to have a convergent series. The neighborhood V considered here will be sufficiently small to make sense for the products appearing in the five steps method described below.

1. Let us consider ${}^*\!\Delta_H^{dN+1} f_\mathfrak{g}(X)$ for $X, H_1, \dots, H_{dN+1} \in V$. It can be written as

$$\begin{aligned} {}^*\!\Delta_N^{dN+1} f_\mathfrak{g}(X) &= \sum_{l=0}^{dN+1} (-1)^{dN+1-l} \sum_{i_1 < \cdots < i_l} f_\mathfrak{g}(X * H_{i_1} * \cdots * H_{i_l}) \\ &= \sum_{l=0}^{dN+1} (-1)^{dN+1-l} \sum_{i_1 < \cdots < i_l} f_\mathfrak{g}(X + H_{i_1} + \cdots + H_{i_l} + R_{i_1 \cdots i_l}) \end{aligned}$$

where $R_{i_1\cdots i_l}$ are the remaining terms of the series made of iterated commutators. Moreover, the series can be truncated to the terms made of iterated Lie brackets of at most N vectors. Since T is multiadditive (hence $\mathbf{Q}$-multilinear, remember that coefficients in the Baker-Campbell-Hausdorff series are rational numbers), we can write

$$\begin{aligned}&{}^*\!\Delta_H^{dN+1} f_{\mathfrak{g}}(X)\\ &= \sum_{l=0}^{dN+1} (-1)^{dN+1-l} \sum_{i_1<\cdots<i_l} (T\circ\delta_d)(X+H_{i_1}+\cdots+H_{i_l}) + R\\ &= \Delta_H^{dN+1} f_{\mathfrak{g}}(X) + R\end{aligned}$$

where are R are the remaining terms that we do not write explicitly and Δ_H^{dN+1} is the difference operator of the group $(\mathfrak{g},+,0)$. Since $f_{\mathfrak{g}}$ is a Fréchet polynomial of ordinary degree d, we have $\Delta_H^{dN+1} f_{\mathfrak{g}}(X)=0$ and

$${}^*\!\Delta_H^{dN+1} f_{\mathfrak{g}}(X) = R.$$

Using multilinearity, we can write out the remaining terms R as

$$\sum_{k=1}^{N}\sum_{j\in J_k} (T_{k,j}^{(1)}\circ\delta_{d_j})(X), \tag{9}$$

where, given k, $T_{k,j}^{(1)}$ is some multilinear mapping obtained as a composition of T with mappings of the form

$$(Y_1,\ldots,Y_l)\mapsto [Y_1,\ldots,[\ldots,Y_2,[\ldots,[Y_l,\ldots]\cdots],$$

in some of its components such that the minimal number of bracket iterations appearing in this composition is k; we will call them terms of type k. Let us denote by d_j the number of entries in $T_{k,j}^{(1)}$. It is clear that d_j cannot exceeds dN because of the vanishing property of T on $\mathfrak{g}^{(N)}$.

2. We have

$${}^*\!\Delta_H^{dN+1} f_{\mathfrak{g}}(X) = \sum_{k=1}^{N}\sum_{j\in J_k} (T_{k,j}^{(1)}\circ\delta_{d_j})(X).$$

The elements $T_{1,j}^{(1)}$ are given by the composition of T with a single Lie bracket in one of its entries. Let us apply ${}^*\!\Delta_{H^{(1)}}^{dN+1}$ on both sides for some other $H_1^{(1)},\ldots,H_{dN+1}^{(1)}$ in V. For $k_0\in\{1,\ldots,N\}$ and $j\in J_{k_0}$, we have

$${}^*\!\Delta_{H^{(1)}}^{dN+1} (T_{k_0,j}^{(1)}\circ\delta_{d_j})(X) = \Delta_{H^{(1)}}^{dN+1} (T_{k_0,j}^{(1)}\circ\delta_{d_j})(X) + (*).$$

The first term vanishes and the remaining terms $(*)$ can be written as the same sort of sum as (9), with the sum starting at $k = k_0$. Doing this procedure to each k_0 and $j \in J_{k_0}$, we may write

$$^{*}\!\Delta_{H^{(1)}}^{dN+1}\,{}^{*}\!\Delta_{H}^{dN+1} f_{\mathfrak{g}}(X) = \sum_{k=1}^{N} \sum_{j \in J_k^{(2)}} (T_{k,j}^{(2)} \circ \delta_{d_j})(X),$$

where, this time, $T_{1,j}^{(2)}$ contains simple brackets at two distinct entries.

3. Apply Step 2 again $d-2$ more times to get

$$^{*}\!\Delta_{H^{(d-1)}}^{dN+1} \cdots {}^{*}\!\Delta_{H^{(1)}}^{dN+1}\,{}^{*}\!\Delta_{H}^{dN+1} f_{\mathfrak{g}}(X) = \sum_{k=1}^{N} \sum_{j \in J_k^{(d)}} (T_{k,j}^{(d)} \circ \delta_{d_j})(X),$$

where $T_{1,j}^{(d)}$ is a composition of T with simple brackets in all of its d entries for all $j \in J_1^{(d)}$.

By applying once again $^{*}\!\Delta_{H^{(d+1)}}^{dN+1}$, we obtain a sum of type

$$\sum_{k=2}^{N} \sum_{j \in J_k^{(d+1)}} (T_{k,j}^{(d+1)} \circ \delta_{d_j})(X),$$

since the Euclidean finite difference will cancel out $T_{1,j}^{(d)} \circ \delta_{d_j}$ for all $j \in J_1^{(d)}$ and the remaining terms will add more brackets. As all the entries have been exhausted with brackets for terms of type 1, this will only give the above sum with iteration of two brackets at least appearing in the entries of T.

4. Repeat $2(d-1)$ times Step 2 to obtain

$$\sum_{k=2}^{N} \sum_{j \in J_k^{(d+1+2(d-1))}} (T_{k,j}^{(d+1+2(d-1))} \circ \delta_{d_j})(X),$$

with $T_{2,j}^{(d+1+2(d-1))}$ given by a composition of T with double brackets on each of its component. So, applying Step 2 again gives only terms of type bigger than 2. We have a sum of type

$$\sum_{k=3}^{N} \sum_{j \in J_k^{(d+2+2(d-1))}} (T_{k,j}^{(d+2+2(d-1))} \circ \delta_{d_j})(X).$$

5. Repeat $(3(d-1)+1)+\cdots+(N(d-1)+1)$ times Step 2 to obtain 0 (using arguments from Step 4 to cancel out each term of each type).

In the end, the number of times we have to apply a finite difference of order $dN+1$ is equal to $N + \frac{1}{2}(N(N+1)(d-1))$; so, in total, we obtain

$$^*\!\Delta_H^{(dN+1)(N+\frac{N(N+1)(d-1)}{2})} f_{\mathfrak{g}}(X) = 0,$$

for all $X, H_1, \ldots, H_{(dN+1)(N+\frac{N(N+1)(d-1)}{2})}$ in a small neighborhood of 0.

□

It is clear from the previous results that the minimal order $m+1$ of the equation (1) satisfied by smooth generalized polynomials on G cannot be easily linked to the degree of those smooth functions seen as ordinary polynomials. This comes from the fact that equations

$$\Delta_Y^{m+1} f_{\mathfrak{g}}(X) = 0$$

and

$$^*\!\Delta_Y^{m+1} f_{\mathfrak{g}}(X) = 0$$

are not equivalent as soon as non-commutativity is involved. Indeed, the possibility to go from one equation to another stems from the fact that the exponential map satisfies, at least in a neighborhood of 0,

$$\exp(X)\exp(Y) = \exp(X+Y),$$

which does not always happen in the non-Abelian setting.

Let us now show that a smooth generalized polynomial belongs to $P^\infty(X)$.

Theorem 16. *If f is a smooth solution of Equation* (1) *for all x and h in a neighborhood of* 1*, then $f \in P^\infty(G)$.*

Proof. We assume that

$$\Delta_h^{m+1} f(x) = 0,$$

for all x and h in some neighborhood V of 1. Consider $X_1, \ldots, X_{m+1} \in \mathfrak{g}$. For $t_1, \ldots, t_{m+1} \in \mathbf{R} \setminus \{0\}$ sufficiently close to 0, we obtain

$$\frac{\Delta^{m+1}_{\exp(t_1 X_1), \ldots, \exp(t_{m+1} X_{m+1})} f(x)}{t_1 \cdots t_{m+1}} = 0.$$

If we successively take the limits for $t_{m+1}, \ldots, t_1 \to 0$, we obtain

$$\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_{m+1}} f(x) = 0,$$

for all $x \in V$ and all $X_1, \ldots, X_{m+1} \in \mathfrak{g}$. However, $X \in \mathfrak{g}^{(m)}$ can be written as a combination of iterated brackets of $m+1$ elements of $\mathfrak{g}$. In terms of left invariant vector field, this allows us to say that for $X \in \mathfrak{g}^{(m)}$, $\mathcal{L}_X$ is a linear combination of operators of type $\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_{m+1}}$. This implies that $\mathcal{L}_X f(x)$ vanishes for all $x \in V$ and $X \in \mathfrak{g}^{(m)}$. Since $\mathfrak{g}^{(N)} \subset \mathfrak{g}^{(m)}$, we infer that $\mathcal{L}_X f(x)$ vanishes for all $x \in V$ and all $X \in \mathfrak{g}^{(N)}$. Using Proposition 5, we see that $f \circ \exp$ is constant on $\mathfrak{g}^{(N)}$ and equal to $f(1)$, so that f belongs to $P^\infty(G)$. □

Corollary 17. *The set of germs at* 1 *of smooth generalized polynomials is equal to $P^\infty(G)$.*

11 The Fréchet functional equation on homogeneous spaces

Let M be a smooth manifold endowed with an action of a Lie group G; without loss of generality, we will assume that the action is done on the right. Given $p \in M$, G_p will denote the stabilizer of p. Using the quotient manifold theorem,

$$\pi_p : G/G_p \to M \quad [x] \mapsto px,$$

where $x, y \in G$ are equivalent for the quotient if $xy^{-1} \in G_p$, is a diffeomorphism for any $p \in M$. The natural right action of G on G/G_p is obviously a right action and π_p is equivariant. On the other hand, the projection

$$\Pi_p : G \to G/G_p \quad x \mapsto [x]$$

is an equivariant submersion. For a function $f : M \to \mathbf{R}$, one naturally defines the operator $\underline{\Delta}_h^m$ with $h \in G$ by

$$\underline{\Delta}_h^m f(p) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(ph^j),$$

for any $p \in M$. We will say that $f : M \to \mathbf{R}$ satisfies the local Fréchet functional equation of order $m+1$ at $p_0 \in M$ if

$$\underline{\Delta}_h^{m+1} f(p) = 0, \tag{10}$$

for any p in a neighborhood of p_0 and any h in a neighborhood of the identity in G.

The local solutions of (10) are obtained from the solutions of (3). Let us denote by $T_p\mathcal{M}$ the tangent space of a differentiable manifold $\mathcal{M}$ at p.

Theorem 18. *Let M be a naturally reductive Riemannian homogeneous space where the geodesics are orbits of one-parameter subgroups of G. If $f : M \to \mathbf{R}$ is a solution of* (10) *at p_0 that is bounded in a neighborhood of p_0, then there exist functions c_0 and $c_{k_1,\ldots,k_j}$ on M ($k_j \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$) that are smooth in a neighborhood of p_0 such that*

$$f(\exp_p X) = c_0(p) + \sum_{j=1}^{m} \sum_{1 \le k_1,\ldots,k_j \le n} c_{k_1,\ldots,k_j}(p) X_{k_1} \cdots X_{k_j},$$

for X in a neighborhood of 0 *in T_pM and p in a neighborhood of p_0 in M.*

Proof. One may reduce the problem to the case $M = G/H$, where H is a closed subgroup of G and $p_0 = [1]$; G is the group of isometries of M and $\Pi : G \to G/H$ will denote the projection. Let us consider the function $f_G : G \to \mathbf{R}$ defined by $f_G = f \circ \Pi$; one has

$$\underline{\Delta}_h^{m+1} f_G(x) = \underline{\Delta}_h^{m+1} f(\Pi(x)) = 0,$$

if x and h are in a neighborhood of the identity element. As a consequence, f_G is smooth in a neighborhood of 1. We may develop f_G into a Taylor polynomial; there thus exist smooth functions a_0 and $a_{k_1,\ldots,k_j}$ on a neighborhood V_1 of 1 ($j \in \{1,\ldots,m\}$) such that

$$f_G(x \exp X) = a_0(x) + \sum_{j=1}^{m} \sum_{1\leq k_1,\ldots,k_j \leq n} a_{k_1,\cdots,k_j}(x) X_{k_1} \cdots X_{k_j}.$$

Since $f_G(x \exp X) = f(Hx \exp X)$, the polynomial is independent of the representative of Hx, so that $f_G(hx \exp X) = f_G(x \exp X)$ for all $h \in H$ and all X in a neighborhood of 0 in $\mathfrak{g}$. The functions a_0 and $a_{k_1,\ldots,k_j}$ thus induce functions c_0 and $c_{k_1,\ldots,k_j}$ on G/H, so that we have

$$f(p \exp X) = c_0(p) + \sum_{j=1}^{m} \sum_{1\leq k_1,\ldots,k_j \leq n} c_{k_1,\ldots,k_j}(p) X_{k_1} \cdots X_{k_j},$$

for p in a neighborhood of p_0 and X in a neighborhood of 0 in $\mathfrak{g}$. As f is smooth, so are the coefficients. Since it is also reductive, there exists a subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ (where Ad denotes the adjoint representation of the group) and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. We also have the isomorphism $T_{p_0}M \simeq \mathfrak{m}$ induced by the projection $d\Pi_{p_0} : \mathfrak{g} \to T_H(G/H)$. Since the Riemannian exponential at p_0 is

$$X \mapsto p_0 \exp((d\Pi_{p_0})|_{\mathfrak{m}}^{-1} X),$$

for $X \in T_{p_0}M$, it suffices to write $p \exp X$ in terms of Riemannian exponential for $X \in \mathfrak{m}$. □

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