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QuantOM

GRAPH AND OPTIMIZATION MODELS FOR KIDNEY EXCHANGES

joint work with Marie BARATTO, Valentin BARTIER, João Pedro PEDROSO, Bart SMEULDERS, Frits SPIEKSMA, Ana VIANA

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Outline

Outline of presentation:

1. Optimization of kidney exchanges
2. Kidney exchanges with uncertainty
3. Cycle selections
4. Stable exchanges

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Note: Main stress on original graph and optimization models!

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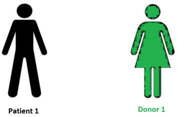
Kidney exchanges

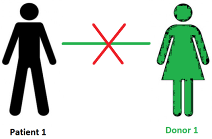
Patient with a serious kidney disease may resort to:

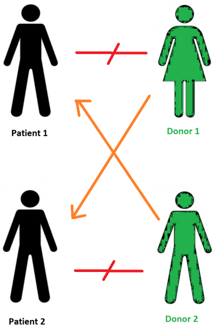
- Dialysis
- Transplant from a deceased donor
- Transplant from a willing donor



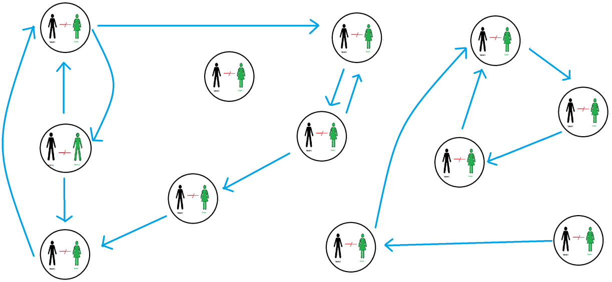
Patient might not be compatible with the donor: e.g.,

- Blood incompatibility
- Tissue type incompatibility

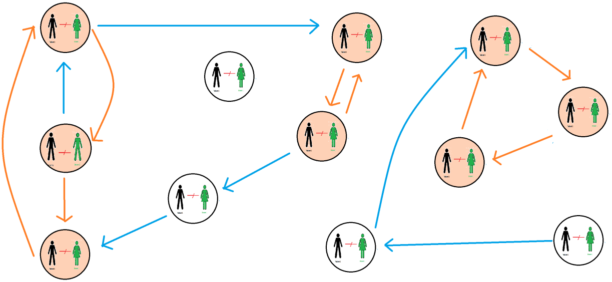



Patient 1
Donor 1
Patient 2
Donor 2

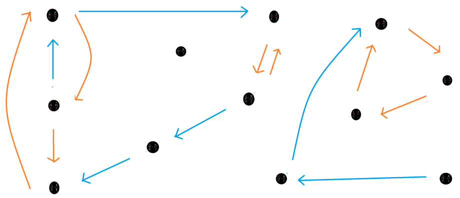
Pool of pairs



Feasible exchange



Compatibility graph



$G = (V, A)$ where:

- $V = \{1, ..., n\}$: set of vertices representing all patient-donor pairs.
- A: set of arcs representing compatibilities between the pairs: $(i, j) \in A$ if the donor in pair i is compatible with the patient in pair j.

Kidney exchange problems

Definition

Given a digraph $G = (V, A)$, an *exchange* is a set of vertex-disjoint cycles in G (i.e., a cycle packing).

- Variants: paths may be allowed, and/or long cycles may be ruled out.

Kidney exchange problems

Definition

Given a digraph $G = (V, A)$, an *exchange* is a set of vertex-disjoint cycles in G (i.e., a cycle packing).

- Variants: paths may be allowed, and/or long cycles may be ruled out.
- Assume *weight function* w on A, where $w_{i,j}$ represents the "quality" of a transplant between donor i and patient j, for each arc $(i, j) \in A$.

Definition

Given a digraph $G = (V, A, w)$, the *kidney exchange problem* is to find an exchange of maximum total weight in G.

- Intensively studied since early 2000's (including 2012 Nobel-prize work by A.E. Roth).
- NP-hard even if $w_{i,j} = 1$ for each $(i, j) \in A$. Polynomial when restricted to cycles of length 2 or when length is unrestricted.
- Several ILP formulations, efficiently solved by branch-and-price.

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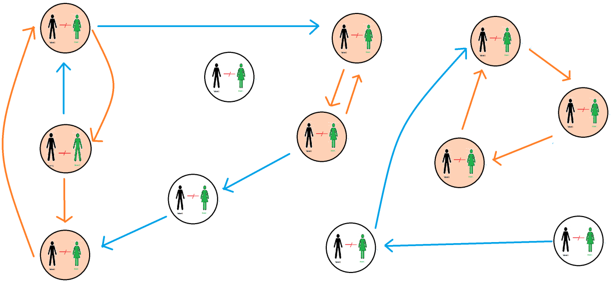
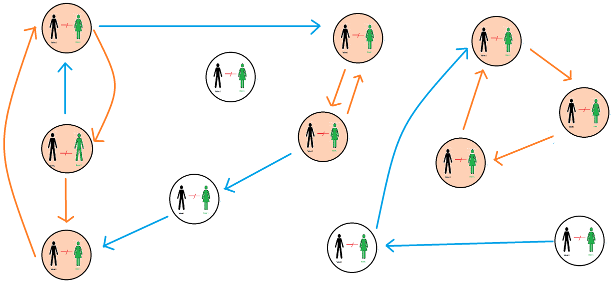
Uncertainty



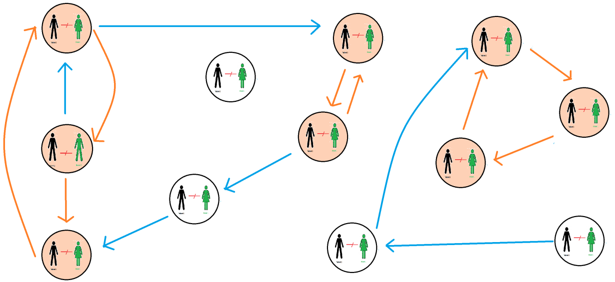
- Arc $(i, j) \in A$ if i and j are initially assumed to be compatible based on preliminary tests.

Uncertainty



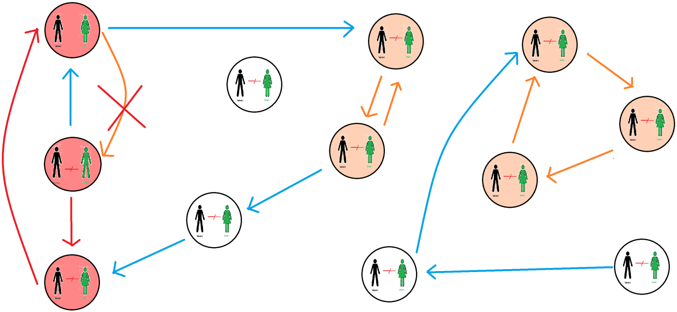
- Arc $(i, j) \in A$ if i and j are initially assumed to be compatible based on preliminary tests.
- After an exchange cycle has been identified, more elaborate testing (*crossmatching*) is carried out, and i, j may turn out to be incompatible!

Uncertainty



- Arc $(i, j) \in A$ if i and j are initially assumed to be compatible based on preliminary tests.
- After an exchange cycle has been identified, more elaborate testing (*crossmatching*) is carried out, and i, j may turn out to be incompatible!
- As a result, all transplants associated with the cycle may become infeasible after crossmatching.

Uncertainty



Robust exchanges

- Assume: probability p_{ij} that arc (i, j) is feasible (independently).
- Various models by different authors:
  - maximize expected weight of the exchange (Dickerson et al. 2013, 2018; no recourse);

Robust exchanges

- Assume: probability p_{ij} that arc (i, j) is feasible (independently).
- Various models by different authors:
  - maximize expected weight of the exchange (Dickerson et al. 2013, 2018; no recourse);
  - internal recourse: identify an exchange; if cycle C fails, allow for reoptimizing over the vertices involved in C (Pedroso 2014);
  - subset recourse: identify small, disjoint subsets of vertices; after crossmatching, optimize over the remaining feasible arcs (Klimentova et al. 2016).

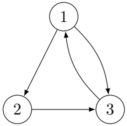


Figure: Internal recourse.

Two-stage stochastic model

Smeulders, Bartier, Crama, Spieksma, Recourse in kidney exchange programs, *INFORMS Journal on Computing* 34 (2022) 1191-1206.

Generic two-stage framework:

- *(Selection)* Select subset of arcs $B \subseteq A$ for crossmatching.
- *(Testing)* Test arcs in B. Say, $R \subseteq B$ pass the crossmatching test.
- *(Recourse)* Solve the kidney exchange problem on $G_R = (V, R)$.

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Smeulders at al. apply the framework with a budget constraint b on the size of B.

Two-stage stochastic model

Resulting two-stage stochastic programming model:

$$\max \sum_{s \in \mathcal{S}} q_s \, weight(x_s) \tag{1}$$

$$\text{subject to} \sum_{(i,j) \in A} \beta_{i,j} \leq b \tag{2}$$

$$x_s \in P(G_{s,\beta}) \qquad \forall s \in \mathcal{S} \tag{3}$$

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j) \in A \tag{4}$$

where

- $\beta_{i,j}$: variables identifying the arcs to be crossmatch-tested;
- $\mathcal{S}$: set of all possible scenarios, i.e., all subsets of arcs;
- q_s: probability of scenario s;
- x_s: exchange implemented under scenario s;
- $P(G_{s,\beta})$: ILP description of the set of feasible kidney exchanges in the graph restricted to arcs identified by $\beta_{i,j}$ and feasible under scenario s.

Two-stage stochastic model

Main results:

- Different variants are NP-hard (even when cycles are restricted to length 2).
- Approximation: restrict to a random subset of scenarios.
- Different solution approaches are tested:
  - CPLEX branch-and-cut, with or without relaxed exchange variables (x_s);
  - CPLEX Benders decomposition (fix $\beta_{i,j}$, solve slave kidney exchange problems).
- Experimental observations:
  - Model is difficult to solve.
  - Expected number of transplants slightly better than with earlier models.

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Two-stage stochastic model

$$\max \sum_{s \in \mathcal{S}} q_s \, weight(x_s) \tag{5}$$

$$\text{subject to} \sum_{(i,j) \in A} \beta_{i,j} \leq b \tag{6}$$

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- Strengthening (Smeulders et al. 2022): express that we are only interested in those subsets of arcs such that each arc is in at least one directed cycle.

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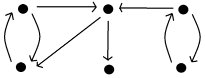
- Benders decomposition: when constraints (7) are omitted, variables $\beta_{i,j}$ may select any subset of b arcs.
- Strengthening (Smeulders et al. 2022): express that we are only interested in those subsets of arcs such that each arc is in at least one directed cycle.
- Leads to the next concepts and research questions (Baratto and Crama, Cycle selections, *Discrete Applied Mathematics* 335 (2023) 4-24).

Cycle selections

Consider a directed graph $G = (V, A)$.



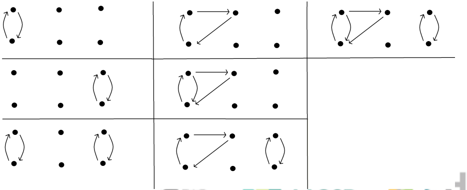
Definition

A *cycle selection* in $G = (V, A)$ is

- a subset $B \subseteq A$ such that, in $G_B = (V, B)$, each arc is in a directed cycle,

or equivalently,

- a union of directed cycles of G.



Cycle selection problem

Definition

The **Maximum Weighted Cycle Selection** (**MWCS**) problem is defined as follows: given a directed graph $G = (V, A)$ and a weight $w_{ij} \in \mathbb{R}$ for each arc $(i, j) \in A$, find a cycle selection B which maximizes $w(B) = \sum_{(i,j) \in B} w_{ij}$.

Objectives:

- Complexity of the maximum weighted cycle selection (MWCS) problem.
- Formulations of the problem.
- Polyhedral study of the convex hull of the set of cycle selections.
- Numerical solution of the MWCS problem.

Complexity of the MWCS problem

Note: the maximization problem is trivial

- when $w_{ij} \geq 0$ for all $(i, j) \in A$,
- when restricted to cycles of length 2.

Complexity of the MWCS problem

Note: the maximization problem is trivial

- when $w_{ij} \geq 0$ for all $(i, j) \in A$,
- when restricted to cycles of length 2.

But in general:

Theorem

The MWCS problem is strongly NP-hard even when restricted to cycles of length at most 3.

Objectives

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Formulations

Baratto and Crama (2023) propose several MILP formulations, including:

Exponential formulations:

- Arc formulation
- Cycle formulation

Extended compact formulations:

- Modified extended arc formulation
- Position indexed formulation

Arc formulation

Variables

$\beta_{i,j} = 1$ if arc (i,j) is selected, 0 otherwise, for all $(i,j) \in A$.

Objective function

$$\max \sum_{(i,j)\in A} w_{i,j}\beta_{i,j}$$

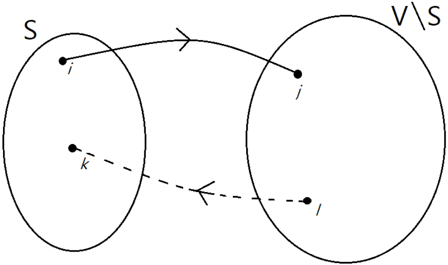
Constraints

$$\beta_{i,j} \leq \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k} \qquad \forall (i,j) \in A, \forall S \subseteq V : i \in S, j \in V \setminus S$$

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j) \in A.$$

Return inequalities:

$$\beta_{i,j} \leq \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k} \qquad \forall (i,j) \in A, \forall S \subseteq V : i \in S, j \in V \setminus S$$



Cycle formulation

Variables:

- $\beta_{i,j} = 1$ if arc (i,j) is selected, 0 otherwise, for all $(i,j) \in A$.
- $x_c = 1$ if cycle c is selected, 0 otherwise, for all $c \in \Gamma$, where Γ is the set of all cycles in G.

Constraints:

$$x_c \leq \beta_{i,j} \qquad \forall c \in \Gamma, \forall (i,j) \in c$$

$$\beta_{i,j} \leq \sum_{c \in \Gamma : (i,j) \in c} x_c \qquad \forall (i,j) \in A$$

$$x_c \in \{0,1\} \qquad \forall c \in \Gamma$$

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j) \in A.$$

Modified extended arc formulation

- View a selection B as a union of exchanges (cycle packings) $C^1, \ldots, C^L$: $B = \cup_{\ell=1}^{L} C^\ell$.
- We view each C^ℓ as the support of a binary *circulation*.
- Since each arc $(u,v) \in B$ is contained in one of the exchanges C^ℓ, we need at most $|A|$ representative exchanges $C^{(u,v)}$, $(u,v) \in A$.

Variables:

$$x_{i,j}^{(u,v)} = 1 \text{ if } (i,j) \in C^{(u,v)}, \qquad \forall (i,j) \in A, \forall (u,v) \in A$$

Constraints:

$$x_{i,j}^{(u,v)} \le x_{i,j}^{(i,j)} = \beta_{i,j} \qquad \forall (i,j) \in A, \forall (u,v) \in A$$

$$\sum_{h:(h,i)\in A} x_{h,i}^{(u,v)} = \sum_{h:(i,h)\in A} x_{i,h}^{(u,v)} \le 1 \qquad \forall i \in V, \forall (u,v) \in A$$

$$x_{i,j}^{(u,v)} \in \{0,1\} \qquad \forall (i,j) \in A, \forall (u,v) \in A$$

Position indexed formulation

Variables:

- $\phi^{\ell}_{i,j,k}$ = 1 if arc (i,j) is in position k in a cycle in graph copy G^{ℓ}, 0 otherwise.
  $$\forall (i,j) \in A, \ell \in V, k \in \kappa(i,j,\ell) \text{ where } \kappa(i,j,\ell) = \begin{cases} \{1\} \text{ if } i = l \\ \{2,\ldots,n\} \text{ if } j = l \\ \{2,\ldots,n-1\} \text{ if } i,j > l \end{cases}$$
- $\beta_{i,j}$ = 1 if arc (i,j) is selected, 0 otherwise. $\qquad \forall (i,j) \in A$

Constraints:

$$\beta_{i,j} \leq \sum_{\ell \in V} \sum_{k \in \lambda(i,j,\ell)} \phi^{\ell}_{i,j,k} \qquad \forall (i,j) \in A$$

$$\phi^{\ell}_{i,j,k} \leq \beta_{i,j} \qquad \forall \ell \in V, (i,j) \in A^{\ell}, k \in \kappa(i,j,\ell)$$

$$\phi^{\ell}_{i,j,k} \leq \sum_{h:(h,i) \in A^{\ell} \wedge k-1 \in \kappa(h,i,l)} \phi^{\ell}_{h,i,k-1} \qquad \forall \ell \in V, (i,j) \in A^{\ell}, k \in \kappa(i,j,\ell), k > 1$$

$$\phi^{\ell}_{i,j,k} \leq \sum_{h:(j,h) \in A^{\ell} \wedge k+1 \in \kappa(j,h,l)} \phi^{\ell}_{j,h,k+1} \qquad \forall \ell \in V, (i,j) \in A^{\ell}, j \neq l, k \in \kappa(i,j,\ell)$$

$$\phi^{\ell}_{i,j,k} \in \{0,1\}, \beta_{i,j} \in \{0,1\} \qquad \forall \ell \in V, (i,j) \in A^{\ell}, k \in \kappa(i,j,\ell)$$

Relative strength of the formulations

Relation between the linear relaxations of the formulations:

Theorem

- Arc formulation = projection of the Modified extended arc formulation.
- Arc formulation $\subseteq$ projection of the Cycle formulation.
- Arc formulation $\subseteq$ projection of the Position indexed formulation.
- The inclusions are strict for complete digraphs.

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- The inclusions are strict for complete digraphs.

Theorem

The return inequalities can be separated in polynomial time, and hence the linear relaxation of the arc formulation can be solved in polynomial time.

$\rightarrow$ Focus on the arc formulation

Objectives

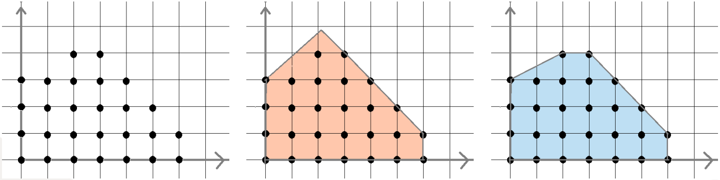
- Complexity of the maximum weighted cycle selection (MWCS) problem. √
- Formulations of the problem. √
- Polyhedral study of the convex hull of the set of cycle selections.
- Numerical solution of the MWCS problem.

Notation

$$S = \left\{ \beta \in \{0,1\}^{|A|} : \beta_{i,j} \leq \sum_{(l,k) \in A: l \in V \setminus S, k \in S} \beta_{l,k} \; \forall S \subseteq V, \; \forall (i,j) \in A : i \in S, j \in V \setminus S \right\}$$

$$SL = \left\{ \beta \in [0,1]^{|A|} : \beta_{i,j} \leq \sum_{(l,k) \in A: l \in V \setminus S, k \in S} \beta_{l,k} \; \forall S \subseteq V, \; \forall (i,j) \in A : i \in S, j \in V \setminus S \right\}$$

$$S^* = conv(S)$$



Polyhedral study

For a complete directed graph on n vertices:

Theorem

S^* is full dimensional: $\dim(S^*) = n(n-1)$ when $n \geq 3$.

Theorem

The trivial inequalities and return inequalities are facet-defining for S^*

$$\beta_{i,j} \leq 1 \qquad \forall (i,j) \in A$$

$$\beta_{i,j} \geq 0 \qquad \forall (i,j) \in A$$

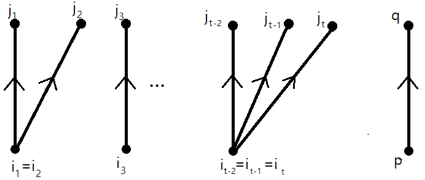
$$\beta_{i,j} \leq \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k} \qquad \forall S \subseteq V,\ \forall (i,j) \in A : i \in S, j \in V \setminus S$$

Additional facet defining inequalities

- Let $E = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\}$ be a subset of arcs, and let $I = \{i_1, i_2, \ldots, i_t\}$, $J = \{j_1, j_2, \ldots, j_t\}$. Assume that $I \cap J = \emptyset$ and $|I| \leq |J| = t$. Let p and q be two distinct vertices not in $I \cup J$. We define the *out-star inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l, j_l} + \beta_{p,q} \leq \sum_{k \in V \setminus I} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k,p}, \quad (9)$$

$$\sum_{l=1}^{t} \beta_{i_l, j_l} + \beta_{p,q} \leq \sum_{k \in V \setminus I} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{q,k}. \quad (10)$$



Theorem

The out-star inequalities are facet defining for S^*.

- Symmetrically, if we assume that that $|J| \leq |I| = t$, we define the *in-star inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V \setminus J} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k,p}, \tag{11}$$

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V \setminus J} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{q,k}. \tag{12}$$

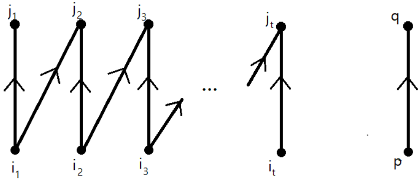
Theorem

The in-star inequalities are facet defining for S^*.

Additional facet defining inequalities

- Let $I = \{i_1, i_2, \ldots, i_t\}$ and $J = \{j_1, j_2, \ldots, j_t\}$ be two subsets of vertices with $I \cap J = \emptyset$ and $|I| = |J| = t$. Let p and q be two distinct vertices not in $I \cup J$, we define the *path inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l, j_l} + \sum_{l=1}^{t-1} \beta_{i_l, j_{l+1}} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k,p}. \tag{13}$$



Theorem

The path inequalities are facet defining for S^*.

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Numerical tests

Maximum weigthed cycle selection

$$\max \sum_{(i,j)\in A} w_{i,j}\beta_{i,j}$$

$$\text{s.t.} \quad \beta_{i,j} \leq \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k} \qquad \forall S \subseteq V,\ \forall (i,j)\in A : i\in S, j\in V\setminus S$$

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j)\in A$$

Relaxation

$$\max \sum_{(i,j)\in A} w_{i,j}\beta_{i,j}$$

$$\text{s.t.} \quad \beta_{i,j} \leq \sum_{k\in V, k\neq i} \beta_{k,i} \qquad \forall (i,j)\in A$$

$$\beta_{i,j} \leq \sum_{k\in V, k\neq j} \beta_{j,k} \qquad \forall (i,j)\in A$$

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j)\in A$$

Numerical tests

Tests on random graphs.

Density	[5,10,20,50,70]
Number of vertices	[50,100,150,200,250,300]

- d% of arcs have a positive weight, uniform in $[0,1]$; $(100-d)$% of arcs have a negative weight, uniform in $[-1,0]$
- Also tested: (random) kidney exchange compatibility digraphs

Numerical tests

- Natural formulation is very strong on random and compatibility graphs.
- No additional cuts are needed.

Numerical tests

- Natural formulation is very strong on random and compatibility graphs.
- No additional cuts are needed.
- **Note**: with high probability (almost surely), each positive arc is contained in a strongly connected component.
- More difficult instances arise when a constraint is placed on the number of arcs in the selection.
- Embedding in the two-stage stochastic kidney exchange problem.

Outline

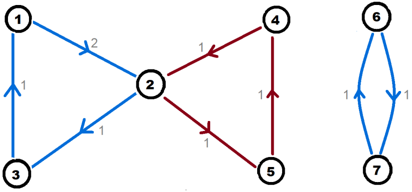
Outline of presentation:

1. Optimization of kidney exchanges
2. Kidney exchanges with uncertainty
3. Cycle selections
4. **Stable exchanges**

Stable exchanges

For each vertex $i \in V$, a **preference ranking** r_i is given on the set of its in-neighbors $N^-(i) := \{j : (j,i) \in A\}$:

$$r_i(j) < r_i(k) \text{ if and only if patient } i \text{ prefers donor } j \text{ to donor } k.$$

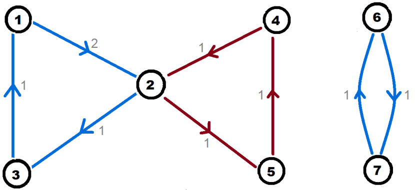


Stable exchanges

Definition

A **blocking cycle** u for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(u)$, i prefers u to $\mathcal{M}$.
We say that vertex i *prefers* the cycle u to the exchange $\mathcal{M}$ if either

- $i \notin V(\mathcal{M})$, or
- $i \in V(\mathcal{M})$, $(k, i) \in A(u)$, $(k', i) \in A(\mathcal{M})$, and i prefers k to k'.



Definitions - Stability

Definition

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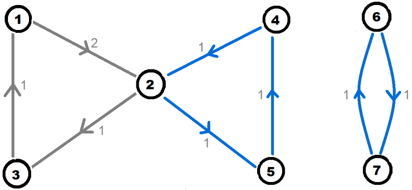
Definition

Given a directed graph $G = (V, A)$, an exchange $\mathcal{M}$ is called **stable** if there is no blocking cycle for $\mathcal{M}$.



Stable exchanges - Weaknesses

Note: Not every digraph has a stable exchange: demanding solution concept! Due to strong definition of blocking cycles.

Definition

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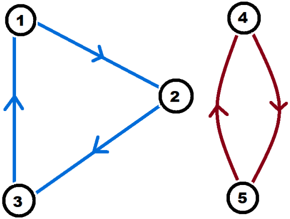


Does the red cycle actually block the blue one?

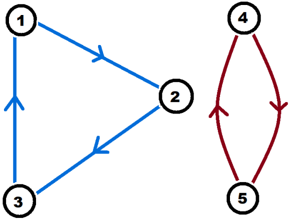
Locally stable exchanges

Definition

A **locally blocking cycle** u for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ **but has a vertex in common with** $\mathcal{M}$ and such that, for every vertex $i \in V(u)$, i prefers u to $\mathcal{M}$.

Definition

Given a digraph $G = (V, A)$, an exchange $\mathcal{M}$ is called **locally stable (L-stable)** if there is no L-blocking cycle for $\mathcal{M}$.



What is new here?

Stability

- Classical concept: stable marriage problem, stable roommate problem

vs.

Local stability

- Every stable exchange is locally stable.
- Local stability has not been previously studied?
- In the context of KE, local stability seems more relevant than stability
- Graph theoretic interpretations
- Integer programming formulation(s) for maximum L-stable exchange
- Comparison with stability

Baratto, Crama, Pedroso and Viana, Local stability in kidney exchange programs, Working paper, ULiège, 2023.

More on this topic: Marie, **Tuesday 11:45am**.

Blocking digraph

Definition

Let $\mathcal{C}_K(G)$ be the set of directed cycles of length at most K in the compatibility digraph $G = (V, A)$.

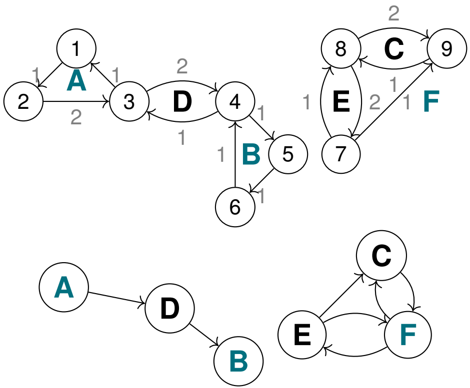
Definition

Define a **blocking digraph** $G' = (V', A')$ such that:

- $V' = \mathcal{C}_K(G)$.
- The arc $(u, v) \in A'$ if
  - either v blocks u,
  - or v does not block u and u does not block v.

Blocking digraph

Example: with $K = 3$.



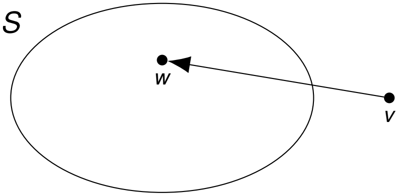
Remark: G' is an orientation of the intersection graph of $\mathcal{C}_K(G)$ (cycles of length at most K in G).

Kernel

Definition

$S \subseteq V'$ is a **kernel** of $G' = (V', A')$ if S is **independent** and **absorbing**:

- for all $(u, v) \in A'$ either $u \notin S$ or $v \notin S$ (independent)
- for every $v \notin S$ there exists $w \in S$ such that $(v, w) \in A'$ (absorbing)



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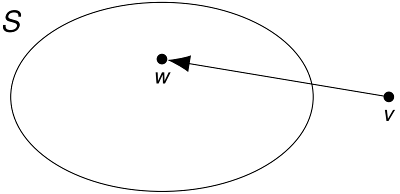

Classical concept: von Neumann and Morgenstern (1953).

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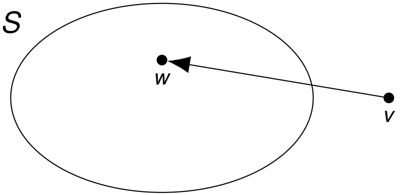

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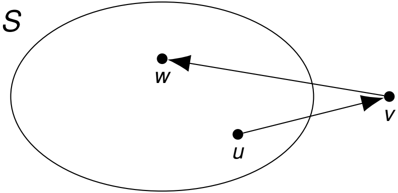
Theorem

The stable exchanges of G are exactly the kernels of G'.

Local kernel

Definition

$S \subseteq V'$ is a **local kernel** of $G' = (V', A')$ if S is independent and if every neighbor of S is absorbed by S.



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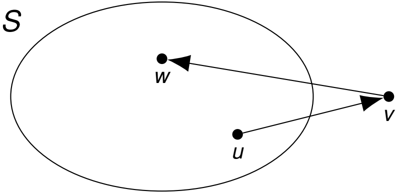

Not so well-known: Galeana-Sánchez and Neumann-Lara (1984), Duchet and Meyniel (1993).

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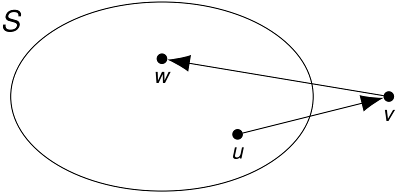

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Theorem

The locally stable exchanges of G are exactly the local kernels of G'.

Local kernels

A few facts:

- Every kernel is a local kernel.
- Some digraphs have no kernel.
- The empty set $S = \emptyset$ is a local kernel. So, every digraph has a local kernel (possibly empty).
- The cardinality of a maximum local kernel is at least as large as the cardinality of a maximum kernel, and it can be strictly larger.

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Deciding whether a digraph has a nonempty local kernel is NP-complete.

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- How different are they from kernels?

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- How difficult is it, in practice, to compute maximum local kernels?
- How different are they from kernels?
- Answers: Tuesday 11:45am.

Conclusions

- Kidney exchange programmes save lives and raise interesting mathematical questions.
- Many connections with classical graph-theoretic concepts, but also with new or more confidential ones: cycle selections, local kernels, intersection graphs of cycles.
- Hard computational challenges remain...

References

B. Smeulders, V. Bartier, Y. Crama, F.C.R. Spieksma, Recourse in kidney exchange programs, *INFORMS Journal on Computing* 34 (2022) 1191-1206.

M. Baratto and Y. Crama, Cycle selections, *Discrete Applied Mathematics* 335 (2023) 4-24.

M. Baratto, Y. Crama, J.P. Pedroso and A. Viana, Local stability in kidney exchange programs, Working paper, ULiège, 2023.