Alternate Base Numeration Systems*

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Abstract. Alternate base numeration systems generalize real base numeration systems as defined by Renyi. Real numbers are represented using a finite number of bases periodically. Such systems naturally appear when considering linear numeration systems without a dominant root. As it happens, many classical results generalize to these numeration systems with multiple bases but some don't. This is a survey of the work done so far concerning combinatorial, algebraic and dynamical aspects. This study has been led in collaboration with several co-authors : Célia Cisternino, Karma Dajani, Savinien Kreczman, Zuzana Masáková and Edita Pelantová.

Keywords: Cantor real bases · Alternate bases · Subshifts · Parry numbers · Pisot numbers · Automata · Normalization · Dynamical systems · Ergodic theory

1 Introduction

Expansions of nonnegative real numbers x with respect to a real base $\beta > 1$ are sequences of integer digits $(a_n)_{n\geq 0}$ such that $x = \sum_{n=0}^{\infty} \frac{a_n}{\beta^{n+1}}$. A distinguished expansion of a given $x \in [0, 1]$, denoted $d_\beta(x) = (d_n)_{n\geq 0}$ and called the *β-expansion* of x, is computed by the greedy algorithm: set $r_0 = x$ and for all $n \geq 0$, let $d_n = \lfloor \beta r_n \rfloor$ and $r_{n+1} = \beta r_n - d_n$. These numeration systems, introduced by Rényi [30], are extensively studied under various points of view and we can only cite a few of the many possible references [3,15,27,29,33].

In parallel, other numeration systems are also widely studied, this time to represent nonnegative integers. We choose an increasing integer sequence $U = (U_n)_{n\geq 0}$ such that $U_0 = 1$ and the quotients between consecutive terms $\frac{U_{n+1}}{U_n}$ are bounded. A nonnegative integer x is then represented by a finite sequence of integer digits $a_0 \cdots a_{\ell-1}$ such that $x = \sum_{n=0}^{\ell-1} a_n U_{\ell-1-n}$. Again, we distinguish the expansion that is obtained thanks to the greedy algorithm: let $\ell \geq 0$ be maximal such that $x < U_\ell$ and set $r_0 = x$; then for $n \in \{0, \ldots, \ell - 1\}$, set $d_n = \lfloor \frac{r_n}{U_{\ell-1-n}} \rfloor$ and $r_{n+1} = r_n - d_n U_{\ell-1-n}$. The so-obtained expansion $d_0 \cdots d_{\ell-1}$ is called the *U-expansion* of x. Similarly, the literature about U-expansions of nonnegative integers is vast, see [4,5,12,13,18,21,26] for the most topic-related ones.

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There exists an intimate link between β-expansions and U-expansions. Its study goes back to the work [4] of Bertrand-Mathis. The case where the base sequence U has a dominant root $\beta > 1$ is quite well understood [21]. More precisely, we say that U has the dominant root $\beta > 1$ whenever $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \beta$. In particular, for sufficiently large n, the U-expansions of $U_n - 1$ share long common prefixes with specific expansions of 1 with respect to the real base β. In the case where the base sequence U has no dominant root, a similar phenomenon occurs with respect to expansions of 1 in a numeration system given by an *alternate base* $(\beta_0, \dots, \beta_{p-1})$ associated with U for some well defined $p \geq 1$: for each $i \in \{0, \dots, p-1\}$, we have $\beta_i = \lim_{n\to\infty} \frac{U_{np-i}}{U_{np-i-1}}$. This discovery was the original motivation for the study of alternate base expansions of real numbers. It turns out that a lot of classical results concerning β-expansions of real numbers extend to this new framework, and sometimes, to the even more general framework of *Cantor real bases*. The purpose of this survey is to give an overview of the results obtained so far in these generalized numeration systems. The study of linear numeration systems without a dominant root will be treated separately, in a subsequent paper.

2 Cantor real bases and alternate bases

Cantor expansions of real numbers were originally introduced by Cantor in 1869 [7]. A real number $x \in [0, 1)$ is represented via a base sequence $(b_n)_{n\geq 0}$ of integers greater than or equal to 2 as follows:

$$x = \sum_{n=0}^{\infty} \frac{a_n}{\prod_{k=0}^{n} b_k}$$

where for each $n \geq 0$, the digit a_n belongs to $\{0, \dots, b_n - 1\}$. Many studies are devoted to Cantor series; see [16,20,22,31] to cite just a few.

In [8], we introduced series expansions of real numbers that are based on a sequence $\boldsymbol{\beta} = (\beta_n)_{n\geq 0}$ of real numbers greater than 1. We call such a base sequence $\boldsymbol{\beta}$ a Cantor real base, and we talk about $\boldsymbol{\beta}$-expansions. In doing so, we generalize both Cantor series and real base expansions. The same framework was introduced simultaneously in [6]. Moreover, other kinds of expansions with multiple real bases were also recently studied, see [23,25,28].

2.1 Cantor real bases

Let $\boldsymbol{\beta} = (\beta_n)_{n\geq 0}$ be a sequence of real numbers greater than 1 such that $\prod_{n=0}^{\infty} \beta_n = \infty$. We call such a sequence $\boldsymbol{\beta}$ a *Cantor real base*. We define the *$\boldsymbol{\beta}$-value (partial) map* $\mathrm{val}_{\boldsymbol{\beta}} \colon (\mathbb{R}_{\geq 0})^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{val}_{\boldsymbol{\beta}}(a) = \sum_{n=0}^{\infty} \frac{a_n}{\prod_{k=0}^{n} \beta_k} \tag{1}$$

for any sequence $a = (a_n)_{n\ge 0}$ over $\mathbb{R}_{\ge 0}$, provided that the series converges. If $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$ then we say that a is an *expansion* of x in base $\boldsymbol{\beta}$. By taking $\beta_n = \beta$ for all $n \ge 0$, we recover Rényi expansions [30].

We will need to represent real numbers not only in a fixed Cantor real base $\boldsymbol{\beta}$ but also in all Cantor real bases obtained by shifting $\boldsymbol{\beta}$. We define

$$\boldsymbol{\beta}^{(n)} = (\beta_n, \beta_{n+1}, \dots) \quad \text{for all } n \ge 0.$$

In particular $\boldsymbol{\beta}^{(0)} = \boldsymbol{\beta}$. We will also need to consider shifted sequences. The *shift operator* is given by

$$\sigma\colon A^{\mathbb{N}} \to A^{\mathbb{N}},\ (a_n)_{n\ge 0} \mapsto (a_{n+1})_{n\ge 0}$$

(where A is any given set).

As a first result, we mention a characterization of those sequences $a \in (\mathbb{R}_{\ge 0})^{\mathbb{N}}$ for which there exists a Cantor real base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$. Note that, unlike what happens in the real base case [27], given a sequence a, the equation $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$ admits more than one Cantor real base $\boldsymbol{\beta}$ as a solution in general.

Theorem 1 ([8]). *Let $a = (a_n)_{n\ge 0}$ be a sequence over $\mathbb{R}_{\ge 0}$. There exists a Cantor real base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$ if and only if $\sum_{n=0}^{\infty} a_n > 1$.*

2.2 The greedy algorithm

A distinguished expansion of a given $x \in [0, 1]$ is obtained thanks to the *greedy algorithm*. We first set $r_0 = x$. Then for all $n \ge 0$, we compute $d_n = \lfloor \beta_n r_n \rfloor$ and $r_{n+1} = \beta_n r_n - d_n$. The obtained expansion is denoted by $d_{\boldsymbol{\beta}}(x) = (d_n)_{n\ge 0}$ and is called the *$\boldsymbol{\beta}$-expansion* of x. Thus for all $\ell \ge 0$, one has

$$x = \sum_{n=0}^{\ell} \frac{d_n}{\prod_{k=0}^{n} \beta_k} + \frac{r_\ell}{\prod_{k=0}^{\ell} \beta_k}$$

where $r_\ell \in [0, 1)$. Note that since a Cantor real base satisfies $\prod_{n=0}^{\infty} \beta_n = \infty$, the latter equality implies the convergence of the greedy algorithm. We let $A_{\boldsymbol{\beta}}$ denote the (possibly infinite) alphabet $\{0, \dots, \sup_{n\ge 0}\lceil \beta_n \rceil - 1\}$ and $D_{\boldsymbol{\beta}}$ denote the subset of $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ of the $\boldsymbol{\beta}$-expansions of the real numbers in the interval $[0, 1)$, that is, $D_{\boldsymbol{\beta}} = \{d_{\boldsymbol{\beta}}(x)\colon x \in [0, 1)\}$.

We can also express the greedy digits d_n and remainders r_n thanks to the β_n-transformations. For $\beta > 1$, the β-transformation is the map

$$T_\beta\colon [0, 1) \to [0, 1),\ x \mapsto \beta x - \lfloor \beta x \rfloor.$$

Then for all $x \in [0, 1)$ and $n \ge 0$, we have

$$d_n = \lfloor \beta_n \left(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x) \right) \rfloor \quad \text{and} \quad r_n = T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x).$$

We call an *alternate base* a periodic Cantor real base: there exists $p \ge 1$ such that for all $n \ge 0$, we have $\beta_n = \beta_{n+p}$. In this case we simply write $\boldsymbol{\beta} = (\beta_0, \dots, \beta_{p-1})$ and the integer p is called the *length* of the alternate base $\boldsymbol{\beta}$.

Example 1.

- Any sequence $\boldsymbol{\beta} = (\beta_n)_{n\geq 0}$ of real numbers greater than 1 that takes only finitely many values is a Cantor real base since in this case, the condition $\prod_{n=0}^{\infty} \beta_n = \infty$ is trivially satisfied.
- For $n \geq 0$, let $\alpha_n = 1 + \frac{1}{2^{n+1}}$ and $\beta_n = 2 + \frac{1}{2^{n+1}}$. The sequence $\boldsymbol{\alpha} = (\alpha_n)_{n\geq 0}$ is not a Cantor real base since $\prod_{n=0}^{\infty} \alpha_n < \infty$. If we perform the greedy algorithm on $x = 1$ for the sequence $\boldsymbol{\alpha}$, we obtain the sequence of digits 10^ω (where the ω notation means an infinite repetition), which is clearly not an expansion of 1 with respect to $\boldsymbol{\alpha}$. However, the sequence $\boldsymbol{\beta} = (\beta_n)_{n\geq 0}$ is indeed a Cantor real base since $\prod_{n=0}^{\infty} \beta_n = \infty$.
- If there exists $n \geq 0$ such that β_n is an integer (without any restriction on the other β_m), then $d_{\boldsymbol{\beta}^{(n)}}(1) = \beta_n 0^\omega$.
- Let $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{5+\sqrt{13}}{6}$.
  - For the alternate base $\boldsymbol{\beta} = (\alpha, \beta)$, we get that $d_{\boldsymbol{\beta}^{(0)}}(1) = 2010^\omega$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 110^\omega$.
  - Consider $\boldsymbol{\beta} = (\beta_n)_{n\geq 0}$ the Cantor real base defined by
$$\beta_n = \begin{cases} \alpha, & \text{if } \mathrm{rep}_2(n) \text{ has an even number of 1's;} \\ \beta, & \text{otherwise} \end{cases}$$
where $\mathrm{rep}_2(n)$ is the binary expansion of n. We get the Thue-Morse sequence $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ over the alphabet $\{\alpha, \beta\}$. We compute $d_{\boldsymbol{\beta}^{(0)}}(1) = 20010110^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 1010110^\omega$ and $d_{\boldsymbol{\beta}^{(2)}}(1) = 110^\omega$. Note that since the base is aperiodic, these computations give us no information on $d_{\boldsymbol{\beta}^{(n)}}(1)$ for $n \geq 3$.
- Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the Golden Ratio and let $\boldsymbol{\beta} = (3, \varphi, \varphi)$. We have $d_{\boldsymbol{\beta}^{(0)}}(1) = 30^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 110^\omega$ and $d_{\boldsymbol{\beta}^{(2)}}(1) = 1(110)^\omega$.
- For the alternate base $\boldsymbol{\beta} = (\sqrt{6}, 3, \frac{2+\sqrt{6}}{3})$, we have that $d_{\boldsymbol{\beta}^{(0)}}(1) = 2(10)^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 30^\omega$ and $d_{\boldsymbol{\beta}^{(2)}}(1) = 110020^\omega$. This shows that the $\boldsymbol{\beta}$-expansion of 1 can have a period less than the length of the base.

The classical properties of the β-expansion theory are still valid for Cantor real bases. Until the end of this section, unless otherwise stated, we consider a fixed Cantor real base $\boldsymbol{\beta} = (\beta_n)_{n\geq 0}$.

Proposition 1. *For all $x \in [0, 1)$ and all integers $n \geq 0$, we have*
$$\sigma^n \circ d_{\boldsymbol{\beta}}(x) = d_{\boldsymbol{\beta}^{(n)}} \circ T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x).$$

Proposition 2. *For all sequences a over $\mathbb{N}$ and all $x \in [0, 1]$, we have $a = d_{\boldsymbol{\beta}}(x)$ if and only if $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$ and*
$$\sum_{n=\ell+1}^{\infty} \frac{a_n}{\prod_{k=0}^{n} \beta_k} < \frac{1}{\prod_{k=0}^{\ell} \beta_k} \quad \text{for all } \ell \geq 0.$$

Proposition 3. *Let a be any expansion of a real number x in $[0, 1]$ in base $\boldsymbol{\beta}$. Then the following four assertions are equivalent.*

1. *The sequence a is the $\boldsymbol{\beta}$-expansion of x.*
2. *For all $n \geq 1$, we have $\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) < 1$.*
3. *The sequence $\sigma(a)$ belongs to $D_{\boldsymbol{\beta}^{(1)}}$.*
4. *For all $n \geq 1$, the sequence $\sigma^n(a)$ belongs to $D_{\boldsymbol{\beta}^{(n)}}$.*

Proposition 4. *A sequence a over $\mathbb{N}$ belongs to the set $D_{\boldsymbol{\beta}}$ if and only if $\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) < 1$ for all $n \geq 0$.*

Proposition 5. *The $\boldsymbol{\beta}$-expansion of a real number $x \in [0,1]$ is lexicographically maximal among all expansions of x in base $\boldsymbol{\beta}$.*

Proposition 6. *The function $d_{\boldsymbol{\beta}} \colon [0,1] \to \mathbb{N}^{\mathbb{N}}$ is increasing: for all $x, y \in [0,1]$, we have $x < y \iff d_{\boldsymbol{\beta}}(x) <_{\mathrm{lex}} d_{\boldsymbol{\beta}}(y)$.*

Corollary 1. *If a is a sequence over $\mathbb{N}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) \leq 1$, then $a \leq_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$. In particular, the sequence $d_{\boldsymbol{\beta}}(1)$ is lexicographically maximal among all expansions of all real numbers in $[0,1]$ in base $\boldsymbol{\beta}$.*

Rényi expansions satisfies the property that considering two real bases α and β, one has $\alpha < \beta$ if and only if $d_\alpha(1) < d_\beta(1)$ [27]. The following proposition is a generalization of a weaker version of this property.

Proposition 7. *Let $\boldsymbol{\alpha} = (\alpha_n)_{n\geq 0}$ and $\boldsymbol{\beta} = (\beta_n)_{n \geq 0}$ be Cantor real bases such that $\prod_{k=0}^n \alpha_k \leq \prod_{k=0}^n \beta_k$ for all $n \geq 0$. Then $d_{\boldsymbol{\alpha}}(x) \leq_{\mathrm{lex}} d_{\boldsymbol{\beta}}(x)$ for all $x \in [0,1]$.*

However, it is not true that $d_{\boldsymbol{\alpha}}(1) <_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$ implies that $\prod_{i=0}^n \alpha_i \leq \prod_{i=0}^n \beta_i$ for all $n \geq 0$, as the following example shows. The same example shows that the lexicographic order on the Cantor real bases is not sufficient either. Here, the term lexicographic order refers to the following order: $\boldsymbol{\alpha} < \boldsymbol{\beta}$ whenever there exists $\ell \geq 0$ such that $\alpha_n = \beta_n$ for $n < \ell$ and $\alpha_\ell < \beta_\ell$.

Example 2. Let $\boldsymbol{\alpha} = (2+\sqrt{3}, 2)$ and $\boldsymbol{\beta} = (2+\sqrt{2}, 5)$. Then $d_{\boldsymbol{\alpha}}(1) = 31^\omega$ and $d_{\boldsymbol{\beta}}(1)$ starts with the prefix 32, hence $d_{\boldsymbol{\alpha}}(1) <_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$.

2.3 Quasi-greedy expansions and admissible sequences

An expansion is said to be *finite* if is ultimately zero, and *infinite* otherwise. The *length* of a finite expansion is the length of the longest prefix ending in a non-zero digit. In the finite case, we usually omit to write the tail of zeros.

When the $\boldsymbol{\beta}$-expansion of 1 is finite, we modify it in order to obtain an infinite expansion of 1 that is lexicographically maximal among all infinite expansions of 1. The obtained expansion is denoted by $d^*_{\boldsymbol{\beta}}(1)$ and is called the *quasi-greedy $\boldsymbol{\beta}$-expansion* of 1. It is defined recursively as follows:

$$d^*_{\boldsymbol{\beta}}(1) = \begin{cases} d_{\boldsymbol{\beta}}(1), & \text{if } d_{\boldsymbol{\beta}}(1) \text{ is infinite;} \\ d_0 \cdots d_{n-2}(d_{n-1}-1) d^*_{\boldsymbol{\beta}^{(n)}}(1), & \text{if } d_{\boldsymbol{\beta}}(1) = d_0 \cdots d_{n-1} \text{ with } d_{n-1} > 0. \end{cases}$$

Example 3. – When $\boldsymbol{\beta} = (\beta, \beta, \ldots)$, we recover the usual definition of the quasi-greedy β-expansion.

– Let $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$ be an alternate base such that each β_i is an integer. Then for all $i \in \{0, \ldots, p-1\}$, we have $d_{\boldsymbol{\beta}^{(i)}}(1) = \beta_i 0^\omega$ and

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = ((\beta_i - 1) \cdots (\beta_{p-1} - 1)(\beta_0 - 1) \ldots (\beta_{i-1} - 1))^\omega.$$

– Let $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. Since $d_{\boldsymbol{\beta}^{(0)}}(1) = 201$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 11$, we obtain $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$ and $d^*_{\boldsymbol{\beta}^{(0)}}(1) = 200 d^*_{\boldsymbol{\beta}^{(1)}}(1) = 200(10)^\omega = 20(01)^\omega$.
– For $\boldsymbol{\beta} = (3, \varphi, \varphi)$, we directly have that $d^*_{\boldsymbol{\beta}^{(2)}}(1) = d_{\boldsymbol{\beta}^{(2)}}(1) = 1(110)^\omega$. In order to compute $d^*_{\boldsymbol{\beta}^{(0)}}(1)$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1)$, we need to go through the definition several times since $d_{\boldsymbol{\beta}^{(0)}}(1) = 3$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 11$ are finite. We compute $d^*_{\boldsymbol{\beta}^{(0)}}(1) = 2 d^*_{\boldsymbol{\beta}^{(1)}}(1) = 210 d^*_{\boldsymbol{\beta}^{(0)}}(1) = (210)^\omega$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1) = 10 d^*_{\boldsymbol{\beta}^{(0)}}(1) = 10(210)^\omega = (102)^\omega$.
– Consider $\boldsymbol{\beta} = (3, \beta, \beta, \beta, \beta, \ldots)$ where $\beta = \sqrt{6}(2 + \sqrt{6})$. We get $d_{\boldsymbol{\beta}^{(0)}}(1) = 3$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = d_\beta(1)$ is infinite not ultimately periodic [2]. Therefore, the quasi-greedy expansion $d^*_{\boldsymbol{\beta}^{(0)}}(1) = 2 d^*_{\boldsymbol{\beta}^{(1)}}(1)$ is not ultimately periodic.

The following propositions list the main properties of the quasi-greedy $\boldsymbol{\beta}$-expansion of 1.

Proposition 8. *The quasi-greedy $\boldsymbol{\beta}$-expansion of* 1 *is an expansion of* 1 *in base $\boldsymbol{\beta}$, i.e., we have* $\mathrm{val}_{\boldsymbol{\beta}}(d^*_{\boldsymbol{\beta}}(1)) = 1$.

Proposition 9. *If a is a sequence over $\mathbb{N}$ such that* $\mathrm{val}_{\boldsymbol{\beta}}(a) < 1$*, then* $a <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}}(1)$*. Furthermore, $d^*_{\boldsymbol{\beta}}(1)$ is lexicographically maximal among all infinite expansions of all real numbers in* $[0,1]$ *in base $\boldsymbol{\beta}$.*

Proposition 10. *The quasi-greedy $\boldsymbol{\beta}$-expansion of* 1 *can also be obtained as the following limit:* $d^*_{\boldsymbol{\beta}}(1) = \lim_{x \to 1^-} d_{\boldsymbol{\beta}}(x)$.

In [29], Parry characterized those sequences over $\mathbb{N}$ that belong to D_β. Such sequences are sometimes called *β-admissible sequences*. Analogously, sequences in $D_{\boldsymbol{\beta}}$ are said to be the *$\boldsymbol{\beta}$-admissible sequences*. The following theorem generalizes Parry's theorem to Cantor real bases.

Theorem 2 ([8]). *A sequence a over $\mathbb{N}$ belongs to $D_{\boldsymbol{\beta}}$ if and only if* $\sigma^n(a) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$ *for all* $n \geq 0$.

Example 4. Let $\boldsymbol{\beta} = (3, \varphi, \varphi)$. We obtain from Theorem 2 that $a = 210(110)^\omega$ is the $\boldsymbol{\beta}$-expansion of some $x \in [0,1)$ since $d^*_{\boldsymbol{\beta}^{(0)}}(1) = (210)^\omega$, $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (102)^\omega$ and $d^*_{\boldsymbol{\beta}^{(2)}}(1) = 1(110)^\omega$. This x is given by $\mathrm{val}_{\boldsymbol{\beta}}(a) = \frac{19+9\sqrt{5}}{3(7+3\sqrt{5})}$.

We then obtain a characterization of the $\boldsymbol{\beta}$-expansions of a real number x in the interval $[0,1]$ among all its expansions in base $\boldsymbol{\beta}$.

Theorem 3 ([8]). *An expansion a of some real number* $x \in [0,1]$ *in base $\boldsymbol{\beta}$ is its $\boldsymbol{\beta}$-expansion if and only if* $\sigma^n(a) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$ *for all* $n \geq 1$.

Example 5. Consider $\boldsymbol{\beta} = (\frac{16+5\sqrt{10}}{9}, 9)$. Then $d_{\boldsymbol{\beta}^{(0)}}(1) = d^*_{\boldsymbol{\beta}^{(0)}}(1) = 34(27)^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 9$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1) = 834(27)^\omega$. For all $n \geq 1$, we have $\sigma^{2n}(34(27)^\omega) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(0)}}(1)$ and $\sigma^{2n-1}(34(27)^\omega) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(1)}}(1)$ as prescribed by Theorem 3.

In comparison with the real base expansion theory, considering a Cantor real base $\boldsymbol{\beta}$ and a sequence a over $\mathbb{N}$, Theorem 3 does not provide us with a purely combinatorial condition to check whether a is the $\boldsymbol{\beta}$-expansion of 1. More details will be given in Section 3.2, where we will see that even though an improvement of this result in the context of alternate bases can be proved, a purely combinatorial condition cannot exist.

3 Combinatorial properties of alternate base expansions

Recall that an alternate base is a periodic Cantor real base. The aim of this section is to discuss some results that are specific to these particular Cantor real bases.

In Theorem 1, we gave a characterization of those sequences $a \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ for which there exists a Cantor real base $\boldsymbol{\beta}$ such that $\text{val}_{\boldsymbol{\beta}}(a) = 1$. Here, we are interested in the stronger condition of the existence of an alternate base $\boldsymbol{\beta}$ satisfying $\text{val}_{\boldsymbol{\beta}}(a) = 1$.

Theorem 4 ([8]). *Let a be a sequence over $\mathbb{R}_{\geq 0}$ such that $a_n \in O(n^d)$ for some $d \in \mathbb{N}$ and let $p \in \mathbb{N}_{\geq 1}$. There exists an alternate base $\boldsymbol{\beta}$ of length p such that $\text{val}_{\boldsymbol{\beta}}(a) = 1$ if and only if $\sum_{n=0}^{\infty} a_n > 1$. If moreover $p \geq 2$, then there exist uncountably many such alternate bases.*

From now on, we let $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$ be a fixed alternate base.

3.1 Greedy alternate expansions

The greedy and the quasi-greedy $\boldsymbol{\beta}$-expansions of 1 enjoy specific properties whenever $\boldsymbol{\beta}$ is an alternate base.

Proposition 11. *The $\boldsymbol{\beta}$-expansion of 1 is not purely periodic.*

In the framework of β-expansions, a real base β is called a *Parry number* whenever the quasi-greedy β-expansion of 1 is ultimately periodic. In the context of alternate bases, in order to have an ultimately periodic quasi-greedy $\boldsymbol{\beta}$-expansion of 1, one might think at first that the product $\delta = \prod_{i=0}^{p-1} \beta_i$ should be a Parry number since by grouping terms p by p in the sum

$$\frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \frac{a_2}{\beta_0\beta_1\beta_2} + \cdots$$

we get an expansion of the kind

$$\frac{c_0}{\delta} + \frac{c_1}{\delta^2} + \frac{c_2}{\delta^3} + \cdots .$$

But in the previous expression, the numerators are no longer integers. The following example shows that this intuition is not correct, even whenever all quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 are ultimately periodic for $i \in \{0, \dots, p-1\}$.

Example 6. Consider again the Parry alternate base $\boldsymbol{\beta} = (3, \varphi, \varphi)$. As previously seen, all the corresponding quasi-greedy expansions of 1 are ultimately periodic. However, let us show that the product $\delta = 3\varphi^2$ is not a Parry number, and moreover, none of its powers $\delta^n = (3\varphi^2)^n$ is. It is a well-known property of the Golden Ratio that $\varphi^n = f_n\varphi + f_{n-1}$ for all $n \geq 1$, where $(f_n)_{n\geq 0} = (0, 1, 1, 2, 3, 5, 8, \dots)$ is the Fibonacci sequence starting with the initial conditions $0, 1$. Therefore, denoting $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$, for all $n \geq 1$, the minimal polynomial of $(3\varphi^2)^n$ can be computed as

$$\begin{aligned}
&(X - 3^n(f_{2n}\varphi + f_{2n-1}))(X - 3^n(f_{2n}\overline{\varphi} + f_{2n-1})) \\
&= X^2 - 3^n(f_{2n} + 2f_{2n-1})X + 3^{2n}(-f_{2n}^2 + f_{2n}f_{2n-1} + f_{2n-1}^2) \\
&= X^2 - 3^n(f_{2n+1} + f_{2n-1})X + 3^{2n},
\end{aligned}$$

since it can be easily verified by induction that we have $-f_n^2 + f_nf_{n-1} + f_{n-1}^2 = (-1)^n$ for all $n \geq 1$. We can also check that $f_{n+1} + f_{n-1} \leq 3^{\frac{n}{2}}$ for all $n \geq 1$. But the quadratic Parry numbers are known to be roots of polynomials of the form $X^2 - aX - b$ with $a \geq b \geq 1$ or of the form $X^2 - aX + b$ with $a - 2 \geq b \geq 1$ [2]. Therefore, we get that $(3\varphi^2)^n$ is not a Parry number for all $n \geq 1$.

Proposition 12. *The quasi-greedy expansion $d^*_{\boldsymbol{\beta}}(1)$ is ultimately periodic if and only if either an ultimately periodic expansion is reached or only finite expansions are involved within the first p recursive calls to the definition of $d^*_{\boldsymbol{\beta}}(1)$.*

Ultimately periodic $\boldsymbol{\beta}$-expansions will be investigated further in Sections 3.3, 4.1, 4.3 and 4.4.

3.2 Admissible sequences in alternate bases

The condition given in Theorem 3 does not allow us to check whether a given expansion of 1 is the $\boldsymbol{\beta}$-expansion of 1 without effectively computing the quasi-greedy $\boldsymbol{\beta}$-expansion of 1, and hence the $\boldsymbol{\beta}$-expansion of 1 itself. The following result provides us with such a condition in the case of alternate bases, provided that we are given the quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 for $i \in \{1, \dots, p-1\}$. Note that the shifted sequences starting in positions that are multiple of p are compared with the sequence a itself and not with $d^*_{\boldsymbol{\beta}}(1)$ as in Theorem 3.

Proposition 13. *An expansion a of 1 in the alternate base $\boldsymbol{\beta}$ is the $\boldsymbol{\beta}$-expansion of 1 if and only if $\sigma^{pm}(a) <_{\text{lex}} a$ for all $m \geq 1$ and $\sigma^{pm+i}(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(i)}}(1)$ for all $m \geq 0$ and $i \in \{1, \dots, p-1\}$.*

We have seen in Theorem 4 that considering a sequence a over $\mathbb{N}$, there may exist more than one alternate base $\boldsymbol{\beta}$ of a given length such that $\text{val}_{\boldsymbol{\beta}}(a) = 1$. Moreover, among all of these alternate bases, it may be that some are such that a is greedy and others are such that a is not. Thus, a purely combinatorial condition for checking whether an expansion is greedy cannot exist.

Example 7. Consider $a = 2(10)^\omega$. Then $\mathrm{val}_{\boldsymbol{\alpha}}(a) = \mathrm{val}_{\boldsymbol{\beta}}(a) = 1$ for both $\boldsymbol{\alpha} = (1+\varphi, 2)$ and $\boldsymbol{\beta} = (\frac{31}{10}, \frac{420}{341})$. It can be checked that $d_{\boldsymbol{\alpha}}(1) = a$ and $d_{\boldsymbol{\beta}}(1) \neq a$.

Furthermore, a sequence a can be greedy for more than one alternate base.

Example 8. The sequence 110^ω is the expansion of 1 with respect to the three alternate bases φ, $(\frac{5+\sqrt{13}}{6}, \frac{1+\sqrt{13}}{2})$ and $(1.7, \frac{1}{0.7})$.

3.3 The alternate $\boldsymbol{\beta}$-shift

Let us recall some definitions from symbolic dynamics. For a finite alphabet A, a subset of $A^{\mathbb{N}}$ is called a *subshift* of $A^{\mathbb{N}}$ if it is shift-invariant and closed with respect to the product topology. For a subset S of $A^{\mathbb{N}}$, we let $\mathrm{Fac}(S)$ denote the set of all finite factors of all elements in S. A subshift S of $A^{\mathbb{N}}$ is said to be *sofic* if the language $\mathrm{Fac}(S) \subset A^*$ is accepted by a finite automaton.

In this section, we generalize the notion of β-shift to the context of alternate bases, and study its properties. First, we let $S_{\boldsymbol{\beta}}$ denote the topological closure of $D_{\boldsymbol{\beta}}$ with respect to the product topology.

Proposition 14. *A sequence a over $\mathbb{N}$ belongs to $S_{\boldsymbol{\beta}}$ if and only if $\sigma^n(a) \leq_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$ for all $n \geq 0$.*

Proposition 15. *Let $a, b \in S_{\boldsymbol{\beta}}$.*

1. *If $a <_{\mathrm{lex}} b$ then $\mathrm{val}_{\boldsymbol{\beta}}(a) \leq \mathrm{val}_{\boldsymbol{\beta}}(b)$.*
2. *If $\mathrm{val}_{\boldsymbol{\beta}}(a) < \mathrm{val}_{\boldsymbol{\beta}}(b)$ then $a <_{\mathrm{lex}} b$.*

Proposition 16. *For all $n \geq 0$, if $w \in S_{\boldsymbol{\beta}^{(n)}}$ then $\sigma(w) \in S_{\boldsymbol{\beta}^{(n+1)}}$.*

Since the set $S_{\boldsymbol{\beta}}$ is not shift-invariant, we rather consider the set

$$\Sigma_{\boldsymbol{\beta}} = \bigcup_{i=0}^{p-1} S_{\boldsymbol{\beta}^{(i)}}.$$

Proposition 17. *The sets $\Sigma_{\boldsymbol{\beta}}$ is closed and shift-invariant.*

The subset $\Sigma_{\boldsymbol{\beta}}$ is thus a subshift of $A_{\boldsymbol{\beta}}^{\mathbb{N}}$, which we call the ***β**-shift*.

Proposition 18. *We have $\mathrm{Fac}(D_{\boldsymbol{\beta}}) = \mathrm{Fac}(S_{\boldsymbol{\beta}}) = \mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$.*

We define *Parry alternate bases* as the alternate bases $\boldsymbol{\beta}$ such that all $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ are ultimately periodic for $i \in \{0, \ldots, p-1\}$. We will see that, analogously to what happens for real base expansions, Parry alternate bases are exactly those alternate bases giving rise to a sofic $\boldsymbol{\beta}$-shift, which justifies the terminology.

For a Parry alternate base $\boldsymbol{\beta}$, we define a deterministic finite automaton $\mathcal{A}_{\boldsymbol{\beta}} = (Q, I, F, A_{\boldsymbol{\beta}}, \delta)$. Without loss of generality, we can consider that the involved periods are all multiples of the length p of the base. Thus, let us write

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_{i,0} \cdots t_{i,m_i-1}(t_{i,m_i} \cdots t_{i,m_i+n_ip-1})^\omega.$$

Then the set of states is $Q = \{0, \dots, p-1\} \times \{0, \dots, m_i + n_i p - 1\}$. The set I of initial states and the set F of final states are defined as $I = \{0, \dots, p-1\} \times \{0\}$ and $F = Q$. The (partial) transition function $\delta\colon Q \times A_{\boldsymbol{\beta}} \to Q$ of the automaton $\mathcal{A}_{\boldsymbol{\beta}}$ is defined as follows. For each state $(i,k) \in Q$, we have

$$\delta((i,k), t_{i,k}) = \begin{cases} (i, k+1), & \text{if } k < m_i + n_i p - 1; \\ (i, m_i), & \text{otherwise} \end{cases}$$

and $\delta((i,k), s) = ((i+k+1) \bmod p, 0)$ for all $s \in \{0, \dots, t_{i,k} - 1\}$. By using Theorem 2 and Proposition 18, we get the following result.

Proposition 19. *The automaton $\mathcal{A}_{\boldsymbol{\beta}}$ accepts the language* $\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$.

This implies that the $\boldsymbol{\beta}$-shift associated with a Parry alternate base is sofic. As it turns out, the converse is also true, so that we obtain the following result extending a result of Bertrand-Mathis for real bases [3]. Proving this result turned out to be much more difficult than the original result for $p = 1$.

Theorem 5 ([8]). *The alternate $\boldsymbol{\beta}$-shift is sofic if and only if $\boldsymbol{\beta}$ is a Parry alternate base.*

Example 9. The finite automaton of Figure 1 accepts the set of factors of elements in the $\boldsymbol{\beta}$-shift for $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$; also see Examples 1 and 3.

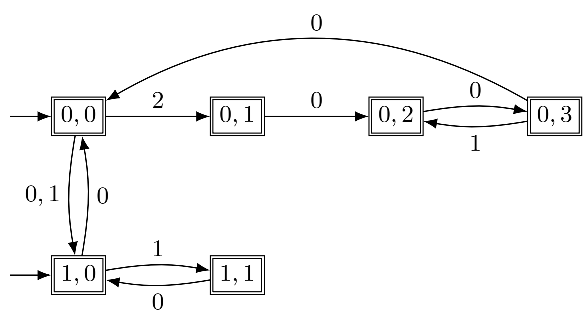

Fig. 1. A deterministic automaton accepting $\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$ for $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

Interestingly, some new phenomena occur in the extended framework of alternate bases when looking at subshifts of finite type. Recall that a subshift is said to be of *finite type* if its minimal set of forbidden factors is finite. For $p = 1$, it is well known that the β-shift is of finite type if and only if $d_\beta(1)$ is finite [3]. However, this result does not generalize to $p \geq 2$ since for the alternate base $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$, we get $d^*_{\boldsymbol{\beta}^{(0)}}(1) = 20(01)^\omega$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$, thus we see that all words in $2(00)^*2$ are factors avoided by $\Sigma_{\boldsymbol{\beta}}$. Therefore, even though the $\boldsymbol{\beta}^{(i)}$-expansions of 1 are finite for $i \in \{1, 2\}$ as we have seen in Example 1, the associated $\boldsymbol{\beta}$-shift is not of finite type.

4 Algebraic properties of alternate base expansions

In this section, we report on the two works [10,11] were we studied the algebraic properties of alternate bases $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$.

The property of being a Parry alternate base was defined in the previous section from a combinatorial point of view. Here we provide some algebraic necessary/sufficient conditions to have a Parry alternate base. Then, in the (stricly) stronger situation where the product $\delta = \prod_{i=0}^{p-1} \beta_i$ is a Pisot number and all the bases β_i belong to the extended field $\mathbb{Q}(\delta)$, we will be able to say much more. On the one hand, we obtain generalizations of some results of Schmidt [33] giving rise to an elementary proof of the original result. On the other hand, we obtain an analogue of Frougny's result [18] concerning normalization of real bases expansions: under these assumptions, the normalization function is computable by a finite Büchi automaton, and furthermore, we effectively construct such an automaton. Results on normalization will be presented in Section 5.

4.1 A necessary condition for being a Parry alternate base

The following result gives a necessary condition on $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$ to be a Parry alternate base, that is, to have eventually periodic $\boldsymbol{\beta}^{(i)}$-expansions of 1 for all $i \in \{0, \ldots, p-1\}$.

Theorem 6 ([11]). *If $\boldsymbol{\beta}$ is a Parry alternate base, then δ is an algebraic integer and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \{0, \ldots, p-1\}$.*

The condition that the bases must be expressible as rational functions of the product δ is a phenomenon that does not show for Rényi numeration systems. Indeed, this condition is trivially satisfied whenever $p = 1$. Therefore, we see once again that new ideas and techniques are necessary in order to understand the properties of alternate bases.

4.2 Alternate spectrum

An important tool in our study is the spectrum of numeration systems associated with alternate bases. The spectrum of a real number $\delta > 1$ and an alphabet $A \subset \mathbb{Z}$ was introduced by Erdős et al [16] and further studied in [1,17]. For our purposes, we use a generalized concept with $\delta \in \mathbb{C}$ and $A \subset \mathbb{C}$ and study its topological properties.

From now on, we consider a p-tuple $\boldsymbol{D} = (D_0, \ldots, D_{p-1})$ where each D_i is an alphabet of integers containing 0. We use the convention that $D_n = D_{n \bmod p}$ and $\boldsymbol{D}^{(n)} = (D_n, \ldots, D_{n+p-1})$ for all $n \geq 0$. Grouping terms p by p, the left-hand side of (1) can be written as

$$\sum_{m=0}^{+\infty} \frac{\sum_{i=0}^{p-1} a_{mp+i}\beta_{i+1} \cdots \beta_{p-1}}{\delta^{m+1}}$$

where $\delta = \prod_{i=0}^{p-1} \beta_i$. If we add the constraint that each letter a_n belongs to D_n, then we obtain an expansion in base δ over the alphabet

$$\mathcal{D} = \left\{ \sum_{i=0}^{p-1} a_i \beta_{i+1} \cdots \beta_{p-1} : \forall i \in \{0, \ldots, p-1\},\ a_i \in D_i \right\}.$$

We define the *alternate spectrum* to be the set

$$X^{\mathcal{D}}(\delta) = \left\{ \sum_{n=0}^{\ell-1} c_n \delta^{\ell-1-n} : \ell \geq 0,\ c_0, c_1, \ldots, c_{\ell-1} \in \mathcal{D} \right\}.$$

For the sake of simplicity, for each $i \in \{0, \ldots, p-1\}$, we let X_i denote the spectrum built from the shifted base $\boldsymbol{\beta}^{(i)}$ and the shifted p-tuple of alphabets $\boldsymbol{D}^{(i)}$. In particular, we have $X_0 = X^{\mathcal{D}}(\delta)$.

Lemma 1. *For each $i \in \{0, \ldots, p-1\}$, we have $X_i \cdot \beta_i + D_i = X_{i+1}$ where it is understood that $X_p = X_0$.*

4.3 A sufficient condition for being a Parry alternate base

In this section, we present a sufficient condition for $\boldsymbol{\beta}$ to be a Parry alternate base. We proceed in two steps, by studying the properties of the spectrum.

Proposition 20. *If $D_i \supseteq \{-\lfloor \beta_i \rfloor, \ldots, \lfloor \beta_i \rfloor\}$ for all $i \in \{0, \ldots, p-1\}$ and the spectrum $X^{\mathcal{D}}(\delta)$ has no accumulation point in $\mathbb{R}$, then $\boldsymbol{\beta}$ is a Parry alternate base.*

Proposition 21. *If δ is a Pisot number and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \{0, \ldots, p-1\}$ then the spectrum $X^{\mathcal{D}}(\delta)$ has no accumulation point in $\mathbb{R}$.*

As a consequence, we get the following theorem, which for the case $p = 1$ is a well-known result of Schmidt [33]. In Section 4.4, we will present an alternative method for proving this result.

Theorem 7 ([11]). *If δ is a Pisot number and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \{0, \ldots, p-1\}$ then $\boldsymbol{\beta}$ is a Parry alternate base.*

Let us make several remarks concerning this theorem. First, the condition of δ being a Pisot number is neither sufficient nor necessary for $\boldsymbol{\beta}$ to be a Parry alternate base. Indeed, it is not necessary even for $p = 1$ since there exist Parry numbers which are not Pisot. To see that it is not sufficient for $p \geq 2$, consider the alternate base $\boldsymbol{\beta} = (\sqrt{\beta}, \sqrt{\beta})$ where β is the smallest Pisot number. The product δ is the Pisot number β. However, the $\boldsymbol{\beta}$-expansion of 1 is equal to $d_{\sqrt{\beta}}(1)$, which is known to be aperiodic.

Furthermore, the bases $\beta_0, \ldots, \beta_{p-1}$ need not be algebraic integers in order to have a Parry alternate base since, for instance, for the Parry alternate base $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$, the second base $\frac{5+\sqrt{13}}{6}$ is not an algebraic integer.

4.4 Ultimately periodic alternate base expansions

Here we present two results from [10] generalizing results on ultimately periodic Renyi expansions [33]. Recall that a *Salem number* is an algebraic integer greater than 1 whose Galois conjugates lie inside the unit disk with at least one of them on the unit circle. Thus, the set of algebraic integers greater than 1 whose Galois conjugates lie inside the unit disk is partioned into the Pisot numbers and the Salem numbers. We study the set $\mathrm{Per}(\boldsymbol{\beta}) = \{x \in [0,1) : d_{\boldsymbol{\beta}}(x) \text{ is ultimately periodic}\}$.

Theorem 8 ([10]).

1. *If* $\mathbb{Q} \cap [0,1) \subseteq \cap_{i=0}^{p-1} \mathrm{Per}(\boldsymbol{\beta}^{(i)})$ *then* $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ *and* δ *is either a Pisot number or a Salem number.*
2. *If* δ *is a Pisot number and* $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ *then* $\mathrm{Per}(\boldsymbol{\beta}) = \mathbb{Q}(\delta) \cap [0,1)$.

It is interesting to note that when adding the hypothesis $p = 1$ in our proof from [10], we obtain a much shorter proof than Schmidt's original one from [33].

Another particularly nice point is that we recover Theorem 7 as a corollary of Theorem 8, whereas the proof technique developed in [10] is independent from the first proof of Theorem 7 since it does not make use of the properties of the spectrum.

Another consequence of Theorem 8 is the following well-known property of Pisot numbers.

Corollary 2. *If* β *is a Pisot number then* $\beta \in \mathbb{Q}(\beta^p)$ *for all integer* $p \geq 1$.

The common proof of this result makes use of algebraic tools such as matrix diagonalization or the Kronecker theorem stating that if the roots of a monic polynomial with integers coefficients all lie inside the unit disc then they must be either zero or roots of unity. No such argument is used in [10].

The second generalization of Schmidt's results we obtain is the following.

Theorem 9 ([10]). *If* δ *is an algebraic integer that is neither a Pisot number nor a Salem number then* $\mathrm{Per}(\boldsymbol{\beta}) \cap \mathbb{Q}$ *is nowhere dense in* $[0,1)$.

5 Normalization of alternate base expansions

The *normalization function* $\nu_{\boldsymbol{\beta},\boldsymbol{D}} \colon (\cup_{i=0}^{p-1} D_i)^{\mathbb{N}} \to (\cup_{i=0}^{p-1} \{0, \ldots, \lceil \beta_i \rceil - 1\})^{\mathbb{N}}$ is the partial function mapping any expansion $a \in \prod_{n=0}^{\infty} D_n$ of a real number $x \in [0,1)$ in base $\boldsymbol{\beta}$ to the $\boldsymbol{\beta}$-expansion of x. We say that $\nu_{\boldsymbol{\beta},\boldsymbol{D}}$ is *computable by a finite automaton* if there exists a finite Büchi automaton accepting the set

$$\left\{(u,v) \in \prod_{n=0}^{\infty} (D_n \times A_{\beta_n}) : \mathrm{val}_{\boldsymbol{\beta}}(u) = \mathrm{val}_{\boldsymbol{\beta}}(v) \text{ and } \exists x \in [0,1),\ v = d_{\boldsymbol{\beta}}(x)\right\}.$$

Büchi automata are defined as classical automata except for the acceptance criterion which has to be adapted in order to deal with infinite words: an infinite word is accepted if it labels an initial path going infinitely many times through a final state. A major difference between Büchi and classical automata (i.e., accepting finite words) is that a set of infinite words accepted by a finite Büchi automaton is not necessarily accepted by a deterministic one.

5.1 Zero automaton

Let us now generalize the notion of zero automaton introduced by Frougny in [18] to the context of alternate bases. The aim is to define a deterministic Büchi automaton accepting the set

$$Z(\boldsymbol{\beta},\boldsymbol{D}) = \left\{ (a_n)_{n\geq 0} \in \prod_{n=0}^{+\infty} D_n : \sum_{n=0}^{+\infty} \frac{a_n}{\prod_{k=0}^{n} \beta_k} = 0 \right\}$$

of all expansions of zero the n-th digit of which belongs to the alphabet D_n. This will be one of the key ingredient in order to compute the normalization function by using a finite (two-tape) Büchi automaton.

We define

$$M = \sum_{n=0}^{+\infty} \frac{\max(D_n)}{\prod_{k=0}^{n} \beta_k} \qquad \text{and} \qquad m = \sum_{n=0}^{+\infty} \frac{\min(D_n)}{\prod_{k=0}^{n} \beta_k}$$

where $\max(D_n)$ and $\min(D_n)$ respectively denote the maximal and minimal digit in the alphabet D_n. Then for each $i \in \{0,\ldots,p-1\}$, we let M_i and m_i denote the numbers M and m corresponding to the shifted base $\boldsymbol{\beta}^{(i)}$ and the shifted p-tuple of alphabets $\boldsymbol{D}^{(i)}$ respectively. In particular, we have $M_0 = M$ and $m_0 = m$. We define the *zero automaton* $\mathcal{Z}(\boldsymbol{\beta},\boldsymbol{D})$ associated with the alternate base $\boldsymbol{\beta}$ and the p-tuple of alphabets $\boldsymbol{D}$ as the deterministic Büchi automaton $\mathcal{Z}(\boldsymbol{\beta},\boldsymbol{D}) = (Q_{\boldsymbol{\beta},\boldsymbol{D}}, (0,0), Q_{\boldsymbol{\beta},\boldsymbol{D}}, \cup_{i=0}^{p-1} D_i, \delta)$ where the set of states is

$$Q_{\boldsymbol{\beta},\boldsymbol{D}} = \cup_{i=0}^{p-1} (\{i\} \times (X_i \cap [-M_i, -m_i]))$$

and the (partial) transition function $\delta \colon Q_{\boldsymbol{\beta},\boldsymbol{D}} \times \cup_{i=0}^{p-1} D_i \to Q_{\boldsymbol{\beta},\boldsymbol{D}}$ is defined as follows: for $(i,s) \in Q_{\boldsymbol{\beta},\boldsymbol{D}}$ and $a \in D_i$, we have $\delta((i,s),a) = ((i+1) \bmod p, \beta_i s + a)$. Observe that since we have assumed that all the alphabets D_i contain the digit 0, the initial state $(0,0)$ is indeed an element of $Q_{\boldsymbol{\beta},\boldsymbol{D}}$. Moreover, if $s \in X_i$ and $a \in D_i$ then $\beta_i s + a \in X_{i+1}$ by Lemma 1.

Proposition 22. *The zero automaton $\mathcal{Z}(\boldsymbol{\beta},\boldsymbol{D})$ accepts the set $Z(\boldsymbol{\beta},\boldsymbol{D})$.*

Example 10. Consider the alternate base $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$ and the pair of alphabets $\boldsymbol{D} = (\{-2,-1,0,1,2\},\{-1,0,1\})$. Then $M_0 = -m_0 = \mathrm{val}_{\boldsymbol{\beta}^{(0)}}((21)^\omega) \simeq 1.67994$ and $M_1 = -m_1 = \mathrm{val}_{\boldsymbol{\beta}^{(1)}}((12)^\omega) \simeq 1.86852$. The zero automaton $\mathcal{Z}(\boldsymbol{\beta},\boldsymbol{D})$ is depicted in Figure 2 where the states with first components 0 and 1 are colored in pink and purple respectively, and where the edges labeled by $-2,-1,0,1$ and 2 are colored in dark blue, dark green, red, light green and light blue respectively. For instance, the sequences $1(\overline{1}0)^\omega$ and $(0\overline{1}21\overline{2}\overline{1})^\omega$ have value 0 in base $\boldsymbol{\beta}$ (where $\overline{1}$ and $\overline{2}$ designate the digits -1 and -2 respectively).

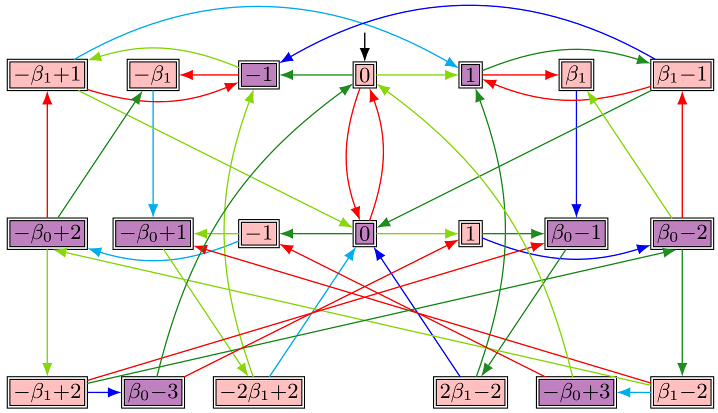

Fig. 2. The zero automaton $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$ for the alternate base $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$ and $\boldsymbol{D} = (\{-2,-1,0,1,2\}, \{-1,0,1\})$. The used colors are described within Example 10.

In general, the zero automaton is infinite, i.e., it has infinitely many states. The following theorem, which generalizes a result from [19], describes when the zero automaton is actually finite in relation with some property of the spectrum.

Theorem 10 ([11]). *The following assertions are equivalent.*

1. *The set $Z(\boldsymbol{\beta}, \boldsymbol{D})$ is accepted by a finite Büchi automaton.*
2. *The spectrum $X^{\mathcal{D}}(\delta)$ has no accumulation point in $\mathbb{R}$.*
3. *The zero automaton $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$ is finite.*

Theorem 10 and Proposition 21 combined give us the following result.

Corollary 3. *If δ is a Pisot number and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \{0, \dots, p-1\}$ then the zero automaton $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$ is finite.*

5.2 Computing the normalization

We show that the normalization in alternate bases is computable by finite automaton under certain conditions, in which case we construct such an automaton.

Following the same lines as in the real base case, we start by constructing a *converter* by using the zero automaton $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$) defined in Section 5.1. Consider two p-tuples of alphabets $\boldsymbol{D} = (D_0, \dots, D_{p-1})$ and $\boldsymbol{D'} = (D'_0, \dots, D'_{p-1})$. We denote the p-tuple of alphabets $(D_0 - D'_0, \dots, D_{p-1} - D'_{p-1})$ by $\boldsymbol{D} - \boldsymbol{D'}$. The *converter from $\boldsymbol{D}$ to $\boldsymbol{D'}$* is the Büchi automaton

$$\mathcal{C}(\boldsymbol{\beta}, \boldsymbol{D}, \boldsymbol{D'}) = (Q_{\boldsymbol{\beta}, \boldsymbol{D}-\boldsymbol{D'}}, (0,0), Q_{\boldsymbol{\beta}, \boldsymbol{D}-\boldsymbol{D'}}, \cup_{i=0}^{p-1}(D_i \times D'_i), E)$$

where E is the set of transitions defined as follows: for $(i,s), (j,t) \in Q_{\boldsymbol{\beta}, \boldsymbol{D}-\boldsymbol{D'}}$ and for $(a,b) \in \cup_{i=0}^{p-1}(D_i \times D'_i)$, there is a transition

$$(i,s) \xrightarrow{(a,b)} (j,t)$$

if and only if $(a,b) \in D_i \times D'_i$ and there is a transition $(i,s) \xrightarrow{a-b} (j,t)$ in $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D} - \boldsymbol{D}')$. Note that the converter is non-deterministic in general.

Proposition 23. *The converter $\mathcal{C}(\boldsymbol{\beta}, \boldsymbol{D}, \boldsymbol{D}')$ accepts the set*

$$\left\{ (u,v) \in \prod_{n=0}^{\infty} (D_n \times D'_n) : \mathrm{val}_{\boldsymbol{\beta}}(u) = \mathrm{val}_{\boldsymbol{\beta}}(v) \right\}.$$

In the case where $\boldsymbol{\beta}$ is a Parry alternate base, we consider a modification of the automaton $\mathcal{A}_{\boldsymbol{\beta}}$ built in Section 3.3 in order to get a Büchi automaton accepting the set $D_{\boldsymbol{\beta}}$. Without loss of generality, we suppose that $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ has a non-zero preperiod for all $i \in \{0, \ldots, p-1\}$, i.e., in the case of a purely periodic expansion $(t_0 \cdots t_{n-1})^\omega$, we rather consider the writing $t_0(t_1 \cdots t_{n-1}t_0)^\omega$. Then we define the deterministic Büchi automaton $\mathcal{B}_{\boldsymbol{\beta}} = (Q, I', F', A_{\boldsymbol{\beta}}, \delta)$ where the states, the alphabet and the transitions are the same as those of the automaton $\mathcal{A}_{\boldsymbol{\beta}}$, but now the sets of initial and final states are given by $I' = \{(0,0)\}$ and $F' = \{0, \ldots, p-1\} \times \{0\}$.

Proposition 24. *If $\boldsymbol{\beta}$ is a Parry alternate base then the Büchi automaton $\mathcal{B}_{\boldsymbol{\beta}}$ accepts the set $D_{\boldsymbol{\beta}}$.*

Example 11. Consider again the alternate base $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. As explained above, since $d^*_{\boldsymbol{\beta}^{(1)}}(1)$ is purely periodic, we consider the writing $1(01)^\omega$ instead of $(10)^\omega$. We obtain the Büchi automaton $\mathcal{B}_{\boldsymbol{\beta}}$ depicted in Figure 3.

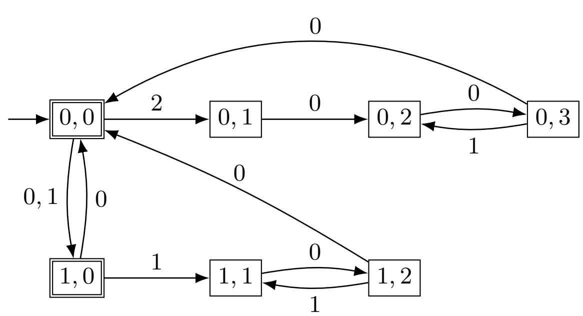

Fig. 3. A deterministic Büchi automaton accepting the set $D_{\boldsymbol{\beta}}$ for $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

By combining the automata $\mathcal{B}_{\boldsymbol{\beta}}$ and $\mathcal{C}(\boldsymbol{\beta}, \boldsymbol{D}, \boldsymbol{D}')$ where $\boldsymbol{D}'$ is the p-tuple $(A_{\beta_0}, \ldots, A_{\beta_{p-1}})$, we can prove the announced result on normalization.

Theorem 11 ([11]). *If δ is a Pisot number and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \{0, \ldots, p-1\}$, then the normalization function $\nu_{\boldsymbol{\beta}, \boldsymbol{D}}$ is computable by a finite automaton.*

6 Ergodic properties of alternate base expansions

In [9], we generalized the β-transformation for a real base β to the setting of alternate bases $\boldsymbol{\beta} = (\beta_0, \dots, \beta_{p-1})$. As in the real base case, this new transformation, denoted $T_{\boldsymbol{\beta}}$, can be iterated in order to generate the digits of the $\boldsymbol{\beta}$-expansions of real numbers. The aim of this section is to report on the results describing the measure theoretical dynamical behavior of $T_{\boldsymbol{\beta}}$. The dynamical properties of a *lazy* $\boldsymbol{\beta}$-transformations $L_{\boldsymbol{\beta}}$ are obtained by showing that the lazy dynamical system is isomorphic to the greedy one. We won't report on the lazy algorithm here, and refer the interested reader to [9,14].

6.1 The alternate $\boldsymbol{\beta}$-transformation

A (measure preserving) dynamical system $(X, \mathcal{F}, \mu, T)$ is said to be *ergodic* if all $B \in \mathcal{F}$ such that $T^{-1}(B) = B$ satisfy $\mu(B) \in \{0,1\}$, and it is said to be *exact* if $\cap_{n=0}^{\infty}\{T^{-n}(B) : B \in \mathcal{F}\}$ only contains sets of measure 0 or 1. Clearly, any exact dynamical system is ergodic.

For two measures μ and ν on the same σ-algebra $\mathcal{F}$, we say that μ is *absolutely continuous with respect to* ν if for all $B \in \mathcal{F}$, $\nu(B) = 0$ implies $\mu(B) = 0$, and we say that μ and ν are *equivalent* if they are absolutely continuous with respect to each other. In what follows, we will be concerned with the Borel σ-algebra $\mathcal{B}([0,1))$.

A fundamental dynamical result of real base expansions is the following. This summarizes results from [29,30,32].

Theorem 12. *There exists a unique T_β-invariant probability measure μ_β on $\mathcal{B}([0,1))$ that is absolutely continuous with respect to the Lebesgue measure λ restricted to $\mathcal{B}([0,1))$. Furthermore, the measures μ_β and λ are equivalent on $\mathcal{B}([0,1))$ and the dynamical system $([0,1), \mathcal{B}([0,1)), \mu_\beta, T_\beta)$ is ergodic and has entropy* $\log(\beta)$.

We now define the $\boldsymbol{\beta}$*-transformation* by

$$T_{\boldsymbol{\beta}} \colon \{0,\dots,p-1\}\times[0,1) \to \{0,\dots,p-1\}\times[0,1),\ (i,x) \mapsto ((i+1) \bmod p, T_{\beta_i}(x)).$$

The $\boldsymbol{\beta}$-transformation generates the digits d_n computed by the greedy algorithm as follows. For all $x \in [0,1)$ and $n \geq 0$, we have $d_n = \lfloor \beta_n \pi_2(T_{\boldsymbol{\beta}}^n(0,x)) \rfloor$ where $\pi_2 \colon \mathbb{N} \times \mathbb{R} \to \mathbb{R},\ (n,x) \mapsto x$.

6.2 Unique absolutely continuous $T_{\boldsymbol{\beta}}$-invariant measure

The following proposition provides us with the main tool for the construction of a $T_{\boldsymbol{\beta}}$-invariant measure. It is proved by using a result of Lasota and Yorke [24].

Proposition 25. *For all integers $n \geq 1$ and all real numbers $\beta_0, \dots, \beta_{n-1} > 1$, there exists a unique $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$-invariant probability measure μ on $\mathcal{B}([0,1))$ that is absolutely continuous with respect to the Lebesgue measure λ restricted to*

$\mathcal{B}([0,1))$. *Furthermore, the measure* μ *is equivalent to* λ *on* $\mathcal{B}([0,1))$, *its density function is bounded and decreasing, and the dynamical system*

$$([0,1), \mathcal{B}([0,1)), \mu, T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$$

is exact and has entropy $\log(\beta_0 \cdots \beta_{n-1})$.

For each $i \in \{0,\ldots,p-1\}$, we let $\mu_{\boldsymbol{\beta},i}$ denote the unique $(T_{\beta_{i+p-1}} \circ \cdots \circ T_{\beta_i})$-invariant absolutely continuous probability measure given by Proposition 25. Let us define a probability measure $\mu_{\boldsymbol{\beta}}$ on the σ-algebra

$$\mathcal{T}_p = \left\{ \bigcup_{i=0}^{p-1} (\{i\} \times B_i) : \forall i \in \{0,\ldots,p-1\},\ B_i \in \mathcal{B}([0,1)) \right\}$$

over $\{0,\ldots,p-1\} \times [0,1)$ as follows. For all $B_0,\ldots,B_{p-1} \in \mathcal{B}([0,1))$, we set

$$\mu_{\boldsymbol{\beta}} \left(\bigcup_{i=0}^{p-1} (\{i\} \times B_i) \right) = \frac{1}{p} \sum_{i=0}^{p-1} \mu_{\boldsymbol{\beta},i}(B_i).$$

Let us also define another measure λ_p over the σ-algebra $\mathcal{T}_p$, which we call the *p-Lebesgue measure*. For all $B_0,\ldots,B_{p-1} \in \mathcal{B}([0,1))$, we set

$$\lambda_p \left(\bigcup_{i=0}^{p-1} (\{i\} \times B_i) \right) = \frac{1}{p} \sum_{i=0}^{p-1} \lambda(B_i). \tag{2}$$

We now state the announced generalization of Theorem 12 to alternate bases.

Theorem 13 ([9]). *The measure* $\mu_{\boldsymbol{\beta}}$ *is the unique* $T_{\boldsymbol{\beta}}$*-invariant probability measure on* $\mathcal{T}_p$ *that is absolutely continuous with respect to* λ_p. *Furthermore,* $\mu_{\boldsymbol{\beta}}$ *is equivalent to* λ_p *on* $\mathcal{T}_p$ *and the dynamical system* $(\{0,\ldots,p-1\} \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}})$ *is ergodic and has entropy* $\frac{1}{p}\log(\beta_0 \cdots \beta_{p-1})$.

For $p \geq 2$, the dynamical system $(\{0,\ldots,p-1\} \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}})$ is not exact even though the dynamical systems $([0,1), \mathcal{B}([0,1)), \mu_{\boldsymbol{\beta},i}, T_{\beta_{i+p-1}} \circ \cdots \circ T_{\beta_i})$ are exact for all $i \in \{0,\ldots,p-1\}$. It suffices to note that the dynamical system $(\{0,\ldots,p-1\} \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}}^p)$ is not ergodic for $p \geq 2$. Indeed, we have $T_{\boldsymbol{\beta}}^{-p}(\{0\} \times [0,1)) = \{0\} \times [0,1)$ whereas $\mu_{\boldsymbol{\beta}}(\{0\} \times [0,1)) = \frac{1}{p}$.

6.3 Density functions

We obtain a formula for the density function $\frac{d\mu_{\boldsymbol{\beta}}}{d\lambda_p}$ by using the density functions $\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda}$ for $i \in \{0,\ldots,p-1\}$.

Proposition 26. *The density function* $\frac{d\mu_{\boldsymbol{\beta}}}{d\lambda_p}$ *of* $\mu_{\boldsymbol{\beta}}$ *with respect to* λ_p *is*

$$\sum_{i=0}^{p-1} \left(\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda} \circ \pi_2 \right) \cdot \chi_{\{i\}\times[0,1)}.$$

6.4 Frequencies

We now turn to the frequencies of the digits in the $\boldsymbol{\beta}$-expansions of real numbers in the interval $[0,1)$. Recall that the frequency of a digit d occurring in the $\boldsymbol{\beta}$-expansion $(a_n)_{n\geq 0}$ of a real number x in $[0,1)$ is equal to

$$\lim_{n\to\infty}\frac{1}{n}\#\{0\leq k<n : a_k=d\},$$

provided that this limit exists.

Proposition 27. *For λ-almost all $x\in[0,1)$, the frequency of any digit d occurring in the $\boldsymbol{\beta}$-expansion of x exists and is equal to*

$$\frac{1}{p}\sum_{i=0}^{p-1}\mu_{\boldsymbol{\beta},i}\left(\left[\tfrac{d}{\beta_i},\tfrac{d+1}{\beta_i}\right)\cap[0,1)\right).$$

Note that, when $p=1$, the previous result gives back the classical formula $\mu_\beta([\frac{d}{\beta},\frac{d+1}{\beta})\cap[0,1))$ where μ_β is the measure given in Theorem 12.

6.5 Isomorphism with the dynamical alternate $\boldsymbol{\beta}$-shift

The aim of this section is to generalize the isomorphism between the β-transformation and the β-shift to the framework of alternate bases. We start by providing some background of the real base case.

For an alphabet A, we let $\mathcal{C}_A$ denote the σ-algebra generated by the *cylinders*

$$C_A(a_0,\ldots,a_{\ell-1})=\{(w_n)_{n\geq 0}\in A^{\mathbb{N}} : w_0=a_0,\ldots,w_{\ell-1}=a_{\ell-1}\}$$

for all $\ell\geq 0$ and $a_0,\ldots,a_{\ell-1}\in A$. Consider the σ-algebra

$$\mathcal{G}_{\boldsymbol{\beta}}=\left\{\bigcup_{i=0}^{p-1}(\{i\}\times C_i) : \forall i\in\{0,\ldots,p-1\},\ C_i\in\mathcal{C}_{A_{\boldsymbol{\beta}}}\cap S_{\boldsymbol{\beta}^{(i)}}\right\}$$

on $\cup_{i=0}^{p-1}(\{i\}\times S_{\boldsymbol{\beta}^{(i)}})$. We define

$$\sigma_p\colon \bigcup_{i=0}^{p-1}(\{i\}\times S_{\boldsymbol{\beta}^{(i)}})\to\bigcup_{i=0}^{p-1}(\{i\}\times S_{\boldsymbol{\beta}^{(i)}}),\ (i,w)\mapsto((i+1)\bmod p,\sigma(w))$$

$$\psi_{\boldsymbol{\beta}}\colon\{0,\ldots,p-1\}\times[0,1)\to\bigcup_{i=0}^{p-1}(\{i\}\times S_{\boldsymbol{\beta}^{(i)}}),\ (i,x)\mapsto(i,d_{\boldsymbol{\beta}^{(i)}}(x)).$$

Note that the transformation σ_p is well defined by Proposition 16.

Proposition 28. *The map $\psi_{\boldsymbol{\beta}}$ defines an isomorphism between the dynamical systems $(\{0,\ldots,p-1\}\times[0,1),\mathcal{T}_p,\mu_{\boldsymbol{\beta}},T_{\boldsymbol{\beta}})$ and $(\cup_{i=0}^{p-1}(\{i\}\times S_{\boldsymbol{\beta}^{(i)}}),\mathcal{G}_{\boldsymbol{\beta}},\mu_{\boldsymbol{\beta}}\circ\psi_{\boldsymbol{\beta}}^{-1},\sigma_p)$.*

However, although $\psi_{\boldsymbol{\beta}}$ is a continuous map, it does not define a topological isomorphism since it is not surjective.

In view of Proposition 28, the set $\cup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}})$ can be thought of as an *alternate $\boldsymbol{\beta}$-shift*, that is, the generalization of the β-shift to alternate bases. However, in Section 3.3, what we called the alternate $\boldsymbol{\beta}$-shift is the topological closure of the union $\cup_{i=0}^{p-1} S_{\boldsymbol{\beta}^{(i)}}$. This definition was motivated by Theorem 5. So we can say that there are two ways to extend the notion of β-shift to alternate bases, depending on the way we look at it: either as a dynamical object or as a combinatorial object.

Thanks to Proposition 28, we obtain an analogue of Theorem 13 for the transformation σ_p.

Theorem 14 ([9]). *The measure $\mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}$ is the unique σ_p-invariant probability measure on $\mathcal{G}_{\boldsymbol{\beta}}$ that is absolutely continuous with respect to $\lambda_p \circ \psi_{\boldsymbol{\beta}}^{-1}$. Furthermore, $\mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}$ is equivalent to $\lambda_p \circ \psi_{\boldsymbol{\beta}}^{-1}$ on $\mathcal{G}_{\boldsymbol{\beta}}$ and the dynamical system $(\cup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}}), \mathcal{G}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}, \sigma_p)$ is ergodic and has entropy $\frac{1}{p}\log(\beta_0 \cdots \beta_{p-1})$.*

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