Polytopes of Absolutely Wigner Bounded Spin States

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Background

Wigner function of a spin state

The SU(2) Wigner kernel of a spin-j system is

$$\Delta : S^2 \to \mathcal{L}(\mathcal{H})$$
$$\Delta(\Omega) = \sqrt{\frac{4\pi}{2j+1}} \sum_{L=0}^{2j} \sum_{M=-L}^{L} Y_{LM}^*(\Omega) T_{LM},$$

where $\Omega = (\theta, \phi) \in S^2$, $Y_{LM}(\Omega)$ are the spherical harmonics, and T_{LM} are the spherical tensor operators associated with spin j.

Wigner function of a spin state ρ: $\boxed{W_\rho(\Omega) = \mathrm{Tr}\left[\rho\Delta(\Omega)\right]}$

Normalization: $\frac{2j+1}{4\pi} \int_{S^2} W_\rho(\Omega)\, d\Omega = 1$

SU(2) covariance: $W_{U_g \rho U_g^\dagger}(\Omega) = W_\rho(g^{-1}\Omega)$

Absolutely Wigner bounded (AWB) states

Definition: A spin-j state ρ is AWB with respect to $W_{\min}$ if the Wigner function of each state unitarily connected to ρ is lower bounded by $W_{\min}$:

$$W_{U\rho U^\dagger}(\Omega) \geq W_{\min} \qquad \begin{array}{l} \forall\, \Omega \in S^2 \\ \forall\, U \in \mathrm{SU}(2j+1). \end{array}$$

When $W_{\min} = 0$ we refer to such states as absolutely Wigner positive (AWP). Hence, an AWP state has only non-negative Wigner function states in its unitary orbit.

Polytopes of AWB states

Proposition: Let $\mathbf{\Delta}^{\uparrow}$ denote the vector of kernel eigenvalues sorted into increasing order. Then a spin state ρ is AWB iff its decreasingly ordered eigenvalues $\boldsymbol{\lambda}^{\downarrow}$ satisfy the following inequality

$$\boxed{\sum_{i=0}^{2j} \lambda_i^{\downarrow} \Delta_i^{\uparrow} \geq W_{\min}}$$

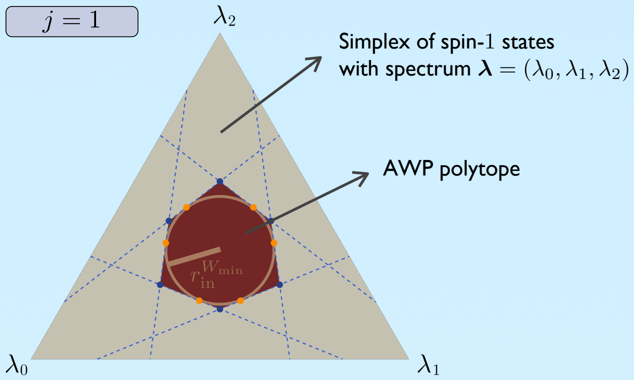
Balls of AWB states

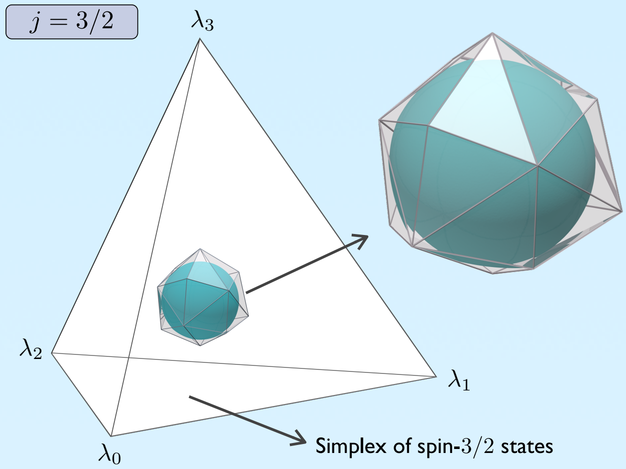
Hilbert-Schmidt distance between a state ρ and the maximally mixed state ρ_0: $r(\rho) = \|\rho - \rho_0\|_{\mathrm{HS}} = \sqrt{\mathrm{Tr}\left[(\rho - \rho_0)^2\right]}$

Proposition: The radius of the largest inner ball of the AWB polytope associated with a $W_{\min}$ value such that the ball is contained within the state simplex is

$$\boxed{r_{\mathrm{in}}^{W_{\min}} = \frac{1 - (2j+1)W_{\min}}{2\sqrt{j(2j+1)(j+1)}}}$$

Low spin quantum numbers





Relationship with entanglement

$$\rho = \frac{2j+1}{4\pi} \int P_\rho(\Omega)\, |\Omega\rangle\langle\Omega|\, d\Omega \quad \text{with } |\Omega\rangle \text{ a spin-coherent state}$$

($P_\rho(\Omega)$: Glauber-Sudarshan P-function)

Maximal negativity in the unitary orbit of a two-qubit symmetric (or equivalently a spin-1) state ρ with spectrum $\lambda_0 \geq \lambda_1 \geq \lambda_2$:

$$\max_{U \in SU(3)} \mathcal{N}(\rho) = \max\left[0, \sqrt{\lambda_0^2 + (\lambda_1 - \lambda_2)^2} - \lambda_1 - \lambda_2\right]$$